Introduction to nonsmooth dynamical systems

Lecture 1. Introduction and motivations.

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Cours. “Systèmes dynamiques.”
ENSIMAG 2A

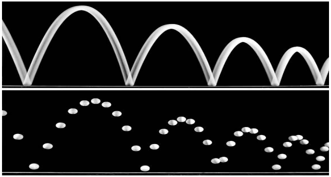
2018–2019

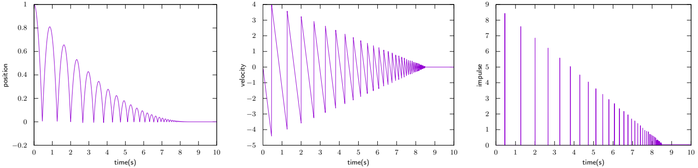
Contents

Outline

Nonsmooth dynamical systems

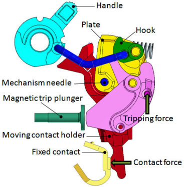
$\boxed{\text{nonsmooth} = \text{lack of continuity/differentiability}}$

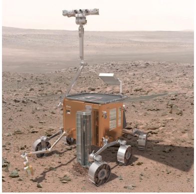




- nonsmooth solutions in time (jumps, kinks, distributions, measures)
- nonsmooth modeling and constitutive laws (set–valued mapping, inequality constraints, complementarity, impact laws)

Application fields.



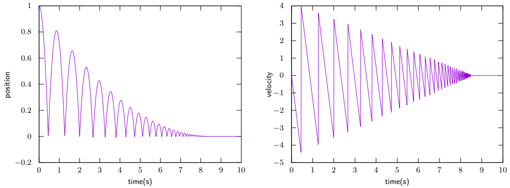


- Computational mechanics. Plasticity. Unilateral contact, Coulomb friction and impacts : multi-body systems, robotic systems, frictional contact oscillators, granular materials.
- Electronics. Switched electrical circuits (digital/analog converters and power electronics, diodes, transistors, switchs).
- Computer science. Hybrid and Cyber–physical systems
- Bio-mathematics. Gene regulatory networks
- Transportation science. Fluid transportation networks with queues.
- Economy and Finance. Oligopolistic market equilibrium

Nonsmooth approach is crucial for a correct modeling and a efficient simulation

Sources of nonsmoothness

- Two largely different time-scales of evolution:
  1. a slow smooth dynamics (free flight of the bouncing ball)
  2. a very fast dynamics (events, transitions, impacts) that can be modeled as a punctual event.



Nonsmooth dynamical systems

Difficulty

Standard tools of numerical analysis and simulation (in finite dimension) are no longer suitable due to the lack of regularity.

Specific tools

Differential measure theory. Convex, nonsmooth and variational Analysis (Clarke, Wets & Rockafellar). Complementarity theory. Maximal monotone operators.

Examples of nonsmooth dynamical systems

- Piecewise smooth systems
- Complementarity systems and differential variational inequality.
- Specific differential inclusions (Filippov, Moreau sweeping process, Normal cone inclusion).

Outline

Example (The RLC circuit with a diode. A half wave rectifier)

A LC oscillator supplying a load resistor through a half-wave rectifier.

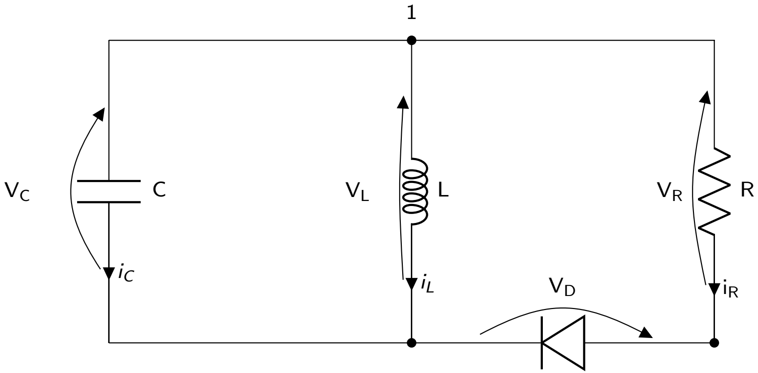
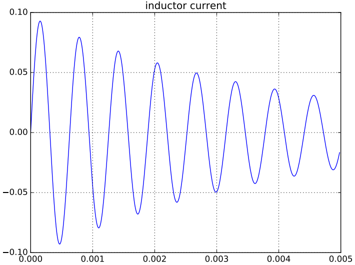
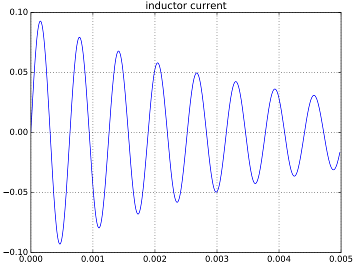


