Introduction to nonsmooth dynamical systems

Lecture 3. Differential Inclusions

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Contents

Outline

General differential inclusion

Concept of differential inclusions

Differential inclusions are a generalization of the concept of differential equations of the form

$$\dot{x}(t) \in A(x(t), t) \tag{1}$$

where $(x, t) \mapsto A(x, t)$ is a multi-valued map, *i.e.* $A(x, t)$ is a set, possibly empty, rather than a single point.

A very general concept

Differential inclusions is a very general concept.

- It contains Ordinary Differential Equations (ODE)
- There are many types of differential inclusions [1].

We will focus on Maximal Monotone Differential Inclusions and Filippov's differential inclusions

LCS as differential inclusion

Complementarity condition as a subdifferential inclusion

$$0 \leqslant y \perp \lambda \geqslant 0 \Longleftrightarrow -y \in \partial\Psi_{\mathbb{R}^m_+}(\lambda) \Longleftrightarrow -\lambda \in \partial\Psi_{\mathbb{R}^m_+}(y) \tag{2}$$

LCS as a differential inclusion with $D = 0$ and $b = 0$

$$\left\{\begin{array}{l} \dot{x}(t) = Ax(t) + B\lambda(t) + a \\ y(t) = Cx(t) \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0 \\ x(0) = x_0. \end{array}\right. \Longleftrightarrow \left\{\begin{array}{l} -(\dot{x}(t) - Ax(t) - a) \in B\partial\Psi_{\mathbb{R}^m_+}(Cx(t)), \\ \quad x(0) = x_0 \end{array}\right. \tag{3}$$

Outline

Maximal monotone operators

Let $2^{\mathbb{R}^n}$ be the set of the subsets of $\mathbb{R}^n$

Definition (Monotone multi-valued operator)

A operator $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is monotone if

$$\forall y_1 \in T(x_1), \quad \forall y_2 \in T(x_2), \quad (y_2 - y_1)^T(x_2 - x_1) \geqslant 0 \tag{4}$$

Definition (Graph)

Let T multi–valued operator $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$. The graph of T is defined by

$$Gr(T) = \{(x, y) \mid y \in T(x)\} \tag{5}$$

Definition (Maximal Monotone multi-valued operator)

A operator T is maximal monotone if it is maximal for all the monotone operators for the inclusion of graphs.

In other words, T is monotone and for all other monotone operator S then $Gr(T) \subset Gr(S) \implies T = S$

Maximal monotone operators

Definition (Domain)

The domain of an operator T is defined by $D(T) = \{x \mid T(x) \neq \emptyset\}$

Definition (Range of T)

Let $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be an operator. The range of T is defined by

$$R(T) = \cup_{x \in \mathbb{R}^n} \{y \mid y \in T(x)\} \tag{6}$$

Definition (Inverse of T)

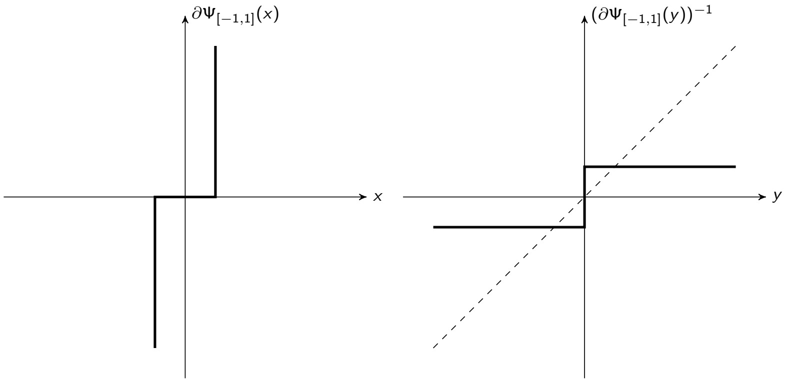
Let $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a maximal monotone operator. Its inverse T^{-1} is defined by

$$y \in T(x) \Longleftrightarrow x \in T^{-1}(y) \tag{7}$$

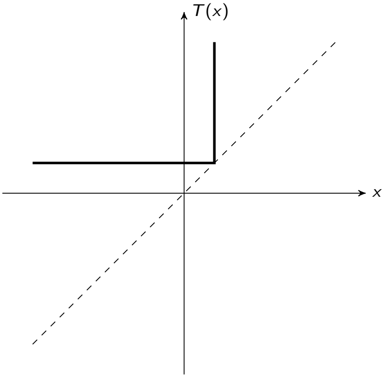
and we have $D(T^{-1}) = R(T)$ and $R(T^{-1}) = D(T)$

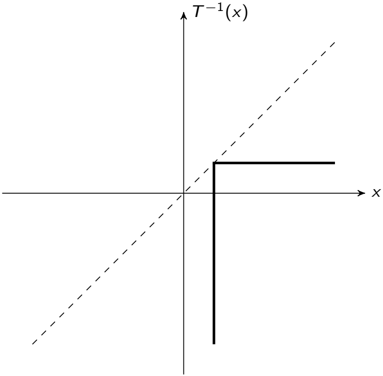
Its inverse is defined by the symmetry of its graph with respect to $y = x$

Maximal monotone operators

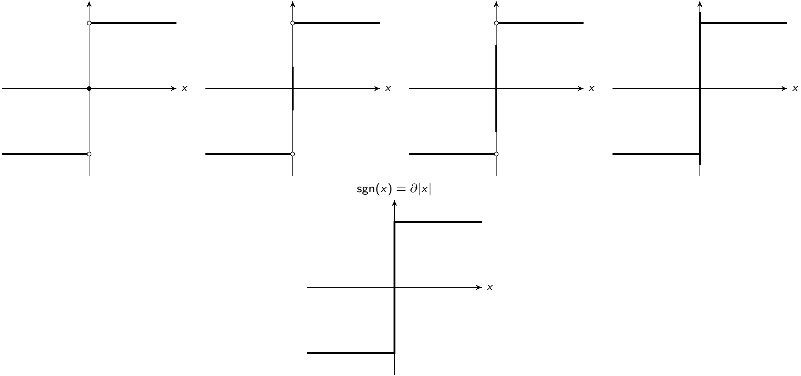


Maximal monotone operators





Maximal monotone operators



Outline

Maximal monotone differential inclusion

Let $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a maximal monotone operator.

Definition (Maximal monotone differential inclusion)

A maximal monotone differential inclusion is defined by

$$-\dot{x}(t) \in T(x(t)). \tag{6}$$

Definition (Perturbed maximal monotone differential inclusion)

A perturbed maximal monotone differential inclusion is defined by

$$-(\dot{x}(t) + f(x,t)) \in T(x(t)), \tag{7}$$

where f is a Lipschitz continuous map w.r.t x.

Maximal monotone differential inclusion

The main instance of maximal monotone operators:

The subdifferential of (some) convex functions.

