Introduction to nonsmooth dynamical systems

Lecture 3. Equilibria and stability

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Contents

Lecture 3. Stability in nonsmooth dynamical systems.

- Computation of equilibria
- Lyapunov stability of LCS
- Lyapunov stability of maximal monotone differential inclusions.

Practical work : Stability and bifurcations in electrical circuits and mechanical systems with friction.

Outline

Equilibria for LCS

Linear Complementarity Systems (LCS)

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + a, \quad x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\ y(t) = Cx(t) + D\lambda(t) + b \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0 \end{cases} \tag{1}$$

Mixed Linear Complementarity Problem (MLCP)

We have to solve a Mixed Linear Complementarity Problem :

$$\begin{cases} 0 = A\tilde{x} + B\lambda + a, \\ y = C\tilde{x} + D\lambda + b \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{2}$$

Equilibria for LCS

Existence of solutions to MLCP

- Trivial case $a = 0, b = 0$. $\tilde{x} = 0$ is an equilibrium.
- If A invertible, then we can substitute $\tilde{x} = -A^{-1}(B\lambda + a)$ to get a LCP

$$0 \leqslant (D - CA^{-1}B)\lambda + A^{-1}a + b \perp \lambda \geqslant 0 \tag{3}$$

If $(D - CA^{-1}B)$ is a P-matrix, it exists a unique solution λ for all a and b. The equilibrium is obtained with $\tilde{x} = -A^{-1}(B\lambda + a)$

Equilibria for LCS

Existence of solutions to MLCP

Reformulation into inclusion

$$-\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} \tilde{x} \\ \lambda \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}\right) \in \partial\Psi_{\mathbb{R}^n\times\mathbb{R}^m_+}\left(\begin{bmatrix} \tilde{x} \\ \lambda \end{bmatrix}\right) \tag{4}$$

as

$$-(Mz+q) \in \partial\Psi_{\mathbb{R}^n\times\mathbb{R}^m_+}(z) \tag{5}$$

Theorem

If M is a semi–definite positive matrix, then the inclusion (5) is solvable if and only if it is feasible, that is

$$\exists z, \quad z \in \mathbb{R}^n \times \mathbb{R}^m_+ \text{ and } Mz+q \in 0^n \times \mathbb{R}^m_+ \tag{6}$$

Application of a more general Theorem 2.4.7 of [1].

Example

Trivial case $a = 0, b \geqslant 0$.

Equilibria for differential inclusion

Computation of equilibria

The equilibria of

$$-(\dot{x}(t) + f(x)) \in T(x(t)) \tag{7}$$

are given by the following generalized equation

$$-f(\tilde{x}) \in T(\tilde{x}) \tag{8}$$

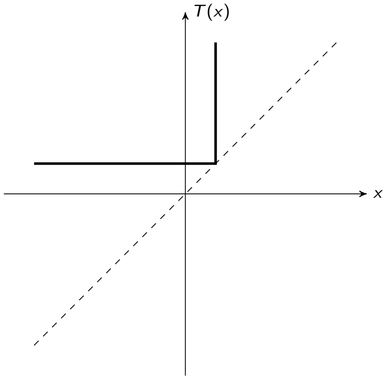
Generalized Equation

$$0 \in f(\tilde{x}) + T(\tilde{x}) \tag{9}$$

Equilibria for differential inclusion. The simple case $f(\tilde{x}) = 0$

$$0 \in T(\tilde{x}) \Longleftrightarrow x \in T^{-1}(0) \tag{10}$$

A condition for $T^{-1}(0) \neq \emptyset$ is $0 \in D(T^{-1}) = R(T)$.



Equilibria for differential inclusion. The simple case $f(\tilde{x}) = 0$

Theorem

Let $\phi : \mathbb{R}^n \to \mathbb{R} \cup +\infty$ be a proper convex function. Then

$$0 \in \partial\phi(\tilde{x}) \iff \tilde{x} \in \operatorname*{argmin}_{z \in \mathbb{R}^n} \phi(z) \tag{10}$$

A solution to $0 \in \partial f(\tilde{x})$ exists if $\min_{z \in \mathbb{R}^n} \phi(z) > -\infty$

