Introduction to nonsmooth dynamical systems

Lecture 2. Mathematical formalisms

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Contents

- Complementarity systems
- Differential inclusions
- Variational inequalities,
- Existence and uniqueness results.

Practical work : study of a slider with friction and basic circuits with a diode.

Outline

Complementarity Systems (CS)

Differential inclusion

Linear Complementarity Systems (LCS)

Linear Complementarity Systems (LCS)

$$\left\{\begin{array}{l}\dot{x}(t) = Ax(t) + B\lambda(t) + a, \quad x(0) = x_0 \\ y(t) = Cx(t) + D\lambda(t) + b \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0.\end{array}\right. \tag{1}$$

Concept of solutions

- The solution to the LCS (1) depends strongly on the quadruplet (A, B, C, D) and the initial conditions
- We will review the simplest cases
  - C^1 solutions, when D is a P-matrix
  - Absolutely continuous (AC) solutions when $D = 0$, $CB \geqslant 0$ and consistent initial solutions

Mathematical nature of the solutions

In order to say more on the mathematical properties of the LCS, we need to characterize the solution λ of

$$\begin{cases} y = Cx + D\lambda + b \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{2}$$

of its equivalent formulation in terms of inclusion into a subdifferential

$$-\,(Cx + D\lambda + b) \in \partial \Psi_{\mathbb{R}^m_+}(\lambda) \tag{3}$$

or in terms of variational inequality

$$(Cx + D\lambda + b)^T(\tau - \lambda) \geqslant 0,\ \text{for all } \tau \in \mathbb{R}^m_+ \tag{4}$$

Linear Complementarity Problem

Definition (LCP)

A *Linear complementarity problem* (LCP) is to find a vector λ that satisfies

$$0 \leqslant \lambda \perp M\lambda + q \geqslant 0$$

Theorem (Fundamental result of complementarity theory)

The LCP $0 \leqslant \lambda \perp M\lambda + q \geqslant 0$ *has a unique solution* λ^* *for any* $q \in \mathbb{R}^m$ *if and only if* M *is a P-matrix.*
In this case the solution λ^* *is a piecewise linear function of* q *(with a finite number of pieces).*

Remarks

- A P-matrix has all its principal minors positive. A positive definite matrix is a P-matrix.
- A symmetric P-matrix is a positive definite matrix.
- There exist non-symmetric P-matrices which are not positive definite. And there exist positive definite matrices which are not symmetric!

Solutions as continuously differentiable functions (C^1 solutions)

ODE with Lipschitz right-hand–side

The substitution of $\lambda(x)$ yields a Ordinary Differential Equation (ODE) with a Lipschitz right–hand–side.
➜ Solutions as continuously differentiable functions (C^1 solutions)

The LCS case

The solution $\lambda(x)$ of the following linear complementarity system

$$0 \leqslant \lambda \perp D\lambda + Cx + b \geqslant 0 \tag{5}$$

is unique for all $Cx + b$ if and only if D is a P-Matrix and moreover $\lambda(x)$ is a Lipschitz function of x.

see the example of the RLCD circuit

Solutions as absolutely continuous functions (AC solutions)

The LCS case with $D = 0$ and $b = 0$

If we consider the LCS (1) with $D = 0$ and $b = 0$, we get

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + a, \quad x(0) = x_0 \\ y(t) = Cx(t) \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0. \end{cases} \tag{6}$$

Regularity: What should we expect ?

The time-derivative of the state $\dot{x}(t)$ and $\lambda(t)$ are expected to be, in this case, discontinuous functions of time.

Indeed, if the output $y(t)$ reaches the boundary of the feasible domain at time t_*, i.e., $y(t_*) = 0$, the time–derivative $\dot{y}(t)$ needs to jump if $\dot{y}(t_*) < 0$

Solutions as absolutely continuous functions (*AC* solutions)

Example (Scalar LCS with $D = 0$)

Let us search for a continuous solution $x(t)$ to

$$\begin{cases} x(0) = x_0 > 0 \\ \dot{x}(t) = -x(t) - 1 + \lambda(t) \\ 0 \leqslant x(t) \perp \lambda(t) \geqslant 0 \end{cases}$$

Two modes :