Theorem

For a lower semi–continuous convex proper function Φ, the subdifferential $\partial\Phi(x)$ is a maximal monotone operator

Remarks

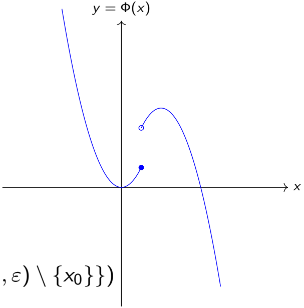
- Obvious in the regular case: $\phi(x) : \mathbb{R} \to \mathbb{R}$ a convex potential C^2 $\phi''(x) \geqslant 0$ and $\phi'(x)$ is monotone (increasing single–valued function)
- For a maximal monotone operator in $\mathbb{R}$, i.e. $T : \mathbb{R} \to 2^{\mathbb{R}}$ it exists a lower semi–continuous convex proper function Φ such that $T = \partial\Phi$

Maximal monotone differential inclusion

Definition (lower semi-continuity)

A function $\Phi : \mathbb{R}^n \to \mathbb{R} \cup +\infty$ is lower semi-continuous if one of the following equivalent assertions is satisfied:

- $$\liminf_{x \to x_0} \Phi(x) \geqslant \Phi(x_0)$$
- Its epigraph is closed



Remarks

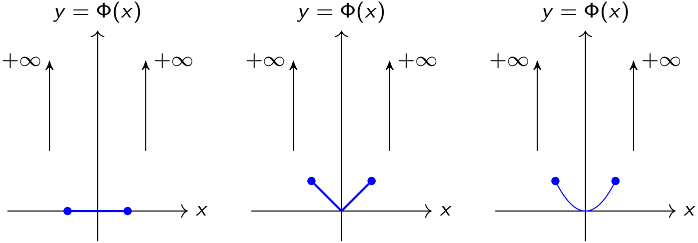
- $\liminf_{x \to x_0} \Phi(x) = \lim_{\varepsilon \to 0}(\inf\{\Phi(x), x \in B(x_0, \varepsilon) \setminus \{x_0\}\})$
- Continuity implies semi-continuity.

Maximal monotone differential inclusion

For a convex proper function Φ, the semi–continuity property has only to be checked on the boundary of the domain of definition

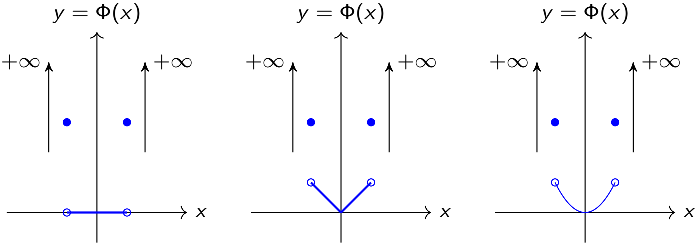
$$\partial D(\Phi) = \overline{D(\Phi)} \setminus \mathring{D}(\Phi).$$

Examples



Maximal monotone differential inclusion

Counter-examples



Maximal monotone differential inclusion

Examples

- $\Phi(x) = 0 = \Psi_{\mathbb{R}}$, $T(x) = 0$
$$-\dot{x} + f(x,t) = 0 \tag{8}$$
- $\Phi(x) = \Psi_c(x)$, $T(x) = \partial\Psi_C(x)$, C a closed non empty convex set
$$-\dot{x} + f(x,t) \in \partial\Psi_C(x) \tag{9}$$
- relay or sign function $\Phi(x) = |x|$, $T(x) = \partial|x|$
$$-\dot{x} \in \partial|x| \iff -\dot{x} \in \mathrm{sgn}(x) \tag{10}$$
- 2-norm $\Phi(x) = \|x\|$, $T(x) = \partial\|x\| = \begin{cases} \frac{x}{\|x\|} & \text{if} \quad x \neq 0 \\ \{s \mid \|s\| \leqslant 1\} & \text{if} \quad x = 0 \end{cases}$

Maximal monotone differential inclusion

Examples

- relay with dead zone

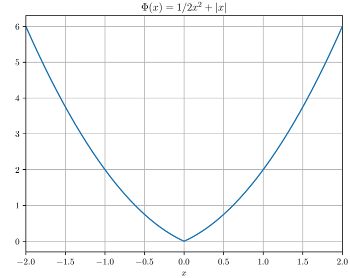
$$\Phi(x) = \begin{cases} -x+1, \text{ if } x \leqslant -1 \\ 0, \text{ if } -1 \leqslant x \leqslant 1 \\ x-1, \text{ if } x \geqslant 1 \end{cases} \tag{8}$$

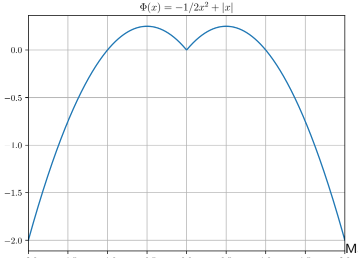
- sum of convex functions. $\Phi(x) = 1/2 * ax^2 + |x|$, $T(x) = ax + \text{sgn}(x)$

$$-\dot{x} \in ax + \partial|x| \Longleftrightarrow -\dot{x} - ax \in \text{sgn}(x) \tag{9}$$

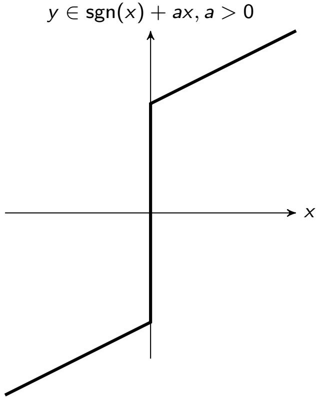
1. $a > 0$. $\Phi(x)$ is convex and $T(x)$ is maximal monotone.
2. $a < 0$. $\Phi(x)$ is not convex and $T(x)$ is not monotone.

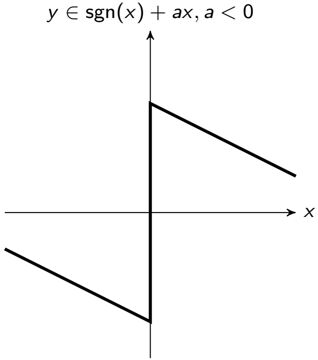
Maximal monotone differential inclusion





Maximal monotone differential inclusion





Maximal monotone differential inclusion

Link with gradient systems with convex potentials

- $\phi(x) : \mathbb{R} \to \mathbb{R}$ a convex potential C^2
  $\phi''(x) \geqslant 0$ and $\phi'(x)$ is monotone (increasing function)

$$-\dot{x} = \phi'(x) \tag{8}$$

- $\Phi(x) : \mathbb{R} \to \mathbb{R}$ a convex potential not necessarily differentiable, but proper and lower semi–continuous $\partial\Phi(x)$ is a maximal monotone operator.