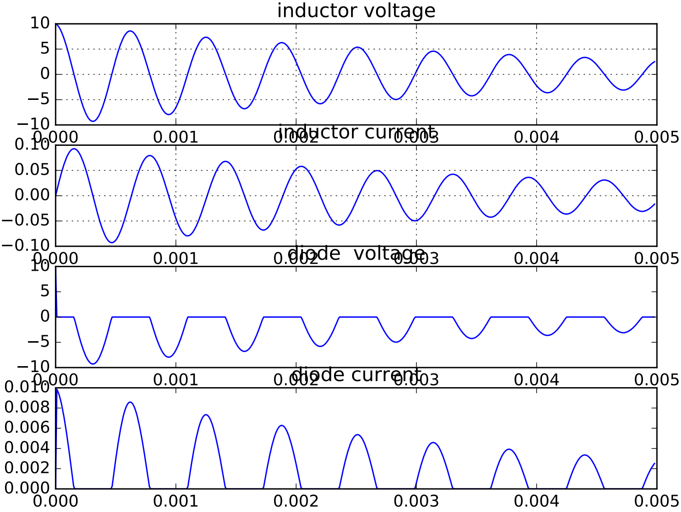
Figure: Electrical oscillator with half-wave rectifier

Example (The RLC circuit with a diode. A half wave rectifier)





Example (The RLC circuit with a diode. A half wave rectifier)



Example (The RLC circuit with a diode. A half wave rectifier)

- Kirchhoff laws :

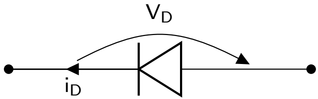
$$\begin{aligned} &v_L = v_C \\ &v_R + v_D = v_C \\ &i_C + i_L + i_R = 0 \\ &i_R = i_D \end{aligned}$$

- Branch constitutive equations for linear devices are :

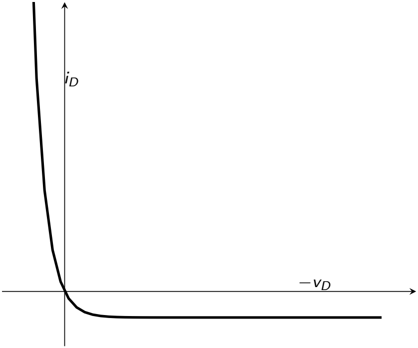
$$\begin{aligned} &i_C = C\dot{v}_C \\ &v_L = L\dot{i}_L \\ &v_R = Ri_R \end{aligned}$$

- "branch constitutive equation" of the ideal diode ?

Example (The RLC circuit with a diode. A half wave rectifier)



(a) A diode



(b) Shockley's law $i_D = i_s(exp(-\frac{v_D}{nv_T}) - 1)$

Figure: A nonlinear model of diode

Example (The RLC circuit with a diode. A half wave rectifier)

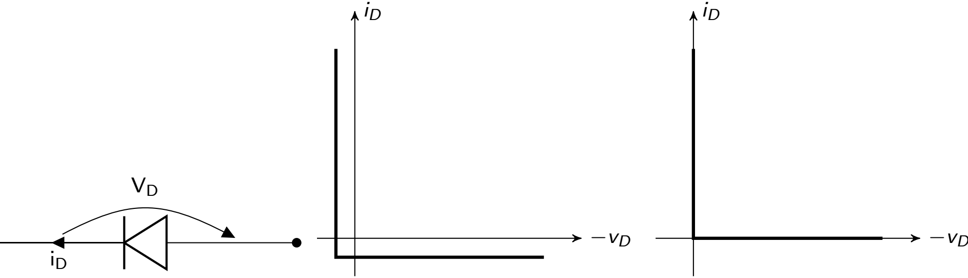


Figure: A ideal diode

Complementarity condition :

$$i_D \geqslant 0, -v_D \geqslant 0, i_D v_D = 0 \Longleftrightarrow 0 \leqslant i_D \perp -v_D \geqslant 0$$