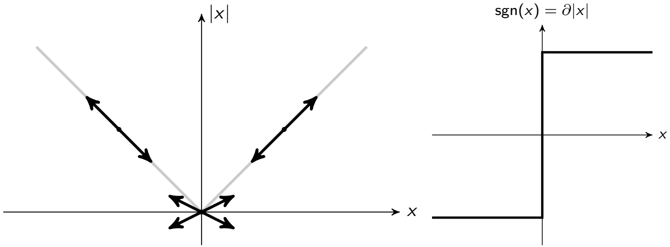


Figure: Absolute value function

Equilibria for differential inclusion. The simple case $f(\tilde{x}) = 0$

Example ($\phi(x) = \Psi_C(x)$)

$$\operatorname*{argmin}_{z \in \mathbb{R}^n} \Psi_C(z) = C \tag{10}$$

Equilibria for differential inclusion. The affine case $f(\tilde{x}) = A\tilde{x} + a$

$T(x) = \Psi_C(x)$ with C a polyhedral set $C = \{Cx + d \geqslant 0\}$

$$\Psi_C(x) = \mathrm{N}_C(x) = \{s = -C^T\lambda, 0 \leqslant \lambda \perp Cx + d \geqslant 0\} \tag{11}$$

The generalized equation

$$-(A\tilde{x} + a) \in \partial\Psi_C(\tilde{x}) \tag{12}$$

is equivalent to the following MLCP

$$\begin{cases} A\tilde{x} + a = C^T\lambda \\ y = Cx + d \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{13}$$

that can be written in turns as an inclusion

$$-\left(\begin{bmatrix} A & -C^T \\ C & 0 \end{bmatrix}\begin{bmatrix} \tilde{x} \\ \lambda \end{bmatrix} + \begin{bmatrix} a \\ d \end{bmatrix}\right) \in \partial\Psi_{\mathbb{R}^n \times \mathbb{R}^m_+}\left(\begin{bmatrix} \tilde{x} \\ \lambda \end{bmatrix}\right) \tag{14}$$

If A is semidefinite positive then $\begin{bmatrix} A & -C^T \\ C & 0 \end{bmatrix}$ is semi-definite positive. If the inclusion is feasible, then it is solvable.

Equilibria for differential inclusion. The affine case $f(\tilde{x}) = A\tilde{x} + a$

$T(x) = \Psi_C(x)$ with C a convex set and A symmetric definite positive

We can define a convex function $\Phi(x) = \Psi_C(x) + \frac{1}{2}x^T Ax + a^T x$. Then

$$\min_{z\in\mathbb{R}^n} \Phi(x) = \min_{z\in\mathbb{R}^n} \Psi_C(x) + \frac{1}{2}x^T Ax + a^T x = \min_{z\in C} \frac{1}{2}x^T Ax + a^T x \tag{15}$$

This is a convex minimization problem that possess a solution and the optimality conditions are

$$0 \in \partial\Phi(\tilde{x}) = A\tilde{x} + a + \partial\Psi_C(\tilde{x}) \tag{16}$$

Remark

If a polyhedral set $C = \{Cx + d \geqslant 0\}$, then the optimality condition are

$$\begin{cases} Ax + a = C^T\lambda \\ y = Cx + d \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{17}$$

Outline

Lyapunov stability (Recap.)

Definition (Lyapunov stability)

The equilibrium $\tilde{x}$ is said to be stable in the sense of Lyapunov if

$$\text{for every } \varepsilon > 0, \exists \delta > 0, \text{ such that } \|x(0) - \tilde{x}\| < \delta \text{ then } \|x(t) - \tilde{x}\| < \varepsilon, \forall t \geqslant 0. \quad (18)$$

Definition (Asymptotic Lyapunov stability)

The equilibrium $\tilde{x}$ is said to be asymptotically stable in the sense of Lyapunov if

- it is stable and
- for every $\varepsilon > 0, \exists \delta > 0,$ such that $\|x(0) - \tilde{x}\| < \delta$ then $\lim_{t \to +\infty} \|x(t) - \tilde{x}\| = 0$

Definition (Exponential Lyapunov stability)

The equilibrium $\tilde{x}$ is said to be asymptotically stable in the sense of Lyapunov if

- it is asymptotically stable and
- $\exists \alpha, \beta, \delta > 0,$ such that $\|x(0) - \tilde{x}\| < \delta$ then $\|x(t) - \tilde{x}\| \leqslant \alpha \|x(0) - \tilde{x}\| e^{-\beta t}, \forall t \geqslant 0$

LTI passive systems

Linear Time Invariant (LTI) systems

Let us consider the following system:

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) \\ y(t) = Cx(t) + D\lambda(t) \end{cases} \tag{19}$$

with a quadratic function $V(x) = \frac{1}{2}x^T Px$ with $P = P^T$.
Let us define the composition:

$$\begin{array}{rccc} \mathcal{V}(t): & \mathbb{R} & \rightarrow & \mathbb{R} \\ & t & \mapsto & V(x(t)) \end{array} \tag{20}$$

LTI passive systems

Derivation of $\mathcal{V}(t)$

$$\dot{\mathcal{V}}(t) = x^T(t)P\dot{x}(t) \tag{19}$$

$$\begin{gathered}
x^T(t)P\dot{x}(t) = x^T(t)PAx(t) + x^T(t)B\lambda(t) \\
\Updownarrow \\
x^T(t)P\dot{x}(t) - \lambda^T(t)y(t) = x^T(t)PAx(t) + x^T(t)PB\lambda(t) - \lambda^T(t)y(t) \\
\Updownarrow \\
x^T(t)P\dot{x}(t) - \lambda^T(t)y(t) = x^T(t)PAx(t) + \lambda^T(t)B^TPx(t) - \lambda^T(t)(Cx(t) + D\lambda(t)) \\
\Updownarrow \\
x^T(t)P\dot{x}(t) - \lambda^T(t)y(t) = x^T(t)PAx(t) + \lambda^T(t)(B^TP - C)x(t) - \lambda^T(t)D\lambda(t)
\end{gathered} \tag{20}$$

LTI passive systems

Derivation of $\mathcal{V}(t)$

$$
\begin{aligned}
V(x(T) - V(x(0)) \quad & - \quad \int_0^T \lambda^T(t)y(t)dt \\
& = \quad \int_0^T x^T(t)PAx(t) + \lambda^T(t)(B^TP - C)x(t) - \lambda^T(t)(D\lambda(t))dt \\
& = \quad \frac{1}{2}\int_0^T \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}^T \begin{bmatrix} A^TP + PA & PB - C^T \\ B^TP - C & -(D + D^T) \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} dt
\end{aligned} \tag{19}
$$

LTI passive systems

Linear Time Invariant (LTI) systems

$$\left\{ \begin{array}{l} \dot{x}(t) = Ax(t) + B\lambda(t) \\ y(t) = Cx(t) + D\lambda(t) \end{array} \right. \tag{20}$$

Definition

The system $\Sigma(A, B, C, D)$ given in (20) is said to be passive (dissipative with respect to the supply rate $\lambda^T y$) is there exists a function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ (a storage function) such that

$$V(x(t_0)) + \int_{t_0}^{t} \lambda^T(t)y(t)dt \geqslant V(x(t)) \tag{21}$$

holds for all t_0 and t with $t \geqslant t_0$ and for all $\mathcal{L}^2$-solutions (x, y, λ).

LTI passive systems

Theorem

The system $\Sigma(A, B, C, D)$ is passive if and only if the following linear matrix inequality (LMI)

$$P = P^T > 0 \text{ and } \begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -(D + D^T) \end{bmatrix} \leqslant 0 \tag{22}$$

has a solution.
In this case, $V(x) = \frac{1}{2} x^T P x$ *is the corresponding* energy storage function.

LTI passive systems

Theorem

The system $\Sigma(A, B, C, D)$ is passive if there exist matrices $L \in \mathbb{R}^{n\times m}$ and $W \in \mathbb{R}^{m\times m}$ and a symmetric positive semi-definite matrix $P \in \mathbb{R}^{n\times n}$, such that:

$$\begin{cases} A^T P + PA = -LL^T & (23) \\ B^T P - C = -W^T L^T & (24) \\ -D - D^T = -W^T W. & (25) \end{cases}$$

LTI passive systems

Reformulation

$$\begin{bmatrix} A^TP + PA & PB - C^T \\ B^TP - C & -(D + D^T) \end{bmatrix} = - \begin{bmatrix} LL^T & LW \\ W^TL^T & W^TW \end{bmatrix} = - \begin{bmatrix} L \\ W \end{bmatrix}^T \begin{bmatrix} L \\ W \end{bmatrix} \triangleq -Q \quad (23)$$

LTI passive systems

Dissipation inequality

The *dissipation equality*

$$V(x(T))-V(x(0))=\frac{1}{2}\int_0^T \lambda^T(t)y(t)+\frac{1}{2}\int_0^T \begin{bmatrix}x(t)\\ \lambda(t)\end{bmatrix}^T Q \begin{bmatrix}x(t)\\ \lambda(t)\end{bmatrix} dt, \quad \forall \quad T \geqslant 0 \quad (23)$$

in terms of the positive semi-definite matrix

$$Q \triangleq \begin{pmatrix} LL^T & W^T L^T \\ LW & W^T W \end{pmatrix}, \quad (24)$$

then implies that

$$V(x(T)) - V(x(0)) - \frac{1}{2}\int_0^T \lambda^T(t)y(t) \leqslant 0. \quad (25)$$

Strictly passive LTI systems

The system is said to be *strictly passive* when Q is positive definite.