$$-\dot{x} = \partial\Phi(x) \tag{9}$$

Existence and uniqueness results

Theorem (Brézis 1973)

Let $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a maximal monotone operator such that $D(\mathring{T}) \neq \emptyset$. Let a function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ such that

1. *the function $f(x, \cdot)$ is Lipschitz continuous on $D(T)$ that is*

$$\exists L \geqslant 0, \forall t \in [0, t_{\max}], \forall x_1, x_2 \in \overline{D(T)}, \quad \|f(t, x_1) - f(t, x_2)\| \leqslant L\|x_1 - x_2\| \tag{10}$$

2. *$\forall x \in \overline{D(T)}$, the mapping $t \mapsto f(x, t)$ belongs to $\mathcal{L}^\infty(0, t_{\max}; \mathbb{R}^n)$*

Then, for all $x_0 \in \overline{D(T)}$, it exists a unique solution $x(t)$ which is absolutely continuous such that

$$\begin{cases} -(\dot{x}(t) + f(x(t), t)) \in T(x(t)), \ \textit{almost everywhere on } [0, t_{\max}] \\ x(0) = x_0 \end{cases} \tag{11}$$

Existence and uniqueness results

Existence. Ideas of proof

Option 1 By using the Moreau-Yosida regularization of T

$$T_\lambda(x) = \frac{1}{\lambda}(I - J_\lambda)(x), \lambda > 0, \tag{12}$$

with $J_\lambda(x)$ the resolvent of $T(x)$ given by

$$J_\lambda(x) = (I + \lambda T)^{-1}(x). \tag{13}$$

For a maximal monotone operator T or $\mathbb{R}$, J_λ is defined over $\mathbb{R}$ and is contracting. The mapping T_λ is a maximal monotone operator and Lipschitz continuous with a Lipschitz constant of $\frac{1}{\lambda}$. We consider that ODE with Lipschitz r.h.s.

$$-(\dot{x}_\lambda(t) + f(x_\lambda(t), t)) = T_\lambda(x_\lambda(t)) \tag{14}$$

and then the limit $\lambda \to 0$ of the sequence of solutions x_λ.

Option 2 By approximation using a discretization scheme, and then compactness results.

Existence and uniqueness results

Uniqueness. Simple case $-\dot{x}(t) \in T(x(t))$. $x \in \mathbb{R}$

Let us consider two solution x_1 and x_2
Since $T(x)$ is monotone, we have

$$(\dot{x}_1(s) - \dot{x}_2(s))^T (x_1(s) - x_2(s)) \leqslant 0 \text{ almost everywhere on } [0, T] \tag{12}$$

By integrating over $[0, t]$, we get

$$\frac{1}{2}(x_2(t) - x_1(t))^2 - \frac{1}{2}(x_2(0) - x_1(0))^2 \leqslant 0 \tag{13}$$

If $x_1(0) = x_2(0)$, we have

$$\frac{1}{2}(x_2(t) - x_1(t))^2 \leqslant 0 \implies x_2 = x_1 \tag{14}$$

Existence and uniqueness results

Uniqueness. Perturbed case $-(\dot{x}(t) + f(x,t)) \in T(x(t))$

Let us consider two solution x_1 and x_2
Since $T(x)$ is monotone, we have

$$(\dot{x}_1(s) + f(x_1(s), s) - \dot{x}_2(s) - f(x_2(s), s))^T (x_1(s) - x_2(s)) \leqslant 0 \tag{12}$$

almost everywhere on $[0, T]$.
By integrating over $[0, t]$, we get

$$\frac{1}{2}(x_2(t) - x_1(t))^2 \leqslant \int_0^t (f(x_2(s), s) - f(x_1(s), s))^T (x_1(s) - x_2(s)) ds \tag{13}$$

Since f is Lipschitz, we have

$$(x_2(t) - x_1(t))^2 \leqslant 2L \int_0^t \|x_1(s) - x_2(s)\|^2 ds \tag{14}$$

Existence and uniqueness results

Gronwall Lemma

Let a a positive constant and m a integrable function, nonnegative almost everywhere on $(0, t_{\max})$ and a function ϕ a continuous function on $[0, t_{\max}]$. If

$$\forall t \in [0, t_{\max}], \phi(t) \leqslant a + \int_0^t m(s)\phi(s)\, ds \tag{12}$$

then

$$\forall t \in [0, t_{\max}], \phi(t) \leqslant a \exp(\int_0^t m(s)\, ds) \tag{13}$$

Applying the Gronwall Lemma, for $a = 0$ and $m(s) = 2L$ and $\phi(s) = \|x_1(s) - x_2(s)\|^2$, we get

$$\|x_2(t) - x_1(t))\|^2 \leqslant 0 \implies x_2 = x_1 \tag{14}$$

Outline

Come back to LCS with $D = 0$ but $B \neq I_d \neq C$

Theorem (LCS as maximal monotone differential inclusion)

Let us consider the following LCS

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + a(t), \quad x(0) = x_0 \\ y(t) = Cx(t) \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0. \end{cases} \tag{15}$$

If there exists P a symmetric definite positive matrix such that

$$PB = C^T \tag{16}$$

then we can perform a change of variable $z = Rx$ with $R^2 = P, R \geqslant 0, R = R^T$

$$-(\dot{z}(t) - RAR^{-1}z(t) - Ra(t)) \in RB\,\partial\Psi_{\mathbb{R}^m_+}(CR^{-1}z(t)) \tag{17}$$

such that (17) *is a maximal monotone differential inclusion.*

Come back to LCS with $D = 0$ but $B \neq I_d \neq C$

We have the following equivalence

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + a(t) \\ y(t) = Cx(t) \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0, \\ x(0) = x_0 \end{cases} \iff \begin{cases} -(\dot{x}(t) - Ax(t) - a(t)) \in B\partial\Psi_{\mathbb{R}^m_+}(Cx(t)), \\ \quad x(0) = x_0 \end{cases} \tag{15}$$

We can perform a change of variable $z = Rx$ with $R^2 = P, R \geqslant 0, R = R^T$

$$-(\dot{z}(t) - RAR^{-1}z(t) - Ra(t)) \in RB\,\partial\Psi_{\mathbb{R}^m_+}(CR^{-1}z(t)) \tag{16}$$

Come back to LCS with $D = 0$ but $B \neq I_d \neq C$

For a matrix E, the function $\phi(x) = \Psi_{\mathbb{R}^m_+}(Ex)$ is a proper convex function and its subdifferential is given by

$$\partial\phi(x) = E^T \partial\Psi_{\mathbb{R}^m_+}(Ex) \tag{15}$$

($\mathrm{Im}(E)$ contains a point of $\mathrm{ri}(D(\partial\Psi_{\mathbb{R}^m_+})))$ (Chain rule)
In our application, we set $E = CR^{-1}$ and we have

$$E^T = R^{-T} C^T = R^{-1} R^2 B = RB \tag{16}$$

The obtained inclusion

$$-(\dot{z}(t) - RAR^{-1}z(t) - Ra) \in \partial\Phi(z(t)) = E^T \partial\Psi_{\mathbb{R}^m_+}(Ez(t)), \tag{17}$$

is a maximal monotone differential inclusion

Outline

Filippov differential inclusions

Ordinary differential equations with discontinuous right hand side

Let us consider the Cauchy problem

$$\begin{cases} \dot{x}(t) = f(x) \\ x(t_0) = x_0 \end{cases} \tag{18}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a bounded function.