- free dynamics for $0 < t < t_*$ with $x(t) > 0$ and $x(t_*) = 0$:

$$\begin{cases} x(0) = x_0 > 0 \\ \dot{x}(t) = -x(t) - 1 \end{cases} \tag{7}$$

  Solution :

$$x(t) = \exp(-t)x_0 + \exp(-t) - 1 \tag{8}$$

$$x(t_*) = 0 \implies t_* = -\ln(\tfrac{1}{1+x_0}) > 0$$

- dynamics for $t \geqslant t_*$

$$\begin{cases} x(t_*) = 0, \\ \dot{x}(t) + 1 = \lambda(t) \geqslant 0 \end{cases} \tag{9}$$

Solutions as absolutely continuous functions (AC solutions)

Example (Scalar LCS with $D = 0$)

Solving the dynamics for $t_* \leqslant t < T$:

$$\begin{cases} x(t_*) = 0 \\ \dot{x}(t) + 1 = \lambda(t) \geqslant 0 \end{cases} \tag{7}$$

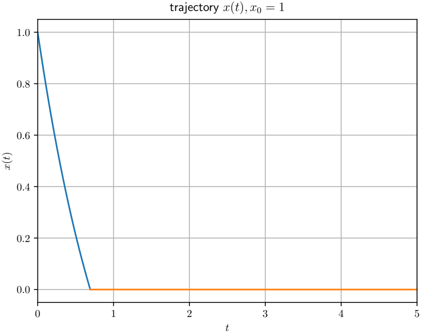
if we are looking for an abs. continuous solution $x(t)$, the abs. continuity and $x(t_*) = 0$ implies that $\dot{x}(t) \geqslant 0, t \in [t_*, t_* + \varepsilon), \varepsilon > 0$, otherwise $x(t_* + \varepsilon) < 0$.

1. $\dot{x}(t) > 0, t \in [t_*, t_* + \varepsilon), \varepsilon > 0$.
   By continuity, $x(t + \epsilon) > 0, \lambda(t + \varepsilon) = 0$ then
   $$\dot{x}(t + \varepsilon) = -x(t + \epsilon) - 1 < 0 \tag{8}$$
   No solution.
2. $\dot{x}(t) = 0, \lambda(t) = 1, x(t) = 0 \qquad \forall t \geqslant t_* \; (T = +\infty \;)$
   The only possible continuous solution.

Solutions as absolutely continuous functions (*AC* solutions)

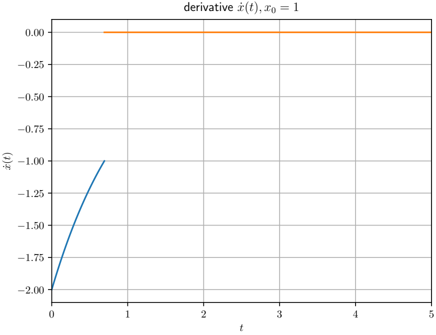
Example (Scalar LCS with $D = 0$)

Conclusion: A continuous $x(t)$ has been computed for all $t \in [0, +\infty)$. The time derivative of the solution $\dot{x}(t)$ jumps at from t_* from $x(t_*^-) = -1$ to $x(t_*^+) = 0$.



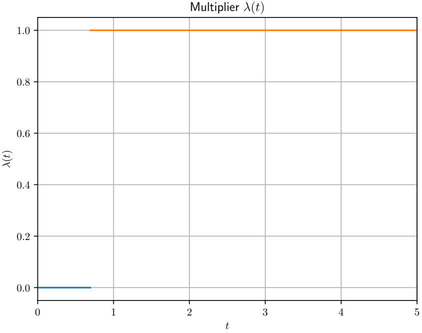
Solutions as absolutely continuous functions (AC solutions)

Example (Scalar LCS with $D = 0$)



Solutions as absolutely continuous functions (AC solutions)

Example (Scalar LCS with $D = 0$)



Solutions as absolutely continuous functions (AC solutions)

Idea of the general statement

If CB is a positive definite matrix (relative degree *one*) and $Cx_0 \geqslant 0$ (consistent initial condition), the unique solution of (31) is an absolutely continuous function.

Why the condition on CB ?