LTI passive systems

A special case

- $D = 0$.

$$(D + D^T) = W^T W = 0 \implies W = 0 \tag{26}$$

$$B^T P - C = -W^T L^T = 0 \implies C = B^T P \implies CB = B^T PB \geqslant 0 \tag{27}$$

The matrix CB is a semi-definite positive matrix

Passive LCS

Assumption

The trajectory $x(t)$ of the LCS is continuous.

Definition (Passive LCS)

The LCS

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) \\ y(t) = Cx(t) + D\lambda(t) \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0 \end{cases} \tag{28}$$

is said to be (strictly) passive if the system $\Sigma(A, B, C, D)$ is (strictly) passive

Supply rate

The complementarity condition implies that $\lambda^T(t)y(t) = 0$ for all $t \geqslant 0$. Then the dissipation inequality reduces to

$$V(x(T)) - V(x(0)) \leqslant 0 \tag{29}$$

Lyapunov stability of LCS

Theorem

- *If the LCS is passive, then the LCS is Lyapunov stable.*
- *If the LCS is strictly passive, then the LCS is globally exponentially stable.*

The energy storage function plays the role of a Lyapunov function.

Lyapunov stability of LCS

- If the LCS is passive, then D is a semi-definite positive matrix

Lyapunov stability of LCS

Example (The RLC circuit with a diode. A half wave rectifier)

A LC oscillator supplying a load resistor through a half-wave rectifier.

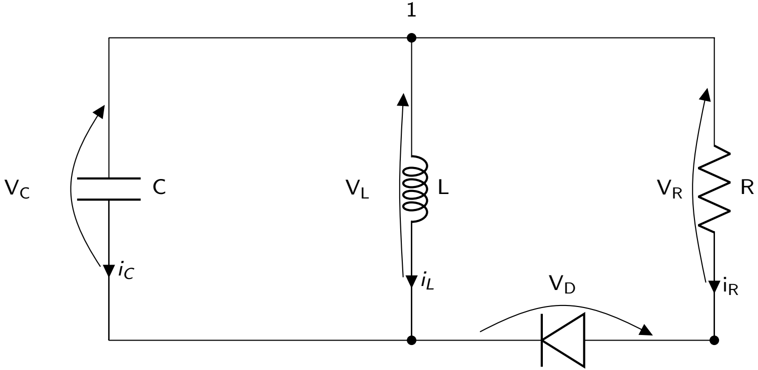


Figure: Electrical oscillator with half-wave rectifier

Lyapunov stability of LCS

Example (The RLC circuit with a diode. A half wave rectifier)

The following linear complementarity system is obtained :

$$\begin{pmatrix} \dot{v}_C \\ \dot{i}_L \end{pmatrix} = \begin{pmatrix} 0 & \frac{-1}{C} \\ \frac{1}{L} & 0 \end{pmatrix} \cdot \begin{pmatrix} v_C \\ i_L \end{pmatrix} + \begin{pmatrix} \frac{-1}{C} \\ 0 \end{pmatrix} \cdot i_D$$

together with a state variable x and one of the complementary variables λ :

$$x = \begin{pmatrix} v_C \\ i_L \end{pmatrix}, \qquad \lambda = i_D, \qquad y = -v_D$$

and

$$y = -v_D = \begin{pmatrix} -1 & 0 \end{pmatrix} x + \begin{pmatrix} R \end{pmatrix} \lambda,$$

Standard form for LCS

$$\begin{cases} \dot{x} = Ax + B\lambda \\ y = Cx + D\lambda \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases}$$

Lyapunov stability of LCS

Example (The RLC circuit with a diode. A half wave rectifier)

- $D = R$ so $D^T + D = 2R > 0$
- We choose $P = \begin{bmatrix} C & 0 \\ 0 & L \end{bmatrix}$

$$V(x) = \frac{1}{2} C v_C^2 + \frac{1}{2} L i_L^2 \tag{30}$$

we get

$$PB - C^T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad A^T P + PA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{31}$$

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2R \end{bmatrix} \tag{32}$$

Outline

Application of standard results for stability and asymptotic behavior

Sufficient assumptions

- Existence of absolutely continuous solution.
- Continuity with respect to initial conditions.
- Lyapunov function $V \in \mathcal{C}^1$
- Invariants in the interior of the domain of maximal monotone operators

With these assumptions, the main result for smooth systems can be proved

- Lyapunov stability theorems.
- Lasalle invariance principle.

