Introduction to nonsmooth dynamical systems

Lecture 2. Complementarity systems

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Cours. "Systèmes dynamiques."
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Contents

Outline

Linear Complementarity Systems (LCS)

Linear Complementarity Systems (LCS)

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + a, \quad x(0) = x_0 \\ y(t) = Cx(t) + D\lambda(t) + b \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0. \end{cases} \tag{1}$$

Concept of solutions

- The solution to the LCS (1) depends strongly on the quadruplet (A, B, C, D) and the initial conditions
- We will review the simplest cases
  - D is a P-matrix
    ➜ C^1 solutions.
  - $D = 0$, $CB \geqslant 0$ and consistent initial solutions
    ➜ Absolutely Continuous (AC) solutions

Mathematical nature of the solutions

In order to say more on the mathematical properties of the LCS, we need to characterize the solution λ of

$$\begin{cases} y = Cx + D\lambda + b \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{2}$$

of its equivalent formulation in terms of inclusion into a subdifferential

$$-(Cx + D\lambda + b) \in \partial \Psi_{\mathbb{R}^m_+}(\lambda) \tag{3}$$

Linear Complementarity Problem

Definition (Linear Complementarity Problem (LCP))

A *Linear complementarity problem* (LCP) is to find a vector $\lambda \in \mathbb{R}^m$ that satisfies

$$0 \leqslant \lambda \perp M\lambda + q \geqslant 0 \tag{4}$$

for a given matrix $M \in \mathbb{R}^{m \times m}$ and a vector $q \in \mathbb{R}^m$.

Comments

- A LCP is often formulated as:

$$\begin{cases} w = M\lambda + q, \\ 0 \leqslant w \perp \lambda \geqslant 0. \end{cases} \tag{5}$$

Linear Complementarity Problem

Link with quadratic programming (QP)

If $M = M^\top \succ 0$, the LCP is the necessary and sufficient optimality condition to the following quadratic problem

$$\begin{array}{ll} \min_\lambda & \frac{1}{2}\lambda^\top M\lambda + \lambda^\top q \\ s.t. & \lambda \geqslant 0 \end{array} \tag{4}$$

or equivalently

$$\min_\lambda \quad \frac{1}{2}\lambda^\top M\lambda + \lambda^\top q + \psi_{\mathbb{R}_+}(\lambda) \tag{5}$$

Hints : Write the optimality condition of a convex QP

Linear Complementarity Problem

Theorem (Fundamental result of complementarity theory)

The LCP

$$0 \leqslant \lambda \perp M\lambda + q \geqslant 0$$

has a unique solution λ^* for any $q \in \mathbb{R}^m$ if and only if M is a P-matrix.
In this case the solution λ^* is a piecewise linear function of q (with a finite number of pieces).

Remarks

- A P-matrix has all its principal minors positive. A positive definite matrix is a P-matrix.
- A symmetric P-matrix is a positive definite matrix.
- There exist non-symmetric P-matrices which are not positive definite. And there exist positive definite matrices which are not symmetric!

Solutions as continuously differentiable functions (C^1 solutions)

ODE with Lipschitz right-hand–side

The substitution of $\lambda(x)$ yields a Ordinary Differential Equation (ODE) with a Lipschitz right–hand–side.
Cauchy-Lipschitz Theorem ➜ Existence and uniqueness of a solution as continuously differentiable functions (C^1 solutions)

The LCS case

The solution $\lambda(x)$ of the following linear complementarity system

$$0 \leqslant \lambda \perp D\lambda + Cx + b \geqslant 0 \qquad (6)$$

is unique for all $Cx + b$ if and only if D is a P-Matrix and moreover $\lambda(x)$ is a Lipschitz function of x.

see the example of the RLCD circuit

Solutions as absolutely continuous functions (AC solutions)

The LCS case with $D = 0$ and $b = 0$

If we consider the LCS (1) with $D = 0$ and $b = 0$, we get

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + a, \quad x(0) = x_0 \\ y(t) = Cx(t) \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0. \end{cases} \tag{7}$$

Regularity: What should we expect ?