Derivation of the output $y(t)$

$$\begin{array}{rcl} y(t) & = & Cx(t) \\ \dot{y}(t) & = & CAx(t) + CB\lambda(t) \text{ if } D = 0 \end{array} \tag{7}$$

If $CB > 0$, we have to solve the following LCP whenever $y(t) = 0$

$$\begin{cases} \dot{y}(t) = CAx(t) + CB\lambda(t) \\ 0 \leqslant \dot{y}(t) \perp \lambda(t) \geqslant 0 \end{cases} \tag{8}$$

The LCP (8) is a LCP for the time derivative $\dot{y}(t)$.

The good framework is the differential inclusion framework (see later)

Existence and uniqueness results for LCS. Summary

Linear Complementarity Systems (LCS)

$$\left\{\begin{array}{l} \dot{x}(t) = Ax(t) + B\lambda(t) + a, \quad x(0) = x_0 \\ y(t) = Cx(t) + D\lambda(t) + b \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0. \end{array}\right. \tag{9}$$

LCS with D a P-matrix

ODE with Lipschitz continuous right-hand side.
Cauchy–Lipschitz Theorem $\implies$ existence and uniqueness of solutions.

LCS with $D = 0$

Existence and uniqueness results based on

- Local (or nonzeno) solution based on the leading Markov parameters assumptions $(D, CB, CAB, CA^2B, ..)$
- or maximal monotone differential inclusion

Extensions of complementarity problems

Let C be a nonempty closed convex set. The subdifferential inclusion continues to hold

$$-y \in \partial \Psi_C(\lambda) \tag{10}$$

The complementarity relation is no longer valid for a set convex that is not a cone, but we can define the following dynamics

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + u(t) \\ y(t) = Cx(t) + D\lambda(t) + a(t) \\ -y(t) \in \partial \Psi_C(\lambda(t)) \end{cases} \tag{11}$$

Extensions of complementarity problems

Relay systems

$C = [-1, 1]$

$$\partial\Psi_{[-1,1]}(\lambda) = \begin{cases} \mathbb{R}_- \text{ if } \lambda = -1 \\ 0 \text{ if } -1 < \lambda < 1 \\ \mathbb{R}_+ \text{ if } \lambda = 1 \end{cases} \tag{12}$$

Equivalent formulations

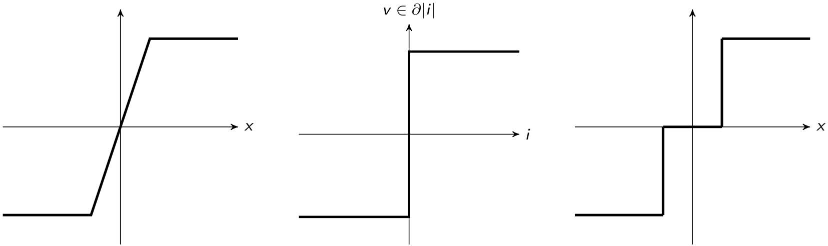
$$y \in \partial\Psi_{[-1,1]}(\lambda) \Longleftrightarrow \lambda \in \mathrm{sgn}(y)$$

Definition (Relay systems)

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + u(t) \\ y(t) = Cx(t) + D\lambda(t) + a(t) \\ \lambda(t) \in \mathrm{sgn}(y(t)) \end{cases} \tag{13}$$

Application in sliding mode control, zener diode modeling or friction in mechanical systems

Piecewise linear systems with monotone graphs



Extensions of complementarity problems

Cone complementarity condition

Let K be a closed non empty convex cone. We can define

$$K^\star \ni y \perp \lambda \in K \iff -y \in \partial\Psi_K(\lambda) \iff -\lambda \in \partial\Psi_{K^\star}(y) \tag{14}$$

where $K^\star$ is the dual cone:

$$K^\star = \{x \in \mathbb{R}^m \mid x^\top y \geqslant 0 \text{ for all } y \in K\}. \tag{15}$$

Definition (Cone Linear complementarity systems (CLCS))

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + u(t) \\ y(t) = Cx(t) + D\lambda(t) + a(t) \\ K^\star \ni y(t) \perp \lambda(t) \in K, \end{cases} \tag{16}$$

Outline

Complementarity Systems (CS)

Differential inclusion

Differential inclusion

Complementarity condition as a subdifferential inclusion

$$0 \leqslant y \perp \lambda \geqslant 0 \iff -y \in \partial \Psi_{\mathbb{R}^m_+}(\lambda) \iff -\lambda \in \partial \Psi_{\mathbb{R}^m_+}(y) \tag{17}$$