Example (The RLC circuit with a diode. A half wave rectifier)

- Kirchhoff laws :

$$\begin{aligned} &v_L = v_C \\ &v_R + v_D = v_C \\ &i_C + i_L + i_R = 0 \\ &i_R = i_D \end{aligned}$$

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$$\begin{aligned} &i_C = C\dot{v}_C \\ &v_L = L\dot{i}_L \\ &v_R = Ri_R \end{aligned}$$

- "branch constitutive equation" of the ideal diode

$$0 \leqslant i_D \perp -v_D \geqslant 0$$

Example (The RLC circuit with a diode. A half wave rectifier)

The following linear complementarity system is obtained :

$$\begin{pmatrix} \dot{v}_L \\ \dot{i}_L \end{pmatrix} = \begin{pmatrix} 0 & \frac{-1}{C} \\ \frac{1}{L} & 0 \end{pmatrix} \cdot \begin{pmatrix} v_L \\ i_L \end{pmatrix} + \begin{pmatrix} \frac{-1}{C} \\ 0 \end{pmatrix} \cdot i_D$$

together with a state variable x and one of the complementary variables λ :

$$x = \begin{pmatrix} v_L \\ i_L \end{pmatrix}, \qquad \lambda = i_D, \qquad y = -v_D$$

and

$$y = -v_D = \begin{pmatrix} -1 & 0 \end{pmatrix} x + \begin{pmatrix} R \end{pmatrix} \lambda,$$

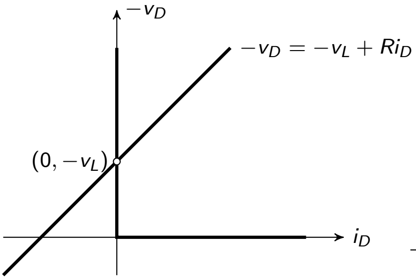
Standard form for LCS

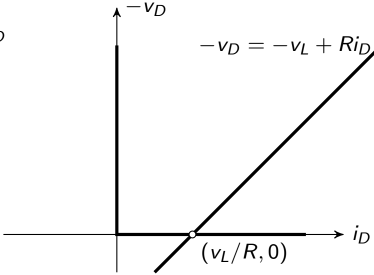
$$\begin{cases} \dot{x} = Ax + B\lambda \\ y = Cx + D\lambda \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases}$$

Example (The RLC circuit with a diode. A half wave rectifier)

$$\left\{\begin{array}{l} y = Cx + D\lambda \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} -v_D = -v_L + R\, i_D \\ 0 \leqslant -v_D \perp i_D \geqslant 0 \end{array}\right. \tag{1}$$

$$\left.\begin{array}{l} \bullet \quad i_D = 0, -v_D = -v_L \geqslant 0, v_L \leqslant 0 \\ \bullet \quad i_D > 0, -v_D = 0, i_D = \frac{V_L}{R}, V_L > 0 \end{array}\right\} \Rightarrow i_D = \max(0, \frac{v_L}{R}) \tag{2}$$





Example (The RLC circuit with a diode. A half wave rectifier)

Note that the lead matrix 0f the LCP $D = \big(R\big) > 0$:

$$\left\{ \begin{array}{l} y = Cx + D\lambda \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{array} \right. \iff \lambda = \mathrm{proj}_{\mathbb{R}_+}(-D^{-1}Cx) = \max(0, -D^{-1}Cx)$$

In the application, $i_D = max(0, \frac{v_L}{R})$ and we get

$$\left(\begin{array}{c} \dot{v}_L \\ \dot{i}_L \end{array} \right) = \left(\begin{array}{cc} 0 & \frac{-1}{C} \\ \frac{1}{L} & 0 \end{array} \right) \cdot \left(\begin{array}{c} v_L \\ i_L \end{array} \right) + \left(\begin{array}{c} \frac{-1}{C} \\ 0 \end{array} \right) \cdot max(0, \frac{v_L}{R})$$

Since max is a Lipschitz operator, we get a standard ODE with Lipschitz r.h.s.