- The Cauchy problem may have no (Carathéodory) solution, if f is discontinuous

Filippov differential inclusions

Example

$$\begin{cases} \dot{x}(t) = a - \mathrm{sgn}(x(t)), \text{ with } 0 < a < 1 \\ x(t_0) = x_0 \end{cases} \tag{19}$$

with the signum function defined as

$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases} \tag{20}$$

- $x_0 = 0$ no solution
- $x_0 > 0$ no global solution

Filippov differential inclusion

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a bounded function.

Definition (Filippov's concept of solutions[2, 5])

The Filippov differential inclusion is defined by

$$\begin{cases} \dot{x}(t) \in F(x) \\ x(t_0) = x_0 \end{cases} \tag{21}$$

with

$$F(x) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}\{f(x+\varepsilon B_n)\}} \tag{22}$$

where

- B_n is the unit ball of $\mathbb{R}^n$
- $\mathrm{co}(X)$ defines the convex hull of X
- $\overline{X}$ is the closure of X

Filippov differential inclusion

Geometrical interpretations

- If f is continuous in x, then $F(x) = \{f(x)\}$
- The solution does not depend on the values of $f(x)$ at the discontinuity points
  An equivalent formulation is given by

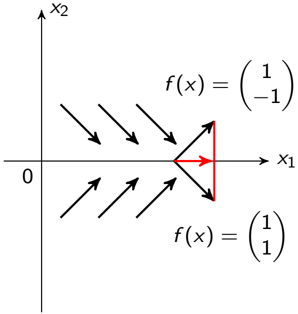
$$F(x) = \bigcap_{\varepsilon>0} \bigcap_{\mu(S)=0} \overline{\text{co}\{f(B(x,\varepsilon) \setminus S)\}} \tag{23}$$

where μ is the Lebesgue measure.
More generally, the definition works also for f essentially bounded (bounded on a bounded neighborhood of every point, excluding sets of measure zero).

Filippov differential inclusion

Geometrical interpretations

- smearing out the vector fields near the discontinuities



Filippov differential inclusions

Computation of the Filippov right-hand-side in practice.

In practice, we often have the following assumptions :

- the set of discontinuities points of f, denoted as $\mathcal{M}$ is given by a set of hypersurfaces of co-dimension 1:

$$\mathcal{M} = \{x \in \mathbb{R}^n \mid \varphi_k(x) = 0, k = [\![1, N]\!]\} \tag{23}$$

  where $\varphi_k : \mathbb{R}^n \to \mathbb{R}$ are differentiable maps that define a finite number of domains $G_j \subset \mathbb{R}^n, j \in [\![1, M]\!]$.
- for $x \in \mathcal{M}$ and a neighborhood Ω of x, we have

$$G_j \cap \Omega \neq \emptyset, \text{for some } i \in \mathrm{I}(x) \tag{24}$$

- for $i \in \mathrm{I}(x)$, the function f admits a limit for all the domains such that

$$\lim_{\substack{y \to x \\ y \in G_i}} f(y, t) = f_i(x, t) \tag{25}$$

- then we get

$$F(x) = \mathrm{co}\{f_i(x, t), i \in \mathrm{I}(x)\} \tag{26}$$

Filippov differential inclusions

Example

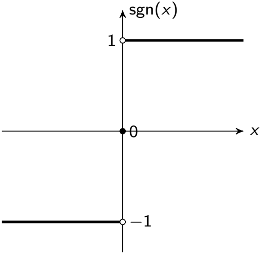
$$\begin{cases} \dot{x}(t) = a - \mathrm{sgn}(x(t)), \text{ with } 0 < a < 1 \\ x(t_0) = x_0 \end{cases} \tag{27}$$

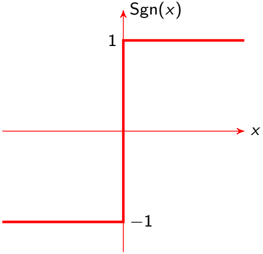
- $x > 0$, we get $F(x) = \{a - 1\}$
- $x < 0$, we get $F(x) = \{a + 1\}$
- $x = 0$, we get $F(x) = \overline{\mathrm{co}\{a - 1, a + 1\}} = [a - 1, a + 1]$

For $x(t_\star) = 0$, we have a solution $\dot{x}(t) = 0, t \in [t_\star, +\infty)$ since

$$0 \in F(x) = [a - 1, a + 1] \tag{28}$$

Filippov differential inclusions





Upper semi-continuous differential inclusion

Definition (Upper semi-continuous set-valued map [5, 4])

A set-valued map $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is upper semi-continuous at $x_0 \in \mathbb{R}^n$ if for any open set M containing $F(x_0)$ there exists a neighborhood Ω of x_0 such that $F(\Omega) \subset M$

Theorem (Upper semi-continuous differential inclusion)

Let us consider the following differential inclusion

$$\dot{x}(t) \in T(x(t)) \tag{27}$$

where $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is upper semi-continuous set valued map with closed convex values. Then the differential inclusion (27) *has always a solution for any $x(t_0) = x_0$.*

Upper-semi continuous differential inclusion

Idea of the proof

- Approximation of T by a sequence of T_k of Lipschitz set-valued operators.
- Passing to the limit by compactness.

Filippov differential inclusion

Theorem (Filippov [2])

The Filippov differential inclusion (22) *has always a solution for any* $x(t_0) = x_0$.

Filippov differential inclusion

Lemma (Closed Graph I)

Let F be an upper semi-continuous multi-valued map with closed values. Then, the graph of F is closed.

Lemma (Closed Graph II)

Assume that the graph of F is closed and the set

$$M = \overline{\{F(x) \mid |x - x_0| < \delta\}} \tag{28}$$

with $\delta > 0$, is compact. Then, F is upper semi-continuous at x_0

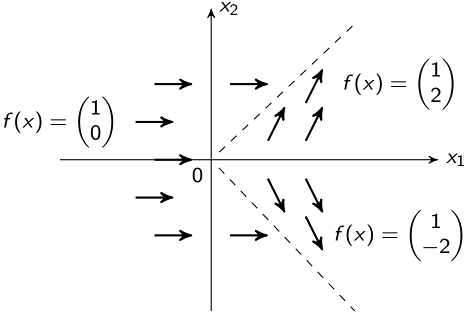
Ideas of the proof for the Filippov DI

- The values of F is closed and convex by definition.
- The graph of $F(x)$ is closed and F is bounded, by the Lemma Closed Graph II, F is a upper-continuous map.