The time-derivative of the state $\dot{x}(t)$ and $\lambda(t)$ are expected to be, in this case, discontinuous functions of time.
Indeed, if the output $y(t)$ reaches the boundary of the feasible domain at time t_*, i.e., $y(t_*) = 0$, the time–derivative $\dot{y}(t)$ needs to jump if $\dot{y}(t_*) < 0$

Solutions as absolutely continuous functions (*AC* solutions)

Example (Scalar LCS with $D = 0$)

Let us search for a continuous solution $x(t)$ to

$$\begin{cases} x(0) = x_0 > 0 \\ \dot{x}(t) = -x(t) - 1 + \lambda(t) \\ 0 \leqslant x(t) \perp \lambda(t) \geqslant 0 \end{cases}$$

Two modes :

- free dynamics for $0 < t < t_*$ with $x(t) > 0$ and $x(t_*) = 0$:

$$\begin{cases} x(0) = x_0 > 0 \\ \dot{x}(t) = -x(t) - 1 \end{cases} \tag{8}$$

  Solution :

$$x(t) = \exp(-t)x_0 + \exp(-t) - 1 \tag{9}$$

  $x(t_*) = 0 \implies t_* = -\ln(\frac{1}{1+x_0}) > 0$

- dynamics for $t \geqslant t_*$

$$\begin{cases} x(t_*) = 0, \\ \dot{x}(t) + 1 = \lambda(t) \geqslant 0 \end{cases} \tag{10}$$

Solutions as absolutely continuous functions (AC solutions)

Example (Scalar LCS with $D = 0$)

Solving the dynamics for $t_* \leqslant t < T$:

$$\begin{cases} x(t_*) = 0 \\ \dot{x}(t) + 1 = \lambda(t) \geqslant 0 \end{cases} \tag{8}$$

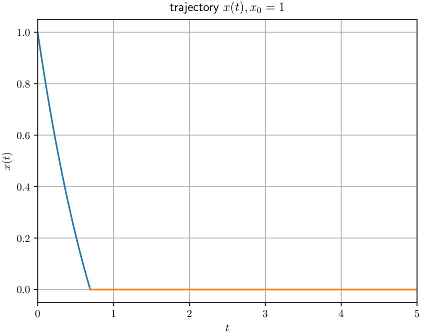
if we are looking for an abs. continuous solution $x(t)$, the abs. continuity and $x(t_*) = 0$ implies that $\dot{x}(t) \geqslant 0, t \in [t_*, t_* + \varepsilon), \varepsilon > 0$, otherwise $x(t_* + \varepsilon) < 0$.

1. $\dot{x}(t) > 0, t \in [t_*, t_* + \varepsilon), \varepsilon > 0$.
   By continuity, $x(t + \epsilon) > 0, \lambda(t + \varepsilon) = 0$ then
   $$\dot{x}(t + \varepsilon) = -x(t + \epsilon) - 1 < 0 \tag{9}$$
   No solution.
2. $\dot{x}(t) = 0, \lambda(t) = 1, x(t) = 0 \qquad \forall t \geqslant t_* \ (T = +\infty\ )$
   The only possible continuous solution.

Solutions as absolutely continuous functions (*AC* solutions)

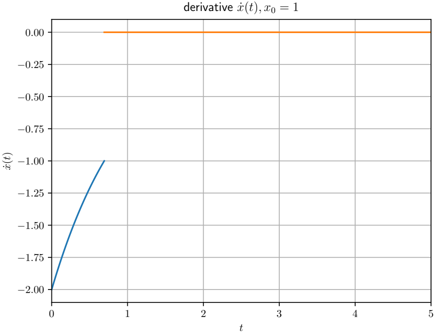
Example (Scalar LCS with $D = 0$)

Conclusion: A unique continuous $x(t)$ has been computed for all $t \in [0, +\infty)$. The time derivative of the solution $\dot{x}(t)$ jumps at from t_* from $x(t_*^-) = -1$ to $x(t_*^+) = 0$.