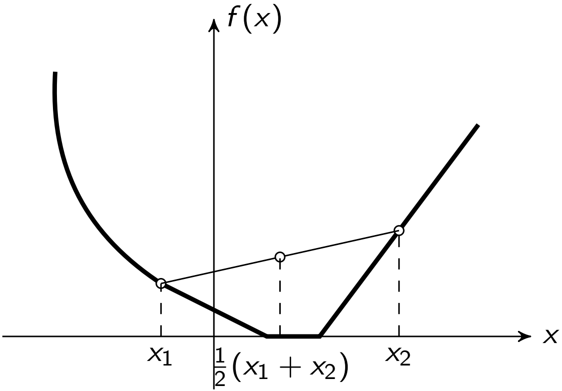
Outline

Convex functions

Definition (Convex function)

A function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup +\infty$ is a convex function if it satisfies

$$f(\alpha x_1 + (1-\alpha)x_2) \leqslant \alpha f(x_1) + (1-\alpha) f(x_2) \text{ for all } x_1, x_2 \in \mathbb{R}^n, \alpha \in [0,1] \quad (1)$$



Convex functions

Definition (Proper convex function)

A convex function $f : \mathbb{R}^n \to \mathbb{R} \cup +\infty$ is proper if $f \not\equiv +\infty$

Definition (Domain of a convex function)

Let $f : \mathbb{R}^n \to \mathbb{R} \cup +\infty$ be a convex function. Its domain $D(f)$ is defined by

$$D(f) = \{x \mid f(x) < +\infty\} \tag{1}$$

Theorem

If $f : \mathbb{R} \to \mathbb{R} \cup +\infty$ is a convex function, then f is Lipschitz continuous on all compact interval $I \subset D(f)$.
If $f : \mathbb{R}^n \to \mathbb{R} \cup +\infty$ is a convex function, then f is locally Lipschitz continuous on all open set $\Omega \subset D(f)$.

Convex functions

The use of functions that may have values in $\mathbb{R} \cup +\infty$ leads to calculations involving $+\infty$ and $-\infty$.
Obvious rules are generally adopted in convex analysis:

$$
\begin{array}{l}
\alpha + \infty = \infty + \alpha = \infty \text{ for } -\infty < \alpha \leqslant \infty \\
\alpha - \infty = -\infty + \alpha = \infty \text{ for } -\infty \leqslant \alpha < \infty \\
\alpha\infty = \infty\alpha = \infty, \quad \alpha(-\infty) = (-\infty)\alpha = -\infty \text{ for } 0 < \alpha \leqslant \infty \\
\alpha\infty = \infty\alpha = -\infty, \quad \alpha(-\infty) = (-\infty)\alpha = \infty \text{ for } -\infty \leqslant \alpha < 0 \\
0\infty = \infty 0 = 0 = 0(-\infty) = (-\infty)0, \quad -(-\infty) = \infty \\
\inf \emptyset = \infty, \sup \emptyset = -\infty
\end{array}
\tag{2}
$$

Some combinations as $+\infty - \infty$ and $-\infty + \infty$ are undefined and forbidden

Convex sets

Definition (Convex set)

A set $C \in \mathbb{R}^n$ is said to be convex if, for all x and y in C and all α in the interval $(0,1)$, the point $(1-\alpha)x + \alpha y$ also belongs to C:

$$\forall \alpha \in (0,1), \forall x \in C, \forall y \in C \implies (1-\alpha)x + \alpha y \in C \tag{3}$$

Convex sets

Definition (Convex set)

A set $C \in \mathbb{R}^n$ is said to be convex if, for all x and y in C and all α in the interval $(0,1)$, the point $(1-\alpha)x + \alpha y$ also belongs to C:

$$\forall \alpha \in (0,1), \forall x \in C, \forall y \in C \implies (1-\alpha)x + \alpha y \in C \tag{3}$$