Relaxed results

In the literature, a large number of results relax the assumptions that are sometimes not necessary. For the sake of simplicity, we assume that there are valid for our applications. In the sequel, we present more specific results for Maximal Monotone differential inclusions

Lyapunov stability of monotone differential inclusions

Monotone differential inclusions, $x(0) = x_0$

- Standard form

$$-\dot{x}(t) \in T(x(t)) \tag{33}$$

- Standard perturbed form

$$-(\dot{x}(t) + f(x(t), t) \in T(x(t)) \tag{34}$$

- Sub-differential of Φ convex, proper and lower-semicontinuous

$$-(\dot{x}(t) + f(x(t), t) \in \partial\Phi(x(t)) \tag{35}$$

Solutions

We assume that there exists an absolutely continuous solution such that one of the previous inclusion is satisfied almost everywhere

Absolutely continuous functions

Definition

Let I be an interval in the real line $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is absolutely continuous on I if for every positive number ε, there exists a positive number δ such that whenever a finite sequence of pairwise disjoint sub-intervals (x_k, y_k) of I satisfies

$$\sum_k (y_k - x_k) < \delta \tag{36}$$

then

$$\sum_k |f(y_k) - f(x_k)| < \varepsilon \tag{37}$$

Absolutely continuous functions

Proposition

The following conditions on a real-valued function f on a compact interval $[a, b]$ are equivalent:

1. f is absolutely continuous
2. f has derivative almost everywhere, the derivative is Lebesque integrable, and
$$f(t) = f(a) + \int_a^t f'(t)dt \tag{36}$$
for all x on $[a, b]$.
3. there exists a Lebesgue integrable function g on $[a, b]$ such that
$$f(t) = f(a) + \int_a^t g(t)dt \tag{37}$$
for all x on $[a, b]$.

If these equivalent conditions are satisfied then necessarily $g = f'$ almost everywhere. Equivalence between (1) and (3) is known as the fundamental theorem of Lebesgue integral calculus, due to Lebesgue.

Absolutely continuous functions

Properties

- The sum and difference of two absolutely continuous functions are also absolutely continuous.
- If the two functions are defined on a bounded closed interval, then their product is also absolutely continuous.
- If an absolutely continuous function is defined on a bounded closed interval and is nowhere zero then its reciprocal is absolutely continuous.
- Every absolutely continuous function is uniformly continuous and, therefore, continuous. Every Lipschitz-continuous function is absolutely continuous.
- If $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then it is of bounded variation on $[a, b]$.
- If $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then it can be written as the difference of two monotonic nondecreasing absolutely continuous functions on [a,b].
- If $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then it has the Luzin N property (that is, for any $L \subseteq [a, b]$ such that $\lambda(L) = 0$, it holds that $\lambda(f(L)) = 0$, where λ stands for the Lebesgue measure on $\mathbb{R}$).
- $f : I \to \mathbb{R}$ is absolutely continuous if and only if it is continuous, is of bounded variation and has the Luzin N property.
- The composition of two absolutely continuous functions is **not** necessarily a absolutely continuous function

Absolutely continuous functions

Proposition

Let f be Lipschitz continuous on $\mathbb{R}$ and g be an absolutely continuous function on $[a, b]$. Then the composition $f \circ g$ is absolutely continuous on $[a, b]$.