Filippov differential inclusion

Comments

Uniqueness of solutions is not guaranteed by Filippov convexifcation.



Filippov differential inclusion

Example (Utkin's example)

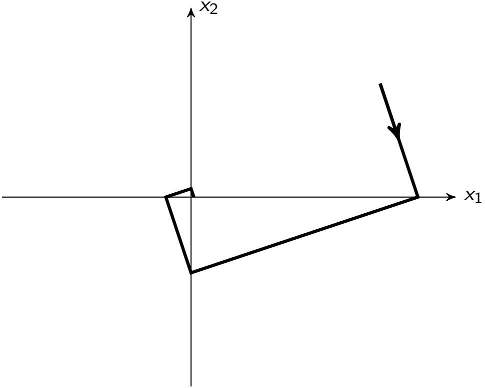
$$\begin{cases} \dot{x}_1(t) = -\operatorname{sgn}(x_1) + 2\operatorname{sgn}(x_2) \\ \dot{x}_2(t) = -2\operatorname{sgn}(x_1) - \operatorname{sgn}(x_2), \end{cases} \tag{29}$$

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad &\text{if} \quad & x \in X_1 = \{x_1 > 0, x_2 > 0\}, \\ \dot{x}(t) &= \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad &\text{if} \quad & x \in X_2 = \{x_1 < 0, x_2 > 0\}, \\ \dot{x}(t) &= \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad &\text{if} \quad & x \in X_3 = \{x_1 < 0, x_2 < 0\}, \\ \dot{x}(t) &= \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \quad &\text{if} \quad & x \in X_4 = \{x_1 > 0, x_2 < 0\}. \end{aligned} \tag{30}$$

Filippov differential inclusion

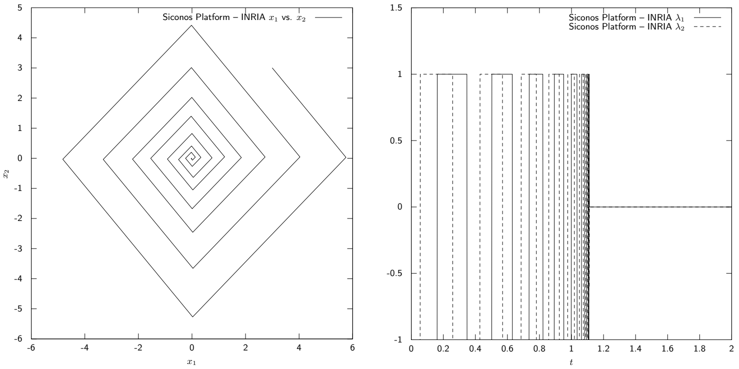
Example (Utkin's example)

Finite accumulation of switches in 0.

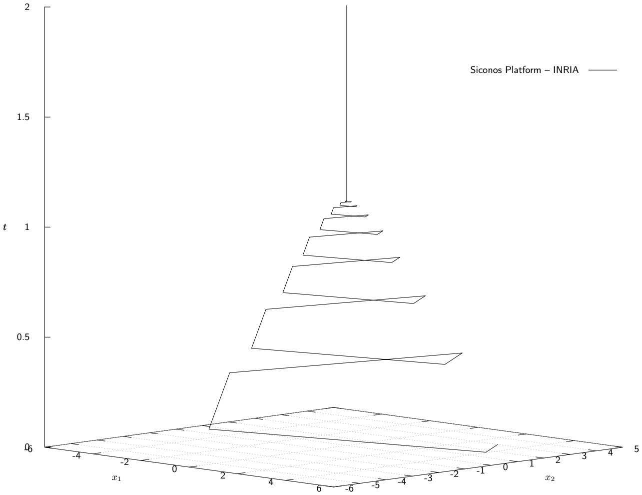


Filippov differential inclusion

$$\begin{cases} \dot{x}_1(t) = -\operatorname{Sgn}(x_1) + (1+c)\operatorname{Sgn}(x_2) \\ \dot{x}_2(t) = -(1+c)\operatorname{Sgn}(x_1) - \operatorname{Sgn}(x_2), \end{cases} \quad \text{with } c = 25 \tag{31}$$



Filippov differential inclusion



Outline

Equilibria for differential inclusions

Computation of equilibria

The equilibria of

$$-(\dot{x}(t) + f(x)) \in T(x(t)) \tag{32}$$

are given by the following generalized equation

$$-f(\tilde{x}) \in T(\tilde{x}) \tag{33}$$

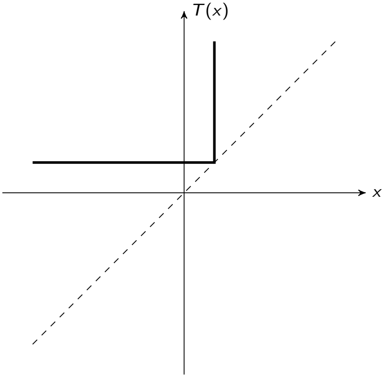
Generalized Equation

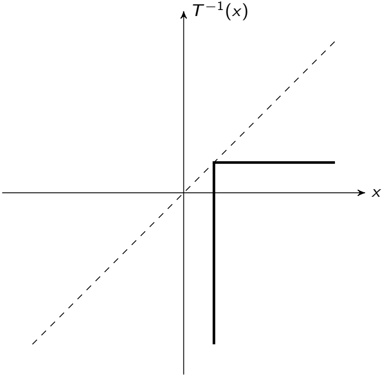
$$0 \in f(\tilde{x}) + T(\tilde{x}) \tag{34}$$

Equilibria for differential inclusion. The simple case $f(\tilde{x}) = 0$

$$0 \in T(\tilde{x}) \Longleftrightarrow x \in T^{-1}(0) \tag{35}$$

A condition for $T^{-1}(0) \neq \emptyset$ is $0 \in D(T^{-1}) = R(T)$.