Properties

- Closed under convex combinations (possible alternative definition)
  If C is a convex set in $\mathbb{R}^n$, then for any collection of r vectors $u_1, \dots u_r$ in C $(r > 1)$ and for any r numbers $\alpha_i \geqslant 0$ such that $\sum_i^r \alpha_i = 1$, we have

$$\sum_i^r \alpha_i u_i \in C \tag{4}$$

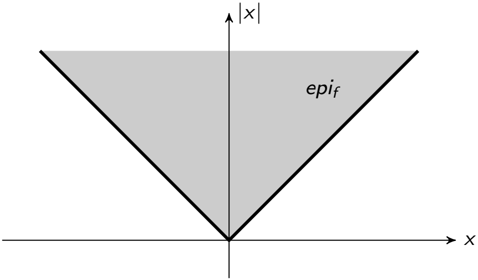
- $\mathbb{R}^n$ and $\emptyset$ are convex
- Any intersection of convex sets is convex

Epigraph

Definition (Epigraph)

$f : \mathbb{R}^n \rightarrow \mathbb{R} \cup +\infty$ a proper function (not necessarily convex)

$$epi_f = \{(y, x) \mid y \geqslant f(x)\} \tag{5}$$



Lemma

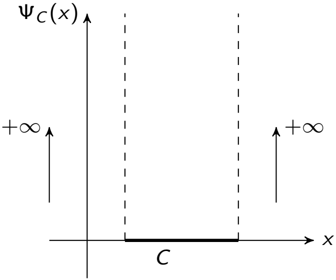
A function is convex if and only if its epigraph is convex

Indicator function of a convex set

Definition (Indicator function of a convex set)

Let C be a nonempty convex set. The indicator of a convex function $\Psi_C(x)$ is defined by

$$\Psi_C(x) = \begin{cases} 0 \text{ if } x \in C \\ +\infty \text{ if } x \notin C \end{cases} \tag{6}$$



Remark

If C is convex, the epigraph of Ψ_C is convex.

Subdifferential of convex functions

Convex functions are not necessarily differentiable. We have only Lipschitz continuity property. How to extend the definition of differentiability to any convex functions?

Definition (Subdifferential of convex functions)

$f : \mathbb{R}^n \to \mathbb{R} \cup +\infty$ a convex function.

$$\partial f(x) = \{\gamma \in \mathbb{R}^n \mid f(y) - f(x) \geqslant \gamma^T(y - x) \text{ for all } y \in \mathbb{R}^n\} \tag{7}$$

Remarks

- The subdifferential can always be computed if the function is proper
- The subdifferential is a set that can be empty.

Standard cases

- If $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, $\partial f(x) = f'(x)$
- If $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable, $\partial f(x) = \nabla f(x)$

Subdifferential of convex functions

Example (Absolute value function $f(x) = |x|$)

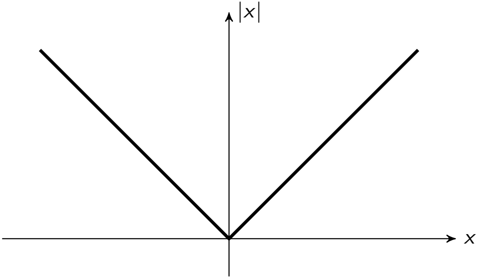


Figure: Absolute value function

Subdifferential of convex functions

Example (Absolute value function $f(x) = |x|$)

$|y| - |x| \geqslant \gamma(y - x)$

- $x > 0, |x| = x, |y| - x \geqslant \gamma(y - x)$

$$\left.\begin{array}{lllll} y = x & & \Rightarrow & \gamma \in \mathbb{R} \\ y > x > 0, & y - x \geqslant \gamma(y - x) & \Rightarrow & \gamma \leqslant 1 \\ x > y > 0, & y - x \geqslant \gamma(y - x) & \Rightarrow & \gamma \geqslant 1 \\ y \leqslant 0, & -y - x \geqslant \gamma(y - x) & \Rightarrow & \gamma = 1 \end{array}\right\} \Rightarrow \gamma = 1 \tag{8}$$

- $x < 0, |x| = x, |y| + x \geqslant \gamma(y - x)$

$$\left.\begin{array}{llll} y = x & & \Rightarrow & \gamma \in \mathbb{R} \\ 0 \geqslant y \geqslant x, & -(y - x) \geqslant \gamma(y - x) & \Rightarrow & \gamma \geqslant -1 \\ y \leqslant x < 0, & -(y - x) \geqslant \gamma(y - x) & \Rightarrow & \gamma \leqslant -1 \\ y \geqslant 0, & y + x \geqslant \gamma(y - x) & \Rightarrow & \gamma = -1 \end{array}\right\} \Rightarrow \gamma = -1 \tag{9}$$