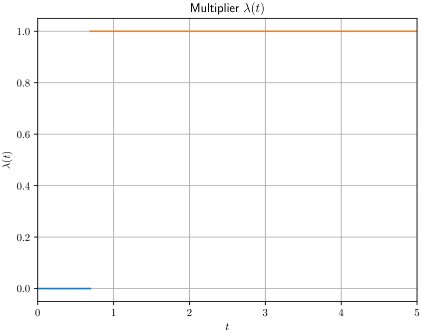
Solutions as absolutely continuous functions (*AC* solutions)

Example (Scalar LCS with $D = 0$)



Solutions as absolutely continuous functions (AC solutions)

Example (Scalar LCS with $D = 0$)



Solutions as absolutely continuous functions (*AC* solutions)

Example (Scalar LCS with $D = 0$)

Let us search for a continuous solution $x(t)$ to

$$\begin{cases} x(0) = x_0 > 0 \\ \dot{x}(t) = -x(t) + 1 - \lambda(t) \\ 0 \leqslant x(t) \perp \lambda(t) \geqslant 0 \end{cases}$$

- $x(t) > 0$ for $0 < t < t_*$ (free dynamics) :

$$\begin{cases} x(0) = x_0 > 0 \\ \dot{x}(t) = -x(t) + 1 \end{cases} \tag{8}$$

Solution :

$$x(t) = \exp(-t)(x_0 - 1) + 1 > 0 \tag{9}$$

solution for all $t \in [0; +\infty]$

Solutions as absolutely continuous functions (*AC* solutions)

Example (Scalar LCS with $D = 0$)

Let us search for a continuous solution $x(t)$ to

$$\begin{cases} x(0) = x_0 = 0 \\ \dot{x}(t) = -x(t) + 1 - \lambda(t) \\ 0 \leqslant x(t) \perp \lambda(t) \geqslant 0 \end{cases}$$

- $x(t) > 0$ for $0 < t < t_*$ (free dynamics) :

$$\begin{cases} x(0) = x_0 = 0 \\ \dot{x}(t) = -x(t) + 1 \end{cases} \tag{8}$$

Solution :

$$x(t) = \exp(-t)(x_0 - 1) + 1 > 0, \text{ for all} t \in [0; +\infty] \tag{9}$$

- $x(t) = 0$ for $0 < t < t_*$ (constrained dynamics):
$\dot{x}(t) = 0, \lambda(t) = 1, x(t) = 0$

Solutions as absolutely continuous functions (AC solutions)

Example (Scalar LCS with $D = 0$)

Conclusion

- A unique continuous $x(t)$ has been computed for $x_0 > 0$ for all $t \in [0, +\infty)$.
- Infinitely many continuous $x(t)$ have been computed for x_0 for all $t \in [0, +\infty)$.

Solutions as absolutely continuous functions (*AC* solutions)

Example (Scalar LCS with $D = 0$)

Let us search for a continuous solution $x(t)$ to

$$\begin{cases} x(0) = x_0 \geqslant 0 \\ \dot{x}(t) = -x(t) - 1 - \lambda(t) \\ 0 \leqslant x(t) \perp \lambda(t) \geqslant 0 \end{cases}$$

Conclusion

- A unique maximal continuous $x(t)$ has been computed for $x_0 > 0$ for $t \in [0, t_\star)$. No solution after $t_\star$
- No continuous solutions for $x_0 = 0$.

Solutions as absolutely continuous functions (*AC* solutions)

Idea of the general statement

If CB is a positive definite matrix (relative degree *one*) and $Cx_0 \geqslant 0$ (consistent initial condition), the unique solution of (10) is an absolutely continuous function.

Why the condition on CB ?

Derivation of the output $y(t)$

$$\begin{array}{rcl} y(t) & = & Cx(t) \\ \dot{y}(t) & = & CAx(t) + CB\lambda(t) \text{ if } D = 0 \end{array} \tag{8}$$

If $CB > 0$, we have to solve the following LCP whenever $y(t) = 0$

$$\begin{cases} \dot{y}(t) = CAx(t) + CB\lambda(t) \\ 0 \leqslant \dot{y}(t) \perp \lambda(t) \geqslant 0 \end{cases} \tag{9}$$

The LCP (9) is a LCP for the time derivative $\dot{y}(t)$.