LCS as a differential inclusion with $D = 0$ and $b = 0$

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + a \\ y(t) = Cx(t) \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0 \\ x(0) = x_0. \end{cases} \iff \begin{cases} -(\dot{x}(t) - Ax(t) - a) \in B\partial\Psi_{\mathbb{R}^m_+}(Cx(t)), \\ \quad x(0) = x_0 \end{cases} \tag{18}$$

General differential inclusion

Concept of differential inclusions

Differential inclusions is a generalization of the concept of differential equations of the form

$$\dot{x}(t) \in A(x(t), t) \tag{19}$$

where $(x, t) \mapsto A(x, t)$ is a multi-valued map, *i.e.* $A(x, t)$ is a set rather than a single point.

A very general concept

Differential inclusions is a very general concept that contains Ordinary Differential Equations (ODE), Differential Algebraic Equations (DAE). There are many types if differential inclusions.

We will focus on Maximal Monotone Differential Inclusion

Maximal monotone operators

Let $2^{\mathbb{R}^n}$ be the set of the subsets of $\mathbb{R}^n$

Definition (Monotone multi-valued operator)

A multi–valued operator $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is monotone if

$$\forall y_1 \in T(x_1), \quad \forall y_2 \in T(x_2), \quad (y_2 - y_1)^T (x_2 - x_1) \geqslant 0 \tag{20}$$

Definition (Graph)

Let T multi–valued operator $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$. The graph of T is defined by

$$Gr(T) = \{(x, y) \mid y \in T(x)\} \tag{21}$$

Definition (Maximal Monotone multi-valued operator)

A operator T is maximal monotone if it is maximal for all the monotone operators for the inclusion of graphs.

In other words, T is monotone and for all other monotone operator S then $Gr(T) \subset Gr(S) \implies T = S$

Maximal monotone operators

Definition (Domain)

The domain of an operator T is defined by $D(T) = \{x \mid T(x) \neq \emptyset\}$

Definition (Range of T)

Let $T : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ be an operator. The range of T is defined by

$$R(T) = \cup_{x \in \mathbb{R}^n} \{y \mid y \in T(x)\} \tag{22}$$

Definition (Inverse of T)

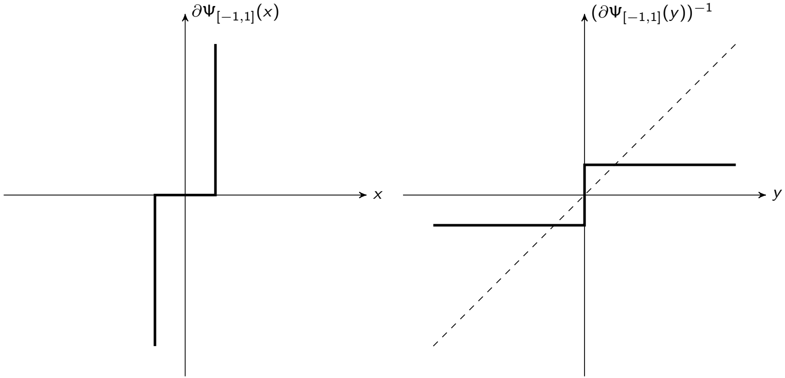
Let $T : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ be a maximal monotone operator. Its inverse T^{-1} is defined by

$$y \in T(x) \Longleftrightarrow x \in T^{-1}(y) \tag{23}$$

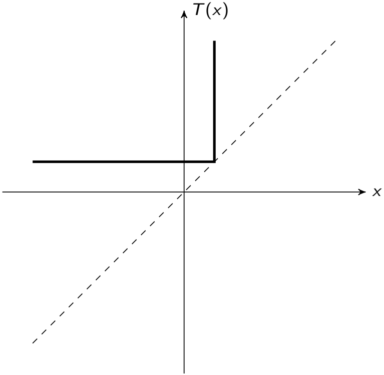
and we have $D(T^{-1}) = R(T)$ and $R(T^{-1}) = D(T)$