- $x = 0 \ |y| \geqslant \gamma y \Rightarrow \gamma \in [-1, 1]$

Subdifferential of convex functions

Example (Absolute value function $f(x) = |x|$)

$$\partial|x| = \begin{cases} -1 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x > 0 \\ [-1, 1] & \text{if} \quad x = 0 \end{cases} = \mathrm{sgn}(x) \tag{8}$$

where sgn() is the multivalued signum function

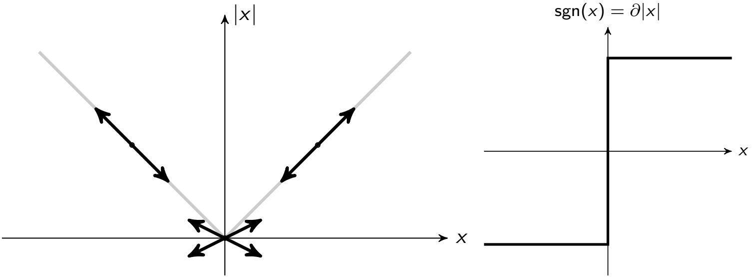


Figure: Absolute value function

Indicator function of a convex set subdifferential

Standard examples

$C = \mathbb{R}_+ \subset \mathbb{R}$.

$$\Psi_{\mathbb{R}_+}(x) = \begin{cases} 0 \text{ if } x \geqslant 0 \\ +\infty \text{ otherwise } . \end{cases} \tag{9}$$

- $x > 0, \quad f(y) \geqslant \gamma(y-x)$

$$\left.\begin{array}{llcl} y \geqslant 0, & 0 \geqslant \gamma(y-x) & \Longrightarrow & \gamma = 0 \\ y < 0, & +\infty \geqslant \gamma(y-x) & \Longrightarrow & \gamma \in \mathbb{R} \end{array}\right\} \Longrightarrow \gamma = 0 \tag{10}$$

- $x = 0, \quad f(y) \geqslant \gamma y$

$$\left.\begin{array}{llcl} y \geqslant 0, & 0 \geqslant \gamma y & \Longrightarrow & \gamma \leqslant 0 \\ y < 0, & +\infty \geqslant \gamma y & \Longrightarrow & \gamma \in \mathbb{R} \end{array}\right\} \Longrightarrow \gamma \leqslant 0 \tag{11}$$

- $x < 0, \quad f(y) - \infty \geqslant \gamma(y-x)$

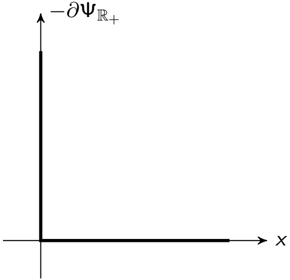
$$\left.\begin{array}{llcl} y \geqslant 0 & -\infty \geqslant \gamma(y-x) & \Longrightarrow & \emptyset \\ y < 0 & \textit{forbidden} & & \end{array}\right\} \Longrightarrow \emptyset \tag{12}$$

Indicator function of a convex set subdifferential

Standard examples

$C = \mathbb{R}_+ \subset \mathbb{R}$.

$$\partial\Psi_{\mathbb{R}_+}(x) = \begin{cases} 0 \text{ if } x > 0 \\ \mathbb{R}_- \text{ if } x = 0 \\ \emptyset \text{ if } x < 0 \end{cases} \tag{9}$$