Equilibria for differential inclusion. The simple case $f(\tilde{x}) = 0$

Theorem

Let $\phi : \mathbb{R}^n \to \mathbb{R} \cup +\infty$ be a proper convex function. Then

$$0 \in \partial\phi(\tilde{x}) \iff \tilde{x} \in \operatorname*{argmin}_{z \in \mathbb{R}^n} \phi(z) \tag{35}$$

A solution to $0 \in \partial f(\tilde{x})$ exists if $\min_{z \in \mathbb{R}^n} \phi(z) > -\infty$

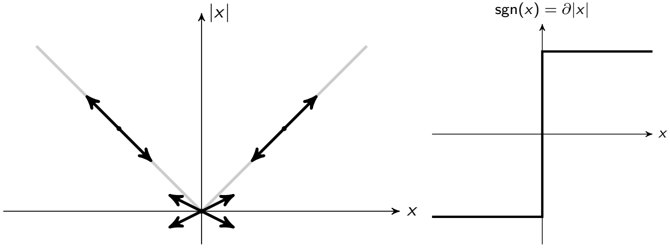


Figure: Absolute value function

Equilibria for differential inclusion. The simple case $f(\tilde{x}) = 0$

Example $(\phi(x) = \Psi_C(x))$

$$\operatorname*{argmin}_{z \in \mathbb{R}^n} \Psi_C(z) = C \tag{35}$$

Equilibria for differential inclusion. The affine case $f(\tilde{x}) = A\tilde{x} + a$

$T(x) = \partial\Psi_C(x)$ with C a polyhedral set $C = \{Cx + d \geqslant 0\}$

$$\partial\Psi_C(x) = \mathrm{N}_C(x) = \{s \mid s = -C^T\lambda, 0 \leqslant \lambda \perp Cx + d \geqslant 0\} \tag{36}$$

The generalized equation

$$-(A\tilde{x} + a) \in \partial\Psi_C(\tilde{x}) \tag{37}$$

is equivalent to the following MLCP

$$\begin{cases} A\tilde{x} + a = C^T\lambda \\ y = Cx + d \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{38}$$

that can be written in turns as an inclusion

$$-\left(\begin{bmatrix} A & -C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \lambda \end{bmatrix} + \begin{bmatrix} a \\ d \end{bmatrix}\right) \in \partial\Psi_{\mathbb{R}^n \times \mathbb{R}^m_+}\left(\begin{bmatrix} \tilde{x} \\ \lambda \end{bmatrix}\right) \tag{39}$$

If A is semi-definite positive then $\begin{bmatrix} A & -C^T \\ C & 0 \end{bmatrix}$ is semi-definite positive. If the inclusion is feasible, then it is solvable.

Equilibria for differential inclusion. The affine case $f(\tilde{x}) = A\tilde{x} + a$

$T(x) = \partial\Psi_C(x)$ with C a convex set and A symmetric definite positive

We can define a convex function $\Phi(x) = \Psi_C(x) + \frac{1}{2}x^T Ax + a^T x$. Then

$$\min_{z\in\mathbb{R}^n} \Phi(x) = \min_{z\in\mathbb{R}^n} \Psi_C(x) + \frac{1}{2}x^T Ax + a^T x = \min_{z\in C} \frac{1}{2}x^T Ax + a^T x \tag{40}$$

This is a convex minimization problem that possess a solution and the optimality conditions are

$$0 \in \partial\Phi(\tilde{x}) = A\tilde{x} + a + \partial\Psi_C(\tilde{x}) \tag{41}$$

Remark

If a polyhedral set $C = \{Cx + d \geqslant 0\}$, then the optimality condition are

$$\begin{cases} Ax + a = C^T\lambda \\ y = Cx + d \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{42}$$

Outline

Application of standard results for stability and asymptotic behavior

Sufficient assumptions

- Existence of absolutely continuous solution.
- Continuity with respect to initial conditions.
- Lyapunov function $V \in \mathcal{C}^1$
- Invariants in the interior of the domain of maximal monotone operators

With these assumptions, the main result for smooth systems can be proved

- Lyapunov stability theorems.
- Lasalle invariance principle.

Relaxed results

In the literature, a large number of results relax the assumptions that are sometimes not necessary. For the sake of simplicity, we assume that there are valid for our applications. In the sequel, we present more specific results for Maximal Monotone differential inclusions

Lyapunov stability of monotone differential inclusions

Monotone differential inclusions, $x(0) = x_0$

- Standard form

$$-\dot{x}(t) \in T(x(t)) \tag{43}$$

- Standard perturbed form

$$-(\dot{x}(t) + f(x(t), t) \in T(x(t)) \tag{44}$$

- Sub-differential of Φ convex, proper and lower-semi-continuous

$$-(\dot{x}(t) + f(x(t), t) \in \partial\Phi(x(t)) \tag{45}$$

Solutions

We assume that there exists an absolutely continuous solution such that one of the previous inclusion is satisfied almost everywhere

Absolutely continuous functions

Definition

Let I be an interval in the real line $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is absolutely continuous on I if for every positive number ε, there exists a positive number δ such that whenever a finite sequence of pairwise disjoint sub-intervals (x_k, y_k) of I satisfies

$$\sum_k (y_k - x_k) < \delta \tag{46}$$

then

$$\sum_k |f(y_k) - f(x_k)| < \varepsilon \tag{47}$$

Absolutely continuous functions

Proposition

The following conditions on a real-valued function f on a compact interval $[a, b]$ are equivalent:

1. f is absolutely continuous
2. f has derivative almost everywhere, the derivative is Lebesgue integrable, and

$$f(t) = f(a) + \int_a^t f'(t)dt \tag{46}$$

   for all x on $[a, b]$.

3. there exists a Lebesgue integrable function g on $[a, b]$ such that

$$f(t) = f(a) + \int_a^t g(t)dt \tag{47}$$

   for all x on $[a, b]$.

If these equivalent conditions are satisfied then necessarily $g = f'$ almost everywhere. Equivalence between (1) and (3) is known as the fundamental theorem of Lebesgue integral calculus, due to Lebesgue.

Absolutely continuous functions

Properties

- The sum and difference of two absolutely continuous functions are also absolutely continuous.
- If the two functions are defined on a bounded closed interval, then their product is also absolutely continuous.
- If an absolutely continuous function is defined on a bounded closed interval and is nowhere zero then its reciprocal is absolutely continuous.
- Every absolutely continuous function is uniformly continuous and, therefore, continuous. Every Lipschitz-continuous function is absolutely continuous.
- If $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then it is of bounded variation on $[a, b]$.
- If $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then it can be written as the difference of two monotonic non-decreasing absolutely continuous functions on [a,b].
- If $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then it has the Luzin N property (that is, for any $L \subseteq [a, b]$ such that $\lambda(L) = 0$, it holds that $\lambda(f(L)) = 0$, where λ stands for the Lebesgue measure on $\mathbb{R}$).
- $f : I \to \mathbb{R}$ is absolutely continuous if and only if it is continuous, is of bounded variation and has the Luzin N property.
- The composition of two absolutely continuous functions is **not** necessarily a absolutely continuous function

Absolutely continuous functions

Proposition

Let f be Lipschitz continuous on $\mathbb{R}$ and g be an absolutely continuous function on $[a, b]$. Then the composition $f \circ g$ is absolutely continuous on $[a, b]$.