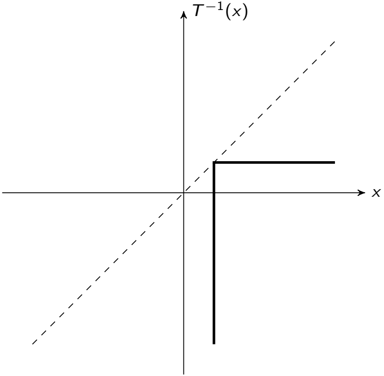
Its inverse is defined by the symmetry of its graph with respect to $y = x$

Maximal monotone operators

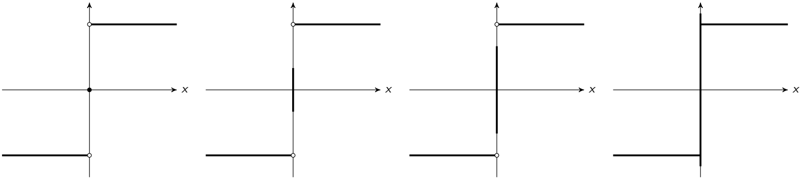


Maximal monotone operators

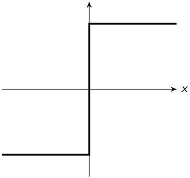




Maximal monotone operators



$\mathrm{sgn}(x) = \partial|x|$



Maximal monotone differential inclusion

Definition (Maximal monotone differential inclusion)

Let T multi–valued operator $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$. A maximal monotone differential inclusion is defined by

$$-\dot{x}(t) \in T(x(t)) \tag{22}$$

Definition (Perturbed maximal monotone differential inclusion)

Let T multi–valued operator $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$. A maximal monotone differential inclusion is defined by

$$-(\dot{x}(t) + f(x,t)) \in T(x(t)) \tag{23}$$

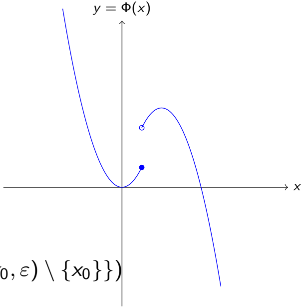
where f is a Lipschitz continuous map w.r.t x.

Maximal monotone differential inclusion

Definition (lower semi-continuity)

A function $\Phi : \mathbb{R}^n \to \mathbb{R} \cup +\infty$ is lower semi-continuous if one of the following equivalent assertions is satisfied:

- $$\liminf_{x \to x_0} \Phi(x) \geqslant \Phi(x_0)$$
- Its epigraph is closed



Remarks

- $\liminf_{x \to x_0} \Phi(x) = \lim_{\varepsilon \to 0}(\inf\{\Phi(x), x \in B(x_0, \varepsilon) \setminus \{x_0\}\})$
- Continuity implies semi-continuity.

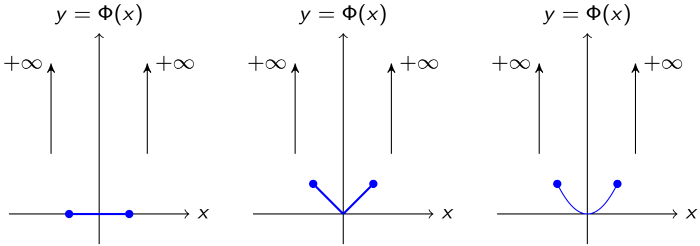
Maximal monotone differential inclusion

For a convex proper function Φ, the semi–continuity property has only to be checked on the boundary of the domain of definition

$$\partial D(\Phi) = \overline{D(\Phi)} \setminus \mathring{D}(\Phi)$$

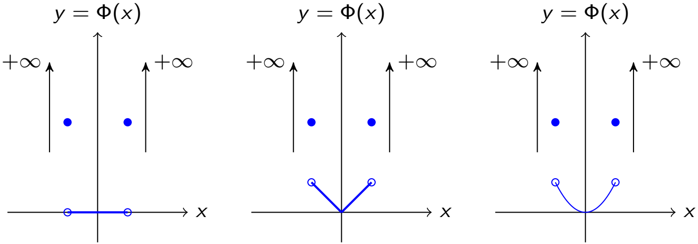
.