Lyapunov stability of monotone differential inclusions

Differentiability of the Lyapunov function

Let us assume that we have a $\mathcal{C}^1$ Lyapunov function, then

$$\begin{array}{rccc} \mathcal{V}(t): & \mathbb{R} & \rightarrow & \mathbb{R} \\ & t & \mapsto & V(x(t)) \end{array} \tag{36}$$

is absolutely continuous if $x(t)$ is also absolutely continuous.
This implies that $\dot{\mathcal{V}}(t)$ exists almost everywhere
Futhermore, if $\dot{\mathcal{V}}(t) \leqslant 0$ almost everywhere then

$$\mathcal{V}(t) - \mathcal{V}(0) = \int_0^t \dot{\mathcal{V}}(t)dt \leqslant 0 \implies \mathcal{V}(t) \text{ is decreasing} \tag{37}$$

Lyapunov stability of monotone differential inclusions

Monotone differential inclusions $-(\dot{x}(t) + f(x(t))) \in T(x(t))$

Let us formulate the autonomous differential inclusion as

$$\begin{cases} \dot{x}(t) + f(x(t)) = \lambda(t) \\ -\lambda(t) \in T(x(t)) \end{cases} \tag{38}$$

If V is $\mathcal{C}^1$, we want to satisfy

$$\dot{\mathcal{V}}(t) = \nabla_x V(x(t)) \cdot [-f(x(t)) + \lambda(t)] \leqslant 0 \text{ with } -\lambda(t) \in T(x(t)) \tag{39}$$

Lyapunov stability of monotone differential inclusions

Monotone differential inclusions, $-\dot{x}(t) \in T(x(t))$

The case when $f(x(t)) = 0$ and we choose $V(x) = \frac{1}{2}\|x - \tilde{x}\|^2$, $\nabla_x V(x) = (x - \tilde{x})$ than we get

$$\dot{\mathcal{V}}(t) = (x(t) - \tilde{x})^T \lambda(t), \text{ with } -\lambda(t) \in T(x(t)) \tag{40}$$

Let us consider and equilibrium point $\tilde{x} \in \mathring{D}(T)$, $0 \in T(\tilde{x})$ then the monotony implies

$$(-\lambda(t) - 0)^T (x(t) - \tilde{x}) \geqslant 0 \tag{41}$$

that is

$$\dot{\mathcal{V}}(t) = (x(t) - \tilde{x})^T \lambda(t) \leqslant 0 \tag{42}$$

For a monotone differential inclusion $-\dot{x}(t) \in T(x(t))$, a equilibrium with $\tilde{x} \in \mathring{D}(T)$ is Lyapunov stable. If T is strictly monotone, $\tilde{x}$ is asymptotically stable.

Lyapunov stability of monotone differential inclusions

Monotone differential inclusions

If $\tilde{x} \in \partial D(T)$, the classical Lyapunov stability theorem does no longer apply immediately, since it is not possible to find a open set Ω that is a neighborhood of $\tilde{x}$.

Lyapunov stability of monotone differential inclusions

Assumption 1

Let us consider the differential inclusion

$$-(\dot{x}(t)+f(x(t)) \in \partial\Phi(x(t)), \quad dt\text{-a.e} \tag{43}$$

with

- $\Phi : \mathbb{R}^n \to \mathbb{R}$ a proper lower semi-continuous convex function
- $f : \mathbb{R}^n \to \mathbb{R}^n$ a Lipschitz continuous function
- an equilibrium point in $0 \in D(\partial\Phi)$, that is

$$-f(0) \in \partial\Phi(0).$$

If Assumption 1 holds then we have a unique absolutely continuous solution whatever $x_0 \in D(\partial\Phi)$.

Lyapunov stability of monotone differential inclusions

The following theorems are extracted from [2].

Theorem

Let us assume the Assumption 1 holds. Suppose that there exist $R > 0, a > 0$ *and* $V \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ *such that*

$$(\forall x \in D(T), \|x\| = R), V(x) \geqslant a \tag{43}$$

and

$$\nabla_x V(x) \cdot f(x) + \Phi(x) - \Phi(x - \nabla V(x)) \geqslant 0. \tag{44}$$

Then, for any $x_0 \in D(\partial\Phi)$ *with* $\|x_0\| \leqslant R$ *and* $V(x_0) < a$*, the solution* $x(t; t_0, x_0)$ *satisfies*

$$\forall t \geqslant t_0, \|x(t; t_0, x_0)\| < R \tag{45}$$

Lyapunov stability of monotone differential inclusions

Idea of the proof:

$$\dot{\mathcal{V}}(t) = \nabla_x V(x(t)) \cdot \dot{x}(t) \text{ a.e} \tag{46}$$

with

$$\dot{x}(t) + f(x(t)) = \lambda(t) \text{ with } -\lambda(t) \in \partial\Phi(x(t)) \text{ a.e} \tag{47}$$