The good framework is the differential inclusion framework (see Lecture 3)

Existence and uniqueness results for LCS. Summary

Linear Complementarity Systems (LCS)

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + a, \quad x(0) = x_0 \\ y(t) = Cx(t) + D\lambda(t) + b \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0. \end{cases} \tag{10}$$

LCS with D a P-matrix

ODE with Lipschitz continuous right-hand side.
Cauchy–Lipschitz Theorem $\Longrightarrow$ existence and uniqueness of solutions.

LCS with $D = 0$

Existence and uniqueness results based on

- Local (or nonzeno) solution based on the leading Markov parameters assumptions $(D, CB, CAB, CA^2B, ..)$
- or maximal monotone differential inclusion

Extensions of complementarity problems

Let C be a nonempty closed convex set. The subdifferential inclusion continues to hold

$$-y \in \partial \Psi_C(\lambda) \tag{11}$$

The complementarity relation is no longer valid for a set convex that is not a cone, but we can define the following dynamics

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + u(t) \\ y(t) = Cx(t) + D\lambda(t) + a(t) \\ -y(t) \in \partial \Psi_C(\lambda(t)) \end{cases} \tag{12}$$

Extensions of complementarity problems

Relay systems

$C = [-1, 1]$

$$\partial\Psi_{[-1,1]}(\lambda) = \begin{cases} \mathbb{R}_- \text{ if } \lambda = -1 \\ 0 \text{ if } -1 < \lambda < 1 \\ \mathbb{R}_+ \text{ if } \lambda = 1 \end{cases} \tag{13}$$

Equivalent formulations

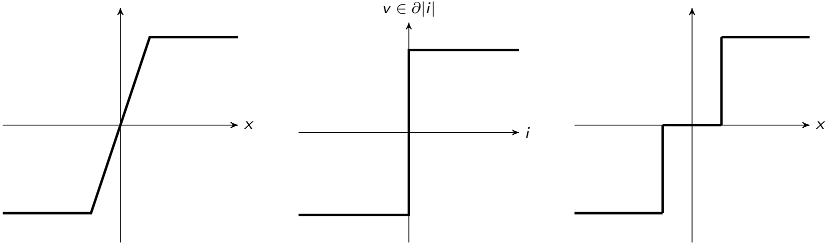
$$y \in \partial\Psi_{[-1,1]}(\lambda) \Longleftrightarrow \lambda \in \text{sgn}(y)$$

Definition (Relay systems)

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + u(t) \\ y(t) = Cx(t) + D\lambda(t) + a(t) \\ \lambda(t) \in \text{sgn}(y(t)) \end{cases} \tag{14}$$

Application in sliding mode control, zener diode modeling or friction in mechanical systems

Piecewise linear systems with monotone graphs



Extensions of complementarity problems

Cone complementarity condition

Let K be a closed non empty convex cone. We can define

$$K^\star \ni y \perp \lambda \in K \iff -y \in \partial \Psi_K(\lambda) \iff -\lambda \in \partial \Psi_{K^\star}(y) \tag{15}$$

where $K^\star$ is the dual cone:

$$K^\star = \{x \in \mathbb{R}^m \mid x^\top y \geqslant 0 \text{ for all } y \in K\}. \tag{16}$$

Definition (Cone Linear complementarity systems (CLCS))

$$\left\{\begin{array}{l} \dot{x}(t) = Ax(t) + B\lambda(t) + u(t) \\ y(t) = Cx(t) + D\lambda(t) + a(t) \\ K^\star \ni y(t) \perp \lambda(t) \in K, \end{array}\right. \tag{17}$$

Outline

Complementarity Systems (CS)

Computations of equilibria for LCS

Stability of Linear Complementarity Systems
- Linear Time Invariant (LTI) passive systems
- Lyapunov stability of LCS

Equilibria for LCS

Linear Complementarity Systems (LCS)

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + a, \quad x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\ y(t) = Cx(t) + D\lambda(t) + b \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0 \end{cases} \tag{18}$$

Mixed Linear Complementarity Problem (MLCP)