Examples



Maximal monotone differential inclusion

Counter-examples



Maximal monotone differential inclusion

Theorem

For a lower semi–continuous convex proper function Φ, the subdifferential $\partial\Phi(x)$ is a maximal monotone operator

Remarks

- Obvious in the regular case: $\phi(x) : \mathbb{R} \rightarrow \mathbb{R}$ a convex potential C^2
  $\phi''(x) \geqslant 0$ and $\phi'(x)$ is monotone (increasing single–valued function)
- For a maximal monotone operator in $\mathbb{R}$, i.e. $T : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ it exists a lower semi–continuous convex proper function Φ such that $T = \partial\Phi$

Maximal monotone differential inclusion

Examples

- $\Phi(x) = 0 = \Psi_{\mathbb{R}}, \, T(x) = 0$

$$-\dot{x} + f(x,t) = 0 \tag{24}$$

- $\Phi(x) = \Psi_c(x), \, T(x) = \partial \Psi_C(x)$

$$-\dot{x} + f(x,t) \in \partial \Psi_C(x) \tag{25}$$

- relay or sign function $\Phi(x) = |x|, \, T(x) = \partial |x|$

$$-\dot{x} \in \partial |x| \iff -\dot{x} \in \mathrm{sgn}(x) \tag{26}$$

- 2-norm $\Phi(x) = \|x\|, \;\; T(x) = \partial \|x\| = \begin{cases} \frac{x}{\|x\|} & \text{if} \quad x \neq 0 \\ \{s \mid \|s\| \leqslant 1\} & \text{if} \quad x = 0 \end{cases}$

Maximal monotone differential inclusion

Examples

- relay with dead zone

$$\Phi(x) = \begin{cases} -x+1, \text{ if } x \leqslant -1 \\ 0, \text{ if } -1 \leqslant x \leqslant 1 \\ x-1, \text{ if } x \geqslant 1 \end{cases} \tag{24}$$

Maximal monotone differential inclusion

Examples

- Sum of (proper) convex functions $\Phi_1 + \Phi_2$ is convex. Moreover, if the relative interior $\mathrm{ri}(D(\partial\Phi_1))$ and $\mathrm{ri}(D(\partial\Phi_2)$ have a common point then

$$\partial(\Phi_1(x) + \Phi_2(x)) = \partial\Phi_1(x) + \partial\Phi_2(x) \tag{24}$$

  Relative interior : $\mathrm{ri}(X) = \{x \in X \mid \exists \varepsilon > 0, B_\varepsilon \cap \mathrm{Aff}(X) \subset X\}$ where $\mathrm{Aff(X)}$ is the affine hull of X, the smallest affine set containing X:

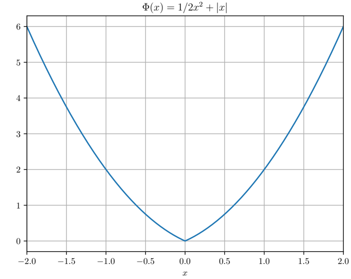
$$\mathrm{Aff}(X) = \{\sum_{i=0}^{k} \alpha_i x_i \mid k > 0, x_i \in X, \alpha_i \in \mathbb{R}, \sum_{i=0}^{k} \alpha_i = 1\} \tag{25}$$

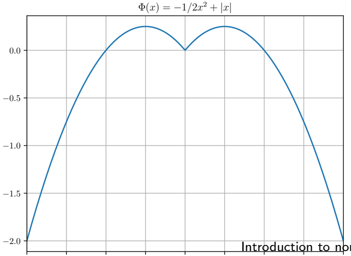
  Ex: $C = \{x \in \mathbb{R}^2 \mid x_1 \in [-1, 1], x_2 = 0\}$ $\mathrm{Aff}(C) = \mathbb{R} \times \{0\}$
- $\Phi(x) = 1/2 * ax^2 + |x|$, $T(x) = ax + \mathrm{sgn}(x)$

$$-\dot{x} \in ax + \partial|x| \iff -\dot{x} - ax \in \mathrm{sgn}(x) \tag{26}$$