Applying the definition of the sub-differential,

$$\begin{gathered} -\lambda(t) \in \partial\Phi(x(t)) \\ \Updownarrow \\ (\lambda(t))^T(v - x(t)) + \Phi(v) - \Phi(x(t)) \geqslant 0, \forall v \in \mathbb{R}^n \end{gathered} \tag{48}$$

we get

$$(\dot{x}(t) + f(x(t)))^T(v - x(t)) + \Phi(v) - \Phi(x(t)) \geqslant 0, \forall v \in \mathbb{R}^n \tag{49}$$

Lyapunov stability of monotone differential inclusions

Idea of the proof:

Let us choose $v = x - \nabla_x V(x(t))$

$$-(\dot{x}(t) + f(x(t)))^T \nabla_x V(x(t)) + \Phi(x(t) - \nabla_x V(x(t))) - \Phi(x(t)) \geqslant 0 \text{ a.e} \tag{46}$$

thus

$$\dot{\mathcal{V}}(t) \leqslant -\left[f(x(t)))^T \nabla_x V(x(t)) + \Phi(x(t)) - \Phi(x(t) - \nabla_x V(t))\right] \text{ a.e} \tag{47}$$

from the assumption we get

$$\dot{\mathcal{V}}(t) \leqslant 0 \text{ a.e} \tag{48}$$

Lyapunov stability of monotone differential inclusions

Let us denote by B_σ the ball of radius $\sigma > 0$, $B_\sigma = \{x \mid \|x\| \leqslant \sigma\}$

Theorem (Stability)

Let us assume the Assumption 1 holds. Suppose that there exists $\sigma > 0$ and $V \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ such that $V(0) = 0$

$$\forall x \in D(\partial\Phi) \cap B_\sigma, V(x) \geqslant a(\|x\|) \tag{49}$$

with $a : [0, \sigma] \to \mathbb{R}$, $a(t) > 0, \forall t \in (0, \sigma)$, and

$$\forall x \in D(\partial\Phi) \cap B_\sigma, \nabla_x V(x) \cdot f(x) + \Phi(x) - \Phi(x - \nabla V(x)) \geqslant 0. \tag{50}$$

Then 0 is a stable equilibrium.

Lyapunov stability of monotone differential inclusions

Example

Let us consider this example:

$$f(x) = \begin{bmatrix} x_2 \\ -\sin(x_1) \end{bmatrix}, \quad x \in \mathbb{R}^2 \tag{51}$$

and

$$\Phi(x) = \Psi_{\mathbb{R}^2_+}(x) \tag{52}$$

We choose

$$V(x) = 1 - \cos(x_1) + \frac{x_2^2}{2} \tag{53}$$

and we obtain

$$\nabla_x V(x) = \begin{bmatrix} \sin(x_1) \\ x_2 \end{bmatrix} \tag{54}$$

and

$$\nabla_x V(x) \cdot f(x) = 0 \tag{55}$$

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Example

There exists $\sigma > 0$ such that

$$\|x\| \geqslant \sigma \implies 1 - \cos(x_1) \geqslant \frac{x_1^2}{4} \tag{51}$$

Thus

$$\|x\| \geqslant \sigma \implies V(x) \geqslant \frac{x_1^2 + x_2^2}{4} \tag{52}$$

We have also

$$x \in \mathbb{R}^2_+ \implies x - \nabla_x V(x) = \begin{bmatrix} x_1 - \sin(x_1) \\ 0 \end{bmatrix} \in \mathbb{R}^2_+ \tag{53}$$

Thus

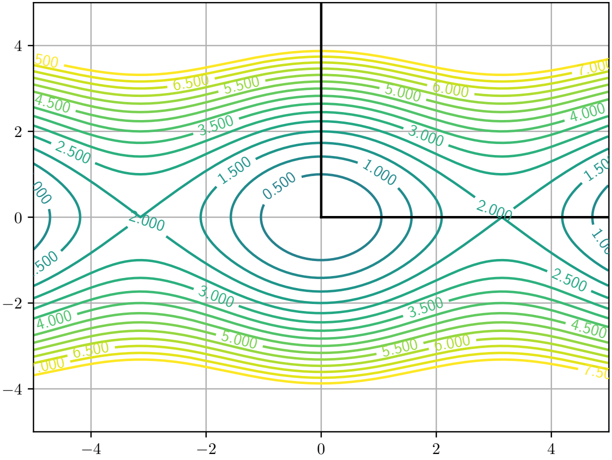
$$x \in \mathbb{R}^2_+, \|x\|\sigma \implies \nabla_x V(x) \cdot f(x) + \Phi(x) - \Phi(x - \nabla V(x)) = 0. \tag{54}$$

With the previous theorem, we can conclude to the stability of the equilibrium $\tilde{x} = 0$.