We have to solve a Mixed Linear Complementarity Problem :

$$\begin{cases} 0 = A\tilde{x} + B\lambda + a, \\ y = C\tilde{x} + D\lambda + b \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{19}$$

Equilibria for LCS

Existence of solutions to MLCP

- Trivial case $a = 0, b = 0$. $\tilde{x} = 0$ is an equilibrium.
- If A invertible, then we can substitute $\tilde{x} = -A^{-1}(B\lambda + a)$ to get a LCP

$$0 \leqslant (D - CA^{-1}B)\lambda + A^{-1}a + b \perp \lambda \geqslant 0 \tag{20}$$

If $(D - CA^{-1}B)$ is a P-matrix, it exists a unique solution λ for all a and b. The equilibrium is obtained with $\tilde{x} = -A^{-1}(B\lambda + a)$

Equilibria for LCS

Existence of solutions to MLCP

Reformulation into inclusion

$$-\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} \tilde{x} \\ \lambda \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}\right) \in \partial\Psi_{\mathbb{R}^n\times\mathbb{R}^m_+}\left(\begin{bmatrix} \tilde{x} \\ \lambda \end{bmatrix}\right) \tag{21}$$

as

$$-(Mz+q) \in \partial\Psi_{\mathbb{R}^n\times\mathbb{R}^m_+}(z) \tag{22}$$

Theorem

If M is a semi–definite positive matrix, then the inclusion (22) is solvable if and only if it is feasible, that is

$$\exists z, \quad z \in \mathbb{R}^n \times \mathbb{R}^m_+ \text{ and } Mz+q \in 0^n \times \mathbb{R}^m_+ \tag{23}$$

Application of a more general Theorem 2.4.7 of [**?**].

Example

Trivial case $a = 0, b \geqslant 0$.

Outline

Lyapunov stability (Recap.)

Definition (Lyapunov stability)

The equilibrium $\tilde{x}$ is said to be stable in the sense of Lyapunov if

$$\text{for every } \varepsilon > 0, \exists \delta > 0, \text{ such that } \|x(0) - \tilde{x}\| < \delta \text{ then } \|x(t) - \tilde{x}\| < \varepsilon, \forall t \geqslant 0. \quad (24)$$

Definition (Asymptotic Lyapunov stability)

The equilibrium $\tilde{x}$ is said to be asymptotically stable in the sense of Lyapunov if

- it is stable and
- for every $\varepsilon > 0, \exists \delta > 0,$ such that $\|x(0) - \tilde{x}\| < \delta$ then $\lim_{t \to +\infty} \|x(t) - \tilde{x}\| = 0$

Definition (Exponential Lyapunov stability)

The equilibrium $\tilde{x}$ is said to be asymptotically stable in the sense of Lyapunov if

- it is asymptotically stable and
- $\exists \alpha, \beta, \delta > 0,$ such that $\|x(0) - \tilde{x}\| < \delta$ then $\|x(t) - \tilde{x}\| \leqslant \alpha \|x(0) - \tilde{x}\| e^{-\beta t}, \forall t \geqslant 0$

LTI passive systems

Linear Time Invariant (LTI) systems

Let us consider the following system:

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) \\ y(t) = Cx(t) + D\lambda(t) \end{cases} \tag{25}$$

with a quadratic function $V(x) = \frac{1}{2}x^T Px$ with $P = P^T$.
Let us define the composition:

$$\begin{array}{rccc} \mathcal{V}(t): & \mathbb{R} & \to & \mathbb{R} \\ & t & \mapsto & V(x(t)) \end{array} \tag{26}$$