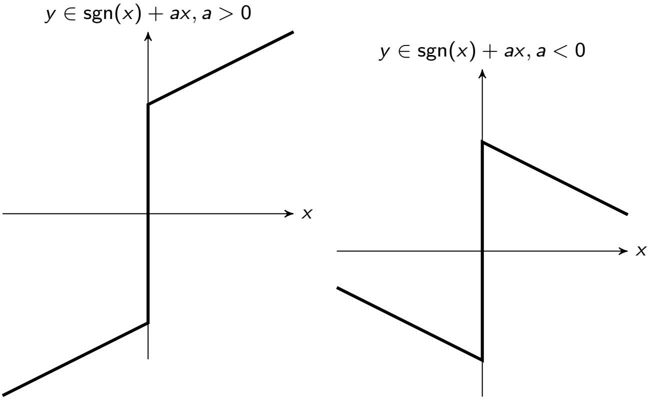
  1. $a > 0$. $\Phi(x)$ is convex and $T(x)$ is maximal monotone.
  2. $a < 0$. $\Phi(x)$ is not convex and $T(x)$ is not monotone.

Maximal monotone differential inclusion





Maximal monotone differential inclusion



Maximal monotone differential inclusion

Link with gradient systems with convex potentials

- $\phi(x) : \mathbb{R} \to \mathbb{R}$ a convex potential C^2
  $\phi''(x) \geqslant 0$ and $\phi'(x)$ is monotone (increasing function)

$$-\dot{x} = \phi'(x) \tag{24}$$

- $\Phi(x) : \mathbb{R} \to \mathbb{R}$ a convex potential not necessarily differentiable, but proper and lower semi–continuous $\partial\Phi(x)$ is a maximal monotone operator.

$$-\dot{x} = \partial\Phi(x) \tag{25}$$

Existence and uniqueness results

Theorem (Brézis 1973)

Let $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a maximal monotone operator such that $D(\mathring{T}) \neq \emptyset$. Let a function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ such that

1. *the function $f(x, \cdot)$ is Lipschitz continuous on $D(T)$ that is*

$$\exists L \geqslant 0, \forall t \in [0, t_{\max}], \forall x_1, x_2 \in \overline{D(T)}, \quad \|f(t, x_1) - f(t, x_2)\| \leqslant L\|x_1 - x_2\| \tag{26}$$

2. $\forall x \in \overline{D(T)}$*, the mapping $t \mapsto f(x, t)$ belongs to $\mathcal{L}^\infty(0, t_{\max}; \mathbb{R}^n)$*

Then, for all $x_0 \in \overline{D(T)}$, it exists a unique solution $x(t)$ which is absolutely continuous such that

$$\begin{cases} -(\dot{x}(t) + f(x(t), t)) \in T(x(t)), \text{ almost everywhere on } [0, t_{\max}] \\ x(0) = x_0 \end{cases} \tag{27}$$

Existence and uniqueness results

Existence

- By using the Moreau-Yosida regularization of T

$$T_\lambda(x) = \frac{1}{\lambda}(I - J_\lambda(x)), \lambda > 0, \tag{28}$$

with $J_\lambda(x)$ the resolvent of $T(x)$ given by

$$J_\lambda(x) = (I + \lambda T(x))^{-1}. \tag{29}$$

For a maximal monotone operator T or $\mathbb{R}$, J_λ is defined over $\mathbb{R}$ and is contracting. The mapping T_λ is a maximal monotone operator and Lipschitz continuous with a Lipschitz constant of $\frac{1}{\lambda}$. We consider that ODE with Lipschitz r.h.s.

$$-(\dot{x}_\lambda(t) + f(x_\lambda(t), t)) = T_\lambda(x_\lambda(t)) \tag{30}$$

and then the limit $\lambda \to 0$ of the sequence of solutions x_λ.

- By approximation using a discretization scheme

Existence and uniqueness results

Uniqueness

Simple case $-\dot{x}(t) \in T(x(t))$. $x \in \mathbb{R}$
Let us consider two solution x_1 and x_2
Since $T(x)$ is monotone, we have

$$(\dot{x}_1(s) - \dot{x}_2(s))^T (x_1(s) - x_2(s)) \leqslant 0 \text{ almost everywhere on } [0, T] \tag{28}$$

By integrating over $[0, t]$, we get

$$\frac{1}{2}(x_2(t) - x_1(t))^2 - \frac{1}{2}(x_2(0) - x_1(0))^2 \leqslant 0 \tag{29}$$