$$y \in -\partial\Psi_{\mathbb{R}_+}(x)$$

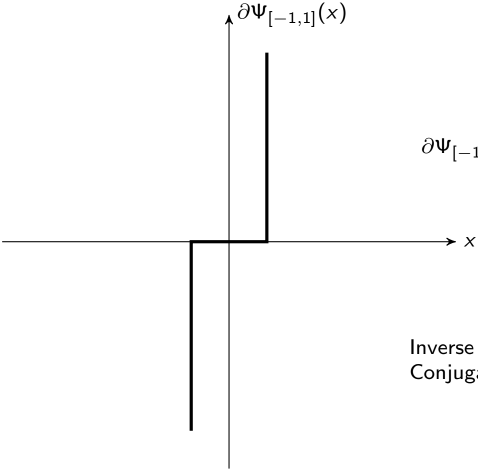
$$\Updownarrow$$

$$0 \leqslant y \perp x \geqslant 0$$

Indicator function of a convex set subdifferential

Standard examples

$C = [-1, 1] \subset \mathbb{R}$



$$\partial\Psi_{[-1,1]}(x) = \begin{cases} \mathbb{R}_- \text{ if } x = -1 \\ 0 \text{ if } -1 < x < 1 \\ \mathbb{R}_+ \text{ if } x = 1 \end{cases} \tag{9}$$

$$y \in \partial\Psi_{[-1,1]}(x)$$

$$\Updownarrow$$

$$x \in \mathrm{sgn}(y)$$

Inverse of the multivalued signum function
Conjugation of convex function

Normal cone to a convex set

Definition (Normal cone to a convex set)

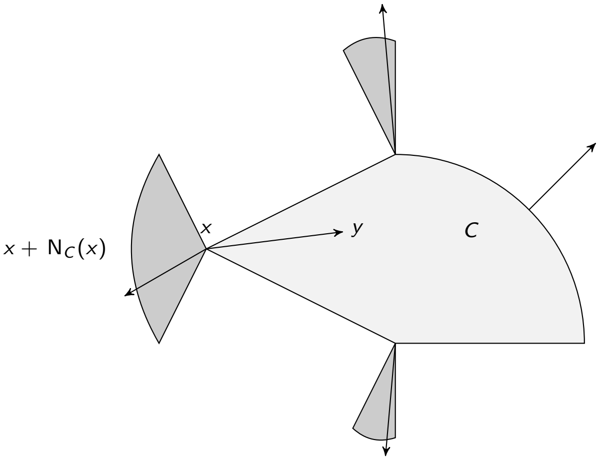
C a nonempty convex set in $\mathbb{R}^n$ and $x \in C$

$$\mathrm{N}_C(x) = \{s \in \mathbb{R}^n \mid s^T(y - x) \leqslant 0 \text{ for all } y \in C\} \tag{10}$$

Properties

- By convention, $\mathrm{N}_X(x) = \emptyset$ for $x \notin C$.
- $x \in int(C) \Rightarrow \mathrm{N}_C(x) = \{0\}$.
- If the boundary is smooth, the normal cone reduces automatically to the standard normal.

Normal cone to a convex set

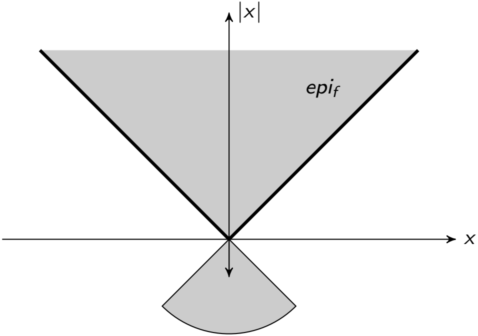


Epigraph and normal cone

Lemma (Epigraph and normal cone)

$f : \mathbb{R}^n \to \mathbb{R} \cup +\infty$ a proper convex function

$$N_{epi_f}(x) = \{(\lambda y, -\lambda) \mid y \in \partial f(x) \text{ and } \lambda \geqslant 0\} \tag{10}$$



Remark

The normal cone is generated by vectors $(y, -1)$ with $y \in \partial f(x)$.

Indicator function of a convex set, normal cone and subdifferential

Lemma

C a nonempty convex set.