Lyapunov stability of monotone differential inclusions

Example

level sets of $V(x) = 1 - \cos(x_1) + 1/2x_2^2$



Lyapunov stability of monotone differential inclusions

Theorem (Asymptotic Stability)

Let us assume the Assumption 1 holds.
Suppose that there exist $\sigma > 0, \lambda > 0$ and $V \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ such that $V(0) = 0$

$$\forall x \in D(\partial\Phi) \cap B_\sigma, V(x) \geqslant a(\|x\|) \tag{51}$$

with $a : [0, \sigma] \to \mathbb{R}$, $a(t) > ct^\tau, \forall t \in (0, \sigma)$ for some $c > 0, \tau > 0$, and

$$\forall x \in D(\partial\Phi) \cap B_\sigma, \nabla_x V(x) \cdot f(x) + \Phi(x) - \Phi(x - \nabla V(x)) \geqslant \lambda V(x). \tag{52}$$

Then 0 is an asymptotic stable equilibrium.

Lyapunov stability of monotone differential inclusions

Definition (Set of stationary points)

$$\mathcal{S}(F,\Phi) = \{x \in D(\partial\Phi) \mid -f(x) \in \partial\Phi(x)\} \tag{53}$$

or equivalently

$$\mathcal{S}(F,\Phi) = \{x \in D(\partial\Phi) \mid f^T(x)(v-z) + \Phi(v) - \Phi(x), \forall v \in \mathbb{R}^n\} \tag{54}$$

Definition

Let $V \in \mathcal{C}^1$. We define

$$\mathcal{E}(F,\Phi,V) = \{x \in D(\partial\Phi) \mid \nabla_x V(x) \cdot f(x) + \Phi(x) - \Phi(x - \nabla_x V(x)) = 0\} \tag{55}$$

Lyapunov stability of monotone differential inclusions

Theorem

Let us assume the Assumption 1 holds.
Let A be a subset of $\mathbb{R}^n$. Suppose that there exists $V \in \mathcal{C}^1(\mathbb{R}^n;\mathbb{R})$ such that

$$\forall x \in D(\partial\Phi) \cap A, \nabla_x V(x) \cdot f(x) + \Phi(x) - \Phi(x - \nabla V(x)) \geqslant 0. \tag{56}$$

Then

$$\mathcal{S}(F,\Phi) \cap A \subset \mathcal{E}(F,\Phi,V) \tag{57}$$

Lyapunov stability of monotone differential inclusions

Theorem

Let us assume the Assumption 1 holds.
Suppose that there exist $\sigma > 0$ *and* $V \in \mathcal{C}^1(\mathbb{R}^n; \mathbb{R})$ *such that*

$$\forall x \in D(\partial\Phi) \cap B_\sigma, \nabla_x V(x) \cdot f(x) + \Phi(x) - \Phi(x - \nabla V(x)) \geqslant 0. \qquad (58)$$

and

$$\mathcal{E}(F, \Phi, V) \cap B_\sigma = \{0\} \qquad (59)$$

Then the stationary solution is isolated in $\mathcal{S}(F, \Phi)$

Lyapunov stability of monotone differential inclusions

Assumption

Theorem

Let us assume the Assumption 1 holds.
Suppose that there exists $V \in \mathcal{C}^1(\mathbb{R}^n;\mathbb{R})$ *such that*

$$\forall x \in D(\partial\Phi), \nabla_x V(x) \cdot f(x) + \Phi(x) - \Phi(x - \nabla V(x)) \geqslant 0. \tag{60}$$

and

$$\mathcal{E}(F, \Phi, V) = \{0\} \tag{61}$$

Then $\mathcal{S}(F, \Phi) = \{0\}$ *that is the stationary solution is the unique equilibrium point.*

[1] F. Facchinei and J. S. Pang. *Finite-dimensional Variational Inequalities and Complementarity Problems*, volume I & II of *Springer Series in Operations Research*. Springer Verlag NY. Inc., 2003.

[2] D. Goeleven. *Complementarity and Variational Inequalities in Electronics*. Academic Press, 2017.