If $x_1(0) = x_2(0)$, we have

$$\frac{1}{2}(x_2(t) - x_1(t))^2 \leqslant 0 \implies x_2 = x_1 \tag{30}$$

Existence and uniqueness results

Uniqueness

$-(\dot{x}(t) + f(x,t)) \in T(x(t))$
Let us consider two solution x_1 and x_2
Since $T(x)$ is monotone, we have

$$(\dot{x}_1(s) + f(x_1(s),s) - \dot{x}_2(s) - f(x_2(s),s))^T(x_1(s) - x_2(s)) \leqslant 0 \tag{28}$$

almost everywhere on $[0, T]$.
By integrating over $[0, t]$, we get

$$\frac{1}{2}(x_2(t) - x_1(t))^2 \leqslant \int_0^t (f(x_2(s),s) - f(x_1(s),s))^T(x_1(s) - x_2(s))ds \tag{29}$$

Since f is lipschitz, we have

$$(x_2(t) - x_1(t))^2 \leqslant 2L\int_0^t \|x_1(s) - x_2(s)\|^2 ds \tag{30}$$

Existence and uniqueness results

Gronwall Lemma

Let a a positive constant and m a integrable function, nonnegative almost everywhere on $(0, t_{\max})$ and a function ϕ a continuous function on $[0, t_{\max}]$. If

$$\forall t \in [0, t_{\max}], \phi(t) \leqslant a + \int_0^t m(s)\phi(s)\, ds \tag{28}$$

then

$$\forall t \in [0, t_{\max}], \phi(t) \leqslant a \exp(\int_0^t m(s)\, ds) \tag{29}$$

Applying the Gronwall Lemma, for $a = 0$ and $m(s) = 2L$ and $\phi(s) = \|x_1(s) - x_2(s)\|^2$, we get

$$\|x_2(t) - x_1(t))\|^2 \leqslant 0 \implies x_2 = x_1 \tag{30}$$

Come back to LCS with $D = 0$ but $B \neq I_d \neq C$

Theorem (LCS as maximal monotone differential inclusion)

Let us consider the following LCS

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + a(t), \quad x(0) = x_0 \\ y(t) = Cx(t) \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0. \end{cases} \tag{31}$$

If there exists P a symmetric definite positive matrix such that

$$PB = C^T \tag{32}$$

then we can perform a change of variable $z = Rx$ with $R^2 = P, R \geqslant 0, R = R^T$

$$-(\dot{z}(t) - RAR^{-1}z(t) - Ra(t)) \in RB\,\partial\Psi_{\mathbb{R}^m_+}(CR^{-1}z(t)) \tag{33}$$

such that (33) is a maximal monotone differential inclusion.

Come back to LCS with $D = 0$ but $B \neq I_d \neq C$

We have the following equivalence

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + a(t) \\ y(t) = Cx(t) \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0, \\ x(0) = x_0 \end{cases} \iff \begin{cases} -(\dot{x}(t) - Ax(t) - a(t)) \in B\partial\Psi_{\mathbb{R}^m_+}(Cx(t)), \\ \quad x(0) = x_0 \end{cases} \tag{31}$$

We can perform a change of variable $z = Rx$ with $R^2 = P, R \geqslant 0, R = R^T$

$$-(\dot{z}(t) - RAR^{-1}z(t) - Ra(t)) \in RB\,\partial\Psi_{\mathbb{R}^m_+}(CR^{-1}z(t)) \tag{32}$$

Come back to LCS with $D = 0$ but $B \neq I_d \neq C$

For a matrix E, the function $\phi(x) = \Psi_{\mathbb{R}^m_+}(Ex)$ is a proper convex function and its subdifferential is given by

$$\partial\phi(x) = E^T \partial\Psi_{\mathbb{R}^m_+}(Ex) \tag{31}$$

($\mathrm{Im}(E)$ contains a point of $\mathrm{ri}(D(\partial\Psi_{\mathbb{R}^m_+})))$ (Chain rule)
In our application, we set $E = CR^{-1}$ and we have

$$E^T = R^{-T} C^T = R^{-1} R^2 B = RB \tag{32}$$

The obtained inclusion

$$-(\dot{z}(t) - RAR^{-1}z(t) - Ra) \in \partial\Phi(z(t)) = E^T \partial\Psi_{\mathbb{R}^m_+}(Ez(t)), \tag{33}$$

is a maximal monotone differential inclusion