LTI passive systems

Derivation of $\mathcal{V}(t)$

$$\dot{\mathcal{V}}(t) = x^T(t)P\dot{x}(t) \tag{25}$$

$$\begin{array}{c}
x^T(t)P\dot{x}(t) = x^T(t)PAx(t) + x^T(t)B\lambda(t) \\
\Updownarrow \\
x^T(t)P\dot{x}(t) - \lambda^T(t)y(t) = x^T(t)PAx(t) + x^T(t)PB\lambda(t) - \lambda^T(t)y(t) \\
\Updownarrow \\
x^T(t)P\dot{x}(t) - \lambda^T(t)y(t) = x^T(t)PAx(t) + \lambda^T(t)B^TPx(t) - \lambda^T(t)(Cx(t) + D\lambda(t)) \\
\Updownarrow \\
x^T(t)P\dot{x}(t) - \lambda^T(t)y(t) = x^T(t)PAx(t) + \lambda^T(t)(B^TP - C)x(t) - \lambda^T(t)D\lambda(t)
\end{array} \tag{26}$$

LTI passive systems

Derivation of $\mathcal{V}(t)$

$$
\begin{aligned}
V(x(T) - V(x(0)) \quad & - \quad \int_0^T \lambda^T(t)y(t)dt \\
& = \quad \int_0^T x^T(t)PAx(t) + \lambda^T(t)(B^TP - C)x(t) - \lambda^T(t)(D\lambda(t))dt \\
& = \quad \frac{1}{2}\int_0^T \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}^T \begin{bmatrix} A^TP + PA & PB - C^T \\ B^TP - C & -(D + D^T) \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} dt
\end{aligned}
\tag{25}
$$

LTI passive systems

Linear Time Invariant (LTI) systems

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) \\ y(t) = Cx(t) + D\lambda(t) \end{cases} \tag{26}$$

Definition

The system $\Sigma(A, B, C, D)$ given in (26) is said to be passive (dissipative with respect to the supply rate $\lambda^T y$) is there exists a function $V : \mathbb{R}^n \to \mathbb{R}_+$ (a storage function) such that

$$V(x(t_0)) + \int_{t_0}^{t} \lambda^T(t)y(t)dt \geqslant V(x(t)) \tag{27}$$

holds for all t_0 and t with $t \geqslant t_0$ and for all $\mathcal{L}^2$-solutions (x, y, λ).

LTI passive systems

Theorem

The system $\Sigma(A, B, C, D)$ is passive if and only if the following linear matrix inequality (LMI)

$$P = P^T > 0 \text{ and } \begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -(D + D^T) \end{bmatrix} \leqslant 0 \tag{28}$$

has a solution.
In this case, $V(x) = \frac{1}{2}x^T Px$ is the corresponding energy storage function.

LTI passive systems

Theorem

The system $\Sigma(A, B, C, D)$ *is passive if there exist matrices* $L \in \mathbb{R}^{n\times m}$ *and* $W \in \mathbb{R}^{m\times m}$ *and a symmetric positive semi-definite matrix* $P \in \mathbb{R}^{n\times n}$*, such that:*

$$\begin{cases} A^T P + PA = -LL^T & (29) \\ B^T P - C = -W^T L^T & (30) \\ -D - D^T = -W^T W. & (31) \end{cases}$$

LTI passive systems

Reformulation

$$\begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -(D + D^T) \end{bmatrix} = - \begin{bmatrix} LL^T & LW \\ W^T L^T & W^T W \end{bmatrix} = - \begin{bmatrix} L \\ W \end{bmatrix}^T \begin{bmatrix} L \\ W \end{bmatrix} \triangleq -Q \quad (29)$$

LTI passive systems

Dissipation inequality

The *dissipation equality*

$$V(x(T))-V(x(0)) = \frac{1}{2}\int_0^T \lambda^T(t)y(t)+\frac{1}{2}\int_0^T \begin{bmatrix}x(t)\\ \lambda(t)\end{bmatrix}^T Q \begin{bmatrix}x(t)\\ \lambda(t)\end{bmatrix} dt, \quad \forall \; T \geqslant 0 \quad (29)$$

in terms of the positive semi-definite matrix

$$Q \triangleq \begin{pmatrix} LL^T & W^T L^T \\ LW & W^T W \end{pmatrix}, \quad (30)$$

then implies that

$$V(x(T)) - V(x(0)) - \frac{1}{2}\int_0^T \lambda^T(t)y(t) \leqslant 0. \quad (31)$$

Strictly passive LTI systems

The system is said to be *strictly passive* when Q is positive definite.