Lyapunov stability of monotone differential inclusions

Differentiability of the Lyapunov function

Let us assume that we have a $\mathcal{C}^1$ Lyapunov function, then

$$\begin{array}{rccc} \mathcal{V}(t): & \mathbb{R} & \rightarrow & \mathbb{R} \\ & t & \mapsto & V(x(t)) \end{array} \tag{46}$$

is absolutely continuous if $x(t)$ is also absolutely continuous.
This implies that $\dot{\mathcal{V}}(t)$ exists almost everywhere
Furthermore, if $\dot{\mathcal{V}}(t) \leqslant 0$ almost everywhere then

$$\mathcal{V}(t) - \mathcal{V}(0) = \int_0^t \dot{\mathcal{V}}(t)dt \leqslant 0 \implies \mathcal{V}(t) \text{ is decreasing} \tag{47}$$

Lyapunov stability of monotone differential inclusions

Monotone differential inclusions $-(\dot{x}(t) + f(x(t))) \in T(x(t))$

Let us formulate the autonomous differential inclusion as

$$\begin{cases} \dot{x}(t) + f(x(t)) = \lambda(t) \\ -\lambda(t) \in T(x(t)) \end{cases} \tag{48}$$

If V is $\mathcal{C}^1$, we want to satisfy

$$\dot{\mathcal{V}}(t) = \nabla_x V(x(t)) \cdot [-f(x(t)) + \lambda(t)] \leqslant 0 \text{ with } -\lambda(t) \in T(x(t)) \tag{49}$$

Lyapunov stability of monotone differential inclusions

Monotone differential inclusions, $-\dot{x}(t) \in T(x(t))$

The case when $f(x(t)) = 0$ and we choose $V(x) = \frac{1}{2}\|x - \tilde{x}\|^2$, $\nabla_x V(x) = (x - \tilde{x})$ than we get

$$\dot{\mathcal{V}}(t) = (x(t) - \tilde{x})^T \lambda(t), \text{ with } -\lambda(t) \in T(x(t)) \tag{50}$$

Let us consider and equilibrium point $\tilde{x} \in \mathring{D}(T)$, $0 \in T(\tilde{x})$ then the monotony implies

$$(-\lambda(t) - 0)^T (x(t) - \tilde{x}) \geqslant 0 \tag{51}$$

that is

$$\dot{\mathcal{V}}(t) = (x(t) - \tilde{x})^T \lambda(t) \leqslant 0 \tag{52}$$

For a monotone differential inclusion $-\dot{x}(t) \in T(x(t))$, a equilibrium with $\tilde{x} \in \mathring{D}(T)$ is Lyapunov stable. If T is strictly monotone, $\tilde{x}$ is asymptotically stable.

Lyapunov stability of monotone differential inclusions

Monotone differential inclusions

If $\tilde{x} \in \partial D(T)$, the classical Lyapunov stability theorem does no longer apply immediately, since it is not possible to find a open set Ω that is a neighborhood of $\tilde{x}$.

Lyapunov stability of monotone differential inclusions

Assumption 1

Let us consider the differential inclusion

$$-(\dot{x}(t) + f(x(t)) \in \partial\Phi(x(t)), \quad dt\text{-a.e} \tag{53}$$

with

- $\Phi : \mathbb{R}^n \to \mathbb{R}$ a proper lower semi-continuous convex function
- $f : \mathbb{R}^n \to \mathbb{R}^n$ a Lipschitz continuous function
- an equilibrium point in $0 \in D(\partial\Phi)$, that is

$$-f(0) \in \partial\Phi(0).$$

If Assumption 1 holds then we have a unique absolutely continuous solution whatever $x_0 \in D(\partial\Phi)$.

Lyapunov stability of monotone differential inclusions

The following theorems are extracted from [3].

Theorem

Let us assume the Assumption 1 holds. Suppose that there exist $R > 0, a > 0$ *and* $V \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ *such that*

$$(\forall x \in D(T), \|x\| = R), V(x) \geqslant a \tag{53}$$

and

$$\nabla_x V(x) \cdot f(x) + \Phi(x) - \Phi(x - \nabla V(x)) \geqslant 0. \tag{54}$$

Then, for any $x_0 \in D(\partial\Phi)$ *with* $\|x_0\| \leqslant R$ *and* $V(x_0) < a$*, the solution* $x(t; t_0, x_0)$ *satisfies*

$$\forall t \geqslant t_0, \|x(t; t_0, x_0)\| < R \tag{55}$$

Lyapunov stability of monotone differential inclusions

Idea of the proof:

$$\dot{\mathcal{V}}(t) = \nabla_x V(x(t)) \cdot \dot{x}(t) \text{ a.e} \tag{56}$$

with

$$\dot{x}(t) + f(x(t)) = \lambda(t) \text{ with } -\lambda(t) \in \partial\Phi(x(t)) \text{ a.e} \tag{57}$$

Applying the definition of the sub-differential,

$$\begin{array}{c} -\lambda(t) \in \partial\Phi(x(t)) \\ \Updownarrow \\ (\lambda(t))^T(v - x(t)) + \Phi(v) - \Phi(x(t)) \geqslant 0, \forall v \in \mathbb{R}^n \end{array} \tag{58}$$

we get

$$(\dot{x}(t) + f(x(t)))^T(v - x(t)) + \Phi(v) - \Phi(x(t)) \geqslant 0, \forall v \in \mathbb{R}^n \tag{59}$$

Lyapunov stability of monotone differential inclusions

Idea of the proof:

Let us choose $v = x - \nabla_x V(x(t))$

$$-(\dot{x}(t) + f(x(t)))^T \nabla_x V(x(t)) + \Phi(x(t) - \nabla_x V(x(t))) - \Phi(x(t)) \geqslant 0 \text{ a.e} \tag{56}$$

thus

$$\dot{\mathcal{V}}(t) \leqslant -\left[f(x(t)))^T \nabla_x V(x(t)) + \Phi(x(t)) - \Phi(x(t) - \nabla_x V(t))\right] \text{ a.e} \tag{57}$$

from the assumption we get

$$\dot{\mathcal{V}}(t) \leqslant 0 \text{ a.e} \tag{58}$$

Lyapunov stability of monotone differential inclusions

Let us denote by B_σ the ball of radius $\sigma > 0$, $B_\sigma = \{x \mid \|x\| \leqslant \sigma\}$

Theorem (Stability)

Let us assume the Assumption 1 holds. Suppose that there exists $\sigma > 0$ and $V \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ such that $V(0) = 0$

$$\forall x \in D(\partial\Phi) \cap B_\sigma, V(x) \geqslant a(\|x\|) \tag{59}$$

with $a : [0, \sigma] \to \mathbb{R}$, $a(t) > 0, \forall t \in (0, \sigma)$, and

$$\forall x \in D(\partial\Phi) \cap B_\sigma, \nabla_x V(x) \cdot f(x) + \Phi(x) - \Phi(x - \nabla V(x)) \geqslant 0. \tag{60}$$

Then 0 is a stable equilibrium.