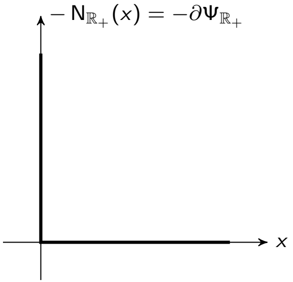
$$\partial \Psi_C(x) = \mathrm{N}_C(x) \tag{11}$$

Indicator function of a convex set, normal cone and subdifferential

Standard examples

$C = \mathbb{R}_+ \subset \mathbb{R}$.

$$\mathrm{N}_{\mathbb{R}_+}(x) = \begin{cases} 0 \text{ if } x > 0 \\ \mathbb{R}_- \text{ if } x = 0 \end{cases} \tag{11}$$

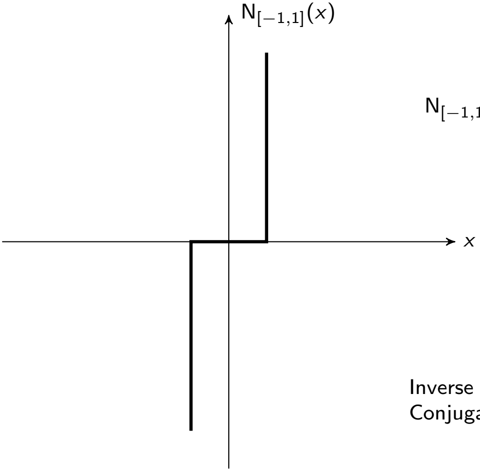


$$-y \in \mathrm{N}_{\mathbb{R}_+}(x)$$
$$\Updownarrow$$
$$-y \in \partial\Psi_{\mathbb{R}_+}(x)$$
$$\Updownarrow$$
$$0 \leqslant y \perp x \geqslant 0$$

Indicator function of a convex set, normal cone and subdifferential

Standard examples

$C = [-1, 1] \subset \mathbb{R}$



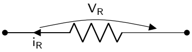
$$\mathrm{N}_{[-1,1]}(x) = \begin{cases} \mathbb{R}_- \text{ if } x = -1 \\ 0 \text{ if } -1 < x < 1 \\ \mathbb{R}_+ \text{ if } x = 1 \end{cases} \tag{11}$$

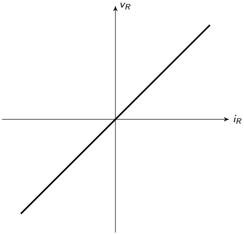
$$y \in \mathrm{N}_{[-1,1]}(x)$$
$$\Updownarrow$$
$$y \in \partial\Psi_{[-1,1]}(x)$$
$$\Updownarrow$$
$$x \in \mathrm{sgn}(y)$$

Inverse of the multivalued signum function
Conjugation of convex function

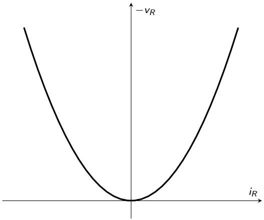
Nonsmooth power and energy

component	characteristic	power function
resistor	linear	quadratic





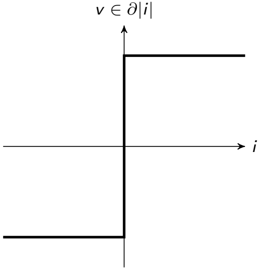
$$v_R = Ri_R$$



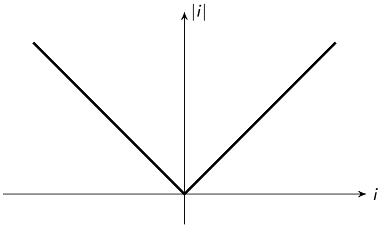
$$P = \frac{1}{2} v_r i_R = \frac{1}{2} R i_R^2$$

Nonsmooth power and energy

component	characteristic	power function
relay	sign function	indicator of $\mathbb{R}_+$



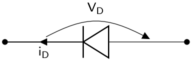
$v \in \mathrm{sgn}(i)$

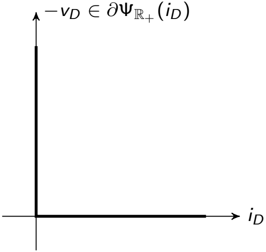


$P = |i|$

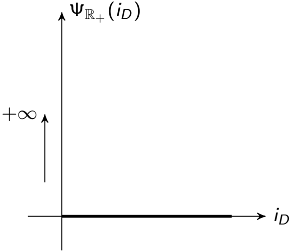
Nonsmooth power and energy

component	characteristic	power function
diode	complementarity	indicator of $\mathbb{R}_+$





$0 \leqslant -v_D \perp i_D \geqslant 0$



$P = \Psi_{\mathbb{R}_+}(i_D)$