Lecture. Formulation of Nonsmooth Dynamical Systems (NSDS). Low relative degree

Vincent Acary

December 2, 2018

Dynamical Complementarity Systems (DCS)

Example (The RLC circuit with a diode. A half wave rectifier)

A LC oscillator supplying a load resistor through a half-wave rectifier.

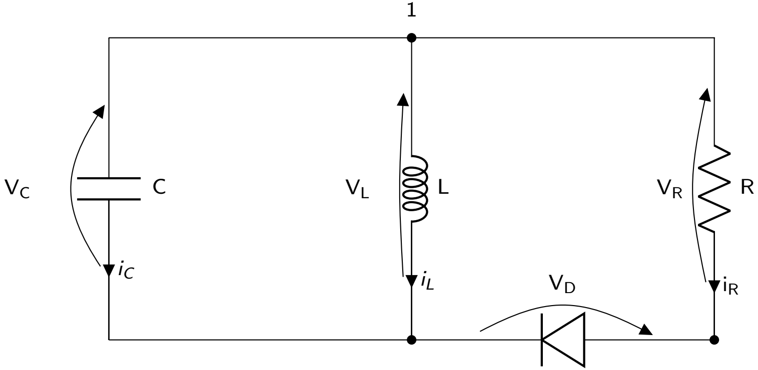
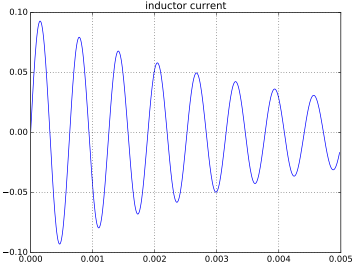
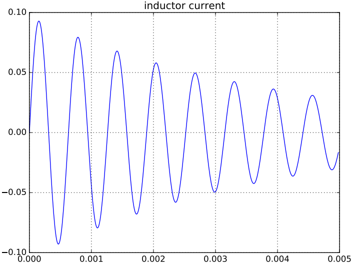


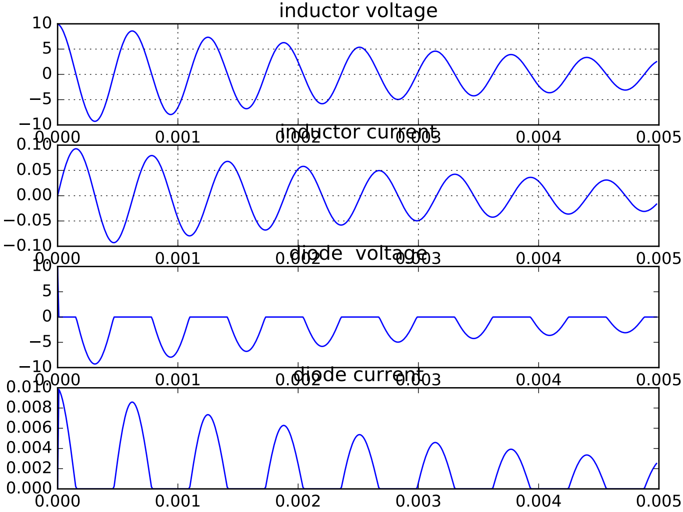
Figure: Electrical oscillator with half-wave rectifier

Example (The RLC circuit with a diode. A half wave rectifier)





Example (The RLC circuit with a diode. A half wave rectifier)



Example (The RLC circuit with a diode. A half wave rectifier)

- Kirchhoff laws :

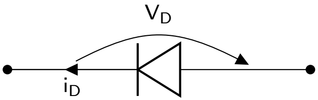
$$\begin{array}{l} v_L = v_C \\ v_R + v_D = v_C \\ i_C + i_L + i_R = 0 \\ i_R = i_D \end{array}$$

- Branch constitutive equations for linear devices are :

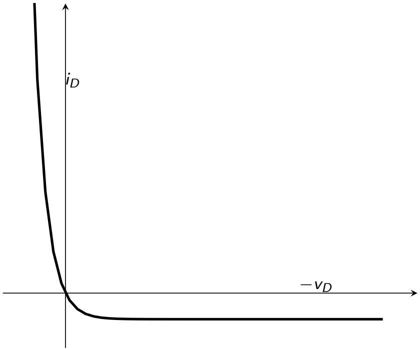
$$\begin{array}{l} i_C = C\dot{v}_C \\ v_L = L\dot{i}_L \\ v_R = Ri_R \end{array}$$

- "branch constitutive equation" of the ideal diode ?