Lyapunov stability of monotone differential inclusions

Example

Let us consider this example:

$$f(x) = \begin{bmatrix} x_2 \\ -\sin(x_1) \end{bmatrix}, \quad x \in \mathbb{R}^2 \tag{61}$$

and

$$\Phi(x) = \Psi_{\mathbb{R}^2_+}(x) \tag{62}$$

We choose

$$V(x) = 1 - \cos(x_1) + \frac{x_2^2}{2} \tag{63}$$

and we obtain

$$\nabla_x V(x) = \begin{bmatrix} \sin(x_1) \\ x_2 \end{bmatrix} \tag{64}$$

and

$$\nabla_x V(x) \cdot f(x) = 0 \tag{65}$$

Lyapunov stability of monotone differential inclusions

Example

There exists $\sigma > 0$ such that

$$\|x\| \geqslant \sigma \implies 1 - \cos(x_1) \geqslant \frac{x_1^2}{4} \tag{61}$$

Thus

$$\|x\| \geqslant \sigma \implies V(x) \geqslant \frac{x_1^2 + x_2^2}{4} \tag{62}$$

We have also

$$x \in \mathbb{R}^2_+ \implies x - \nabla_x V(x) = \begin{bmatrix} x_1 - \sin(x_1) \\ 0 \end{bmatrix} \in \mathbb{R}^2_+ \tag{63}$$

Thus

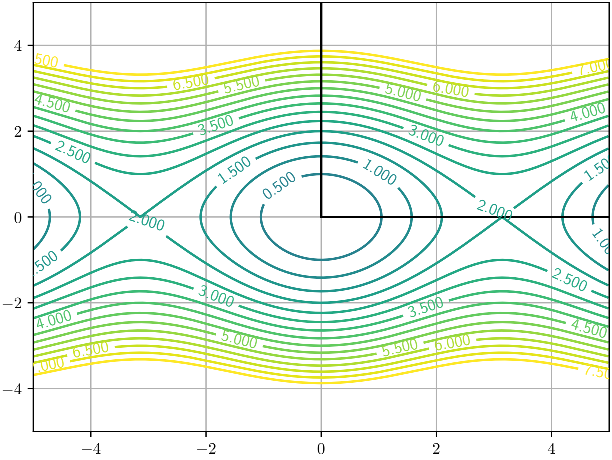
$$x \in \mathbb{R}^2_+, \|x\|\sigma \implies \nabla_x V(x) \cdot f(x) + \Phi(x) - \Phi(x - \nabla V(x)) = 0. \tag{64}$$

With the previous theorem, we can conclude to the stability of the equilibrium $\tilde{x} = 0$.

Lyapunov stability of monotone differential inclusions

Example

level sets of $V(x) = 1 - \cos(x_1) + 1/2x_2^2$



Lyapunov stability of monotone differential inclusions

Theorem (Asymptotic Stability)

Let us assume the Assumption 1 holds.
Suppose that there exist $\sigma > 0, \lambda > 0$ *and* $V \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ *such that* $V(0) = 0$

$$\forall x \in D(\partial\Phi) \cap B_\sigma, V(x) \geqslant a(\|x\|) \tag{61}$$

with $a : [0, \sigma] \to \mathbb{R}$, $a(t) > ct^\tau, \forall t \in (0, \sigma)$ *for some* $c > 0, \tau > 0$, *and*

$$\forall x \in D(\partial\Phi) \cap B_\sigma, \nabla_x V(x) \cdot f(x) + \Phi(x) - \Phi(x - \nabla V(x)) \geqslant \lambda V(x). \tag{62}$$

Then 0 *is an asymptotic stable equilibrium.*

Lyapunov stability of monotone differential inclusions

Definition (Set of stationary points)

$$\mathcal{S}(F,\Phi) = \{x \in D(\partial\Phi) \mid -f(x) \in \partial\Phi(x)\} \tag{63}$$

or equivalently

$$\mathcal{S}(F,\Phi) = \{x \in D(\partial\Phi) \mid f^T(x)(v-z) + \Phi(v) - \Phi(x), \forall v \in \mathbb{R}^n\} \tag{64}$$

Definition

Let $V \in \mathcal{C}^1$. We define

$$\mathcal{E}(F,\Phi,V) = \{x \in D(\partial\Phi) \mid \nabla_x V(x) \cdot f(x) + \Phi(x) - \Phi(x - \nabla_x V(x)) = 0\} \tag{65}$$

Lyapunov stability of monotone differential inclusions

Theorem

Let us assume the Assumption 1 holds.
Let A be a subset of $\mathbb{R}^n$. Suppose that there exists $V \in \mathcal{C}^1(\mathbb{R}^n;\mathbb{R})$ such that

$$\forall x \in D(\partial\Phi) \cap A, \nabla_x V(x) \cdot f(x) + \Phi(x) - \Phi(x - \nabla V(x)) \geqslant 0. \tag{66}$$

Then

$$\mathcal{S}(F,\Phi) \cap A \subset \mathcal{E}(F,\Phi,V) \tag{67}$$

Lyapunov stability of monotone differential inclusions

Theorem

Let us assume the Assumption 1 holds.
Suppose that there exist $\sigma > 0$ *and* $V \in \mathcal{C}^1(\mathbb{R}^n;\mathbb{R})$ *such that*

$$\forall x \in D(\partial\Phi) \cap B_\sigma, \nabla_x V(x) \cdot f(x) + \Phi(x) - \Phi(x - \nabla V(x)) \geqslant 0. \tag{68}$$

and

$$\mathcal{E}(F, \Phi, V) \cap B_\sigma = \{0\} \tag{69}$$

Then the stationary solution is isolated in $\mathcal{S}(F, \Phi)$

Lyapunov stability of monotone differential inclusions

Assumption

Theorem

Let us assume the Assumption 1 holds.
Suppose that there exists $V \in \mathcal{C}^1(\mathbb{R}^n;\mathbb{R})$ *such that*

$$\forall x \in D(\partial\Phi), \nabla_x V(x)\cdot f(x) + \Phi(x) - \Phi(x - \nabla V(x)) \geqslant 0. \tag{70}$$

and

$$\mathcal{E}(F,\Phi,V) = \{0\} \tag{71}$$

Then $\mathcal{S}(F,\Phi) = \{0\}$ *that is the stationary solution is the unique equilibrium point.*

[1] J.P. Aubin and A. Cellina. *Differential Inclusions: Set-valued Maps and Viability Theory*. Springer, Berlin, 1984.

[2] A. F. Filippov. *Differential Equations with Discontinuous Right Hand Sides*. Kluwer, Dordrecht, the Netherlands, 1988.

[3] D. Goeleven. *Complementarity and Variational Inequalities in Electronics*. Academic Press, 2017.

[4] R.T. Rockafellar and R.J.B. Wets. *Variational Analysis*, volume 317. Springer Verlag, New York, 1997.

[5] G. Smirnov. *Introduction to the Theory of Differential Inclusions*, volume 41 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, Rhode Island, 2002.