LTI passive systems

Remarks

- $(D + D^T) \geqslant 0$ implies that D is a semi-definite positive matrix.
- if $D = 0$, then $(D + D^T) = W^T W = 0 \implies W = 0$ and we get

$$B^T P - C = -W^T L^T = 0 \implies C = B^T P \implies CB = B^T PB \geqslant 0 \tag{32}$$

The matrix CB is a semi-definite positive matrix

Passive LCS

Assumption

The trajectory $x(t)$ of the LCS is continuous.

Definition (Passive LCS)

The LCS

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) \\ y(t) = Cx(t) + D\lambda(t) \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0 \end{cases} \tag{33}$$

is said to be (strictly) passive if the system $\Sigma(A, B, C, D)$ is (strictly) passive

Supply rate

The complementarity condition implies that $\lambda^T(t)y(t) = 0$ for all $t \geqslant 0$. Then the dissipation inequality reduces to

$$V(x(T)) - V(x(0)) \leqslant 0 \tag{34}$$

Lyapunov stability of LCS

Theorem

- *If the LCS is passive, then the LCS is Lyapunov stable.*
- *If the LCS is strictly passive, then the LCS is globally exponentially stable.*

The energy storage function plays the role of a Lyapunov function.

Lyapunov stability of LCS

- If the LCS is passive, then D is a semi-definite positive matrix

Lyapunov stability of LCS

Example (The RLC circuit with a diode. A half wave rectifier)

A LC oscillator supplying a load resistor through a half-wave rectifier.

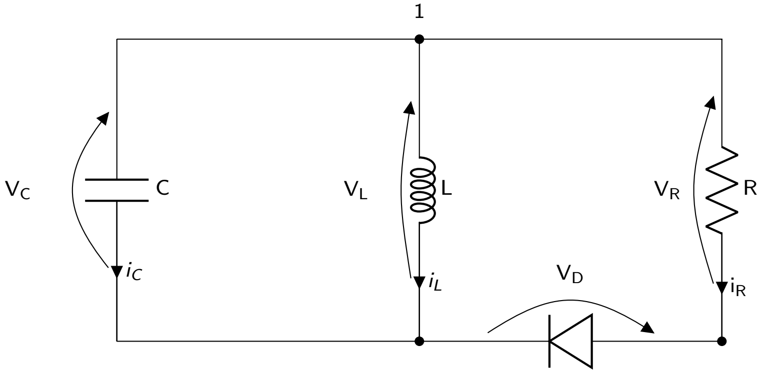


Figure: Electrical oscillator with half-wave rectifier

Lyapunov stability of LCS

Example (The RLC circuit with a diode. A half wave rectifier)

The following linear complementarity system is obtained :

$$\begin{pmatrix} \dot{v}_C \\ \dot{i}_L \end{pmatrix} = \begin{pmatrix} 0 & \frac{-1}{C} \\ \frac{1}{L} & 0 \end{pmatrix} \cdot \begin{pmatrix} v_C \\ i_L \end{pmatrix} + \begin{pmatrix} \frac{-1}{C} \\ 0 \end{pmatrix} \cdot i_D$$

together with a state variable x and one of the complementary variables λ :

$$x = \begin{pmatrix} v_C \\ i_L \end{pmatrix}, \qquad \lambda = i_D, \qquad y = -v_D$$

and

$$y = -v_D = \begin{pmatrix} -1 & 0 \end{pmatrix} x + \begin{pmatrix} R \end{pmatrix} \lambda,$$

Standard form for LCS

$$\begin{cases} \dot{x} = Ax + B\lambda \\ y = Cx + D\lambda \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases}$$

Lyapunov stability of LCS

Example (The RLC circuit with a diode. A half wave rectifier)

- $D = R$ so $D^T + D = 2R > 0$
- We choose $P = \begin{bmatrix} C & 0 \\ 0 & L \end{bmatrix}$

$$V(x) = \frac{1}{2} C v_C^2 + \frac{1}{2} L i_L^2 \tag{35}$$

we get

$$PB - C^T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad A^T P + PA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{36}$$

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2R \end{bmatrix} \tag{37}$$