Example (The RLC circuit with a diode. A half wave rectifier)



(a) A diode



(b) Shockley's law $i_D = i_s(exp(-\frac{v_D}{nv_T}) - 1)$

Figure: A nonlinear model of diode

Example (The RLC circuit with a diode. A half wave rectifier)

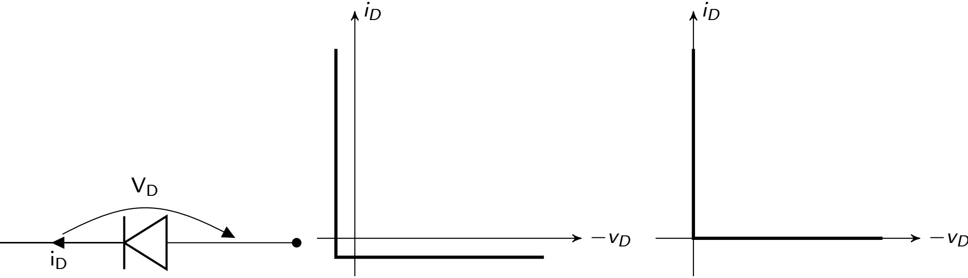


Figure: A ideal diode

Complementarity condition :

$$i_D \geqslant 0, -v_D \geqslant 0, i_D v_D = 0 \Longleftrightarrow 0 \leqslant i_D \perp -v_D \geqslant 0$$

Example (The RLC circuit with a diode. A half wave rectifier)

- Kirchhoff laws :

$$\begin{array}{l} v_L = v_C \\ v_R + v_D = v_C \\ i_C + i_L + i_R = 0 \\ i_R = i_D \end{array}$$

- Branch constitutive equations for linear devices are :

$$\begin{array}{l} i_C = C\dot{v}_C \\ v_L = L\dot{i}_L \\ v_R = Ri_R \end{array}$$

- "branch constitutive equation" of the ideal diode

$$0 \leqslant i_D \perp -v_D \geqslant 0$$

Example (The RLC circuit with a diode. A half wave rectifier)

The following linear complementarity system is obtained :

$$\begin{pmatrix} \dot{v}_L \\ \dot{i}_L \end{pmatrix} = \begin{pmatrix} 0 & \frac{-1}{C} \\ \frac{1}{L} & 0 \end{pmatrix} \cdot \begin{pmatrix} v_L \\ i_L \end{pmatrix} + \begin{pmatrix} \frac{-1}{C} \\ 0 \end{pmatrix} \cdot i_D$$

together with a state variable x and one of the complementary variables λ :

$$x = \begin{pmatrix} v_L \\ i_L \end{pmatrix}, \qquad \lambda = i_D, \qquad y = -v_D$$

and

$$y = -v_D = \begin{pmatrix} -1 & 0 \end{pmatrix} x + \begin{pmatrix} R \end{pmatrix} \lambda,$$

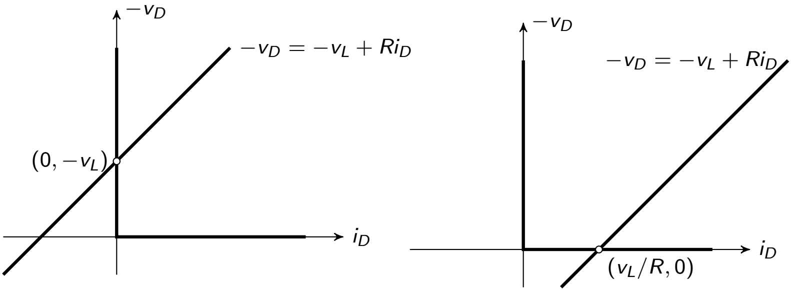
Standard form for LCS

$$\begin{cases} \dot{x} = Ax + B\lambda \\ y = Cx + D\lambda \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases}$$

Example (The RLC circuit with a diode. A half wave rectifier)

$$\left\{\begin{array}{l} y = Cx + D\lambda \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{array}\right. \Rightarrow \left\{\begin{array}{l} -v_D = -v_L + R\, i_D \\ 0 \leqslant -v_D \perp i_D \geqslant 0 \end{array}\right. \tag{1}$$

$$\left.\begin{array}{l} \bullet \quad i_D = 0, -v_D = -v_L \geqslant 0, v_L \leqslant 0 \\ \bullet \quad i_D > 0, -v_D = 0, i_D = \frac{V_L}{R}, V_L > 0 \end{array}\right\} \Rightarrow i_D = \max(0, \frac{v_L}{R}) \tag{2}$$



Example (The RLC circuit with a diode. A half wave rectifier)

Note that the lead matrix 0f the LCP $D = \big(R\big) > 0$:

$$\left\{ \begin{array}{l} y = Cx + D\lambda \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{array} \right. \Longleftrightarrow \lambda = \mathrm{proj}_{\mathbb{R}_+}(-D^{-1}Cx) = \max(0, -D^{-1}Cx)$$

In the application, $i_D = max(0, \frac{v_L}{R})$ and we get

$$\begin{pmatrix} \dot{v}_L \\ \dot{i}_L \end{pmatrix} = \begin{pmatrix} 0 & \frac{-1}{C} \\ \frac{1}{L} & 0 \end{pmatrix} \cdot \begin{pmatrix} v_L \\ i_L \end{pmatrix} + \begin{pmatrix} \frac{-1}{C} \\ 0 \end{pmatrix} \cdot max(0, \frac{v_L}{R})$$

Since max is a Lipschitz operator, we get a standard ODE with Lipschitz r.h.s.

Dynamical Complementarity systems

Notation

Let $I \subset \mathbb{R}$ be an interval.
Let $K \subset \mathbb{R}^m$ be a nonempty closed convex cone and $K^\star$ its dual cone given by

$$K^\star = \{x \in \mathbb{R}^m \mid x^\top y \geqslant 0 \text{ for all } y \in K\}. \tag{1}$$

Dynamical Complementarity systems

Definition (Linear complementarity systems (LCS))

When $K = \mathbb{R}^m_+$, we simply coin the system a linear complementarity system

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + u(t) \\ y(t) = Cx(t) + D\lambda(t) + a(t) \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0, \end{cases} \tag{2}$$

Definition (Linear complementarity systems (LCS) over cones)

A linear complementarity system (LCS) over cones is given as

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + u(t) \\ y(t) = Cx(t) + D\lambda(t) + a(t) \\ K^\star \ni y(t) \perp \lambda(t) \in K, \end{cases} \tag{3}$$

where $t \in I \subset \mathbb{R}$, $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^m$ and $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$.

Dynamical Complementarity systems

Let us consider two smooth ($\mathcal{C}^1$) mappings

$$f : I \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \text{ and } h : I \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m.$$

Definition (Dynamical complementarity systems (DCS) over cones)

A dynamical complementarity system over cones is given as

$$\begin{cases} \dot{x}(t) = f(t, x(t), \lambda(t)) \\ y(t) = h(t, x(t), \lambda(t)) \\ K^\star \ni y(t) \perp \lambda(t) \in K, \end{cases} \tag{4}$$

where $t \in I \subset \mathbb{R}$, $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^m$.

Dynamical Complementarity systems

The notation $y \perp \lambda$ means $y^\top \lambda = 0$. Using basic convex analysis results, standard equivalences

$$K^\star \ni y \perp \lambda \in K \quad \Longleftrightarrow \quad -y \in \mathbb{N}_K(\lambda) \quad \Longleftrightarrow \quad -y \in \partial \Psi_K(\lambda), \tag{5}$$

with the standard definition of the normal cone

$$\mathbb{N}_K(x) = \{ s \in \mathbb{R}^m \mid s^T (y - x) \leqslant 0 \text{ for all } y \in K \} \tag{6}$$

and the definition of the indicator function of K

$$\Psi_K = \begin{cases} 0, & x \in K \\ +\infty & \text{otherwise.} \end{cases} \tag{7}$$

Dynamical Complementarity systems

Definition (Dynamical complementarity systems (DCS))

A dynamical complementarity system (DCS) is given as

$$\left\{\begin{array}{l} \dot{x}(t) = f(t, x(t), \lambda(t)) \\ y(t) = h(t, x(t), \lambda(t)) \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0, \end{array}\right. \tag{8}$$

where $t \in I \subset \mathbb{R}$, $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^m$ is usually called the output vector.

Dynamical Complementarity systems

Let us consider a smooth ($\mathcal{C}^1$) mapping $g : \mathbb{R}^n \to \mathbb{R}^{m\times n}$

Definition (Non Linear complementarity systems (NLCS))

A Non Linear Complementarity System usually (NLCS) is defined by the following system:

$$\begin{cases} \dot{x} = f(x,t) + g(x)^T\lambda \\ y = h(x,\lambda) \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{9}$$

Definition (Gradient Type Complementarity Problem (GTCS))

A Gradient Type Complementarity Problem (GTCS) is defined by the following system:

$$\begin{cases} \dot{x}(t) + f(x(t)) = \nabla_x^T h(x)\lambda \\ y = h(x(t)) \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{10}$$

Dynamical variational inequalities

More general systems may be defined by

$$\begin{cases} \dot{x}(t) = f(t, x(t), \lambda(t)), \\ y(t) = h(t, x(t), \lambda(t)), \\ -y(t) \in \mathbb{N}_X(\lambda(t)), \end{cases} \tag{11}$$

where X is a nonempty closed set of $\mathbb{R}^n$. Some instances where X is not cone are also very interesting in practise. Indeed, note that

$$-y(t) \in \mathbb{N}_{[-1,1]}(\lambda(t)) \quad \Longleftrightarrow \quad -\lambda(t) \in \mathrm{Sgn}(y(t)), \tag{12}$$

For a vector $y \in \mathbb{R}^m$, $\mathrm{Sgn}(y)$ holds component-wise. Let us consider for instance that $X = [-1, 1]^m$ in (11). We end up with a dynamical relay system

$$\begin{cases} \dot{x}(t) = f(t, x(t), \lambda(t)), \\ y(t) = h(t, x(t), \lambda(t)), \\ -\lambda(t) \in \mathrm{sgn}(y(t)). \end{cases} \tag{13}$$

Nature of the solutions

The nature of the solutions is very important for designing consistent time–integration schemes.
Following the properties of the DCS, we can have

- Solutions as continuously differentiable solutions (C^1 solutions)
- Solutions as absolutely continuous functions (AC solutions)
- Solutions as functions of Bounded Variations (BV solutions)
- Solutions as distribution of any order.

Nature of the solutions

In order to say more on the mathematical properties of

$$\begin{cases} \dot{x}(t) = f(t, x(t), \lambda(t)) \\ y(t) = h(t, x(t), \lambda(t)) \\ -y(t) \in \mathbb{N}_X(\lambda(t)), \end{cases} \tag{14}$$

we note that the inclusion into a normal cone is equivalent to the following VI

$$y(t)(\tau - \lambda(t)) \geqslant 0, \text{ for all } \tau \in X, \tag{15}$$

that is

$$h(t, x(t), \lambda(t))(\tau - \lambda(t)) \geqslant 0, \text{ for all } \tau \in X. \tag{16}$$

Let us denote by $\lambda(t) \in \mathrm{SOL}(X, h(t, x(t), \cdot))$ an element of $\mathbb{R}^m$ solution of (16). Depending on the mathematical nature of the mapping $(x, t) \mapsto \mathrm{SOL}(X, h(t, x, \cdot))$, various types of solutions to (14) are obtained.

Solutions as continuously differentiable functions (C^1 solutions)

Solutions as continuously differentiable functions (C^1 solutions)

Solutions as continuously differentiable functions (C^1 solutions)

Assumption

The mapping $(x, t) \mapsto \mathrm{SOL}(X, h(t, x, \cdot))$ is a single–valued Lipschitz function denoted by $\lambda(x, t)$.

ODE with Lipschitz right-hand–side

The substitution of $\lambda(x, t)$ in (14) yields a Ordinary Differential Equation (ODE) with a Lipschitz right–hand–side.
➜ Solutions as continuously differentiable functions (C^1 solutions)

Linear Complementarity Systems (LCS)

Linear Complementarity Systems (LCS)

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + a, \quad x(0) = x_0 \\ y(t) = Cx(t) + D\lambda(t) + b \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0. \end{cases} \tag{17}$$

Concept of solutions

- The solution to the LCS (17) depends strongly on the quadruplet (A, B, C, D) and the initial conditions

Mathematical nature of the solutions

In order to say more on the mathematical properties of the LCS, we need to characterize the solution λ of

$$\begin{cases} y = Cx + D\lambda + b \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{18}$$

of its equivalent formulation in terms of inclusion into a subdifferential

$$-(Cx + D\lambda + b) \in \partial\Psi_{\mathbb{R}^m_+}(\lambda) \tag{19}$$

or in terms of variational inequality

$$(Cx + D\lambda + b)^T(\tau - \lambda) \geqslant 0, \text{ for all } \tau \in \mathbb{R}^m_+ \tag{20}$$

Linear Complementarity Problem

Definition (LCP)

A *Linear complementarity problem* (LCP) is to find a vector λ that satisfies

$$0 \leqslant \lambda \perp M\lambda + q \geqslant 0$$

Theorem (Fundamental result of complementarity theory)

The LCP $0 \leqslant \lambda \perp M\lambda + q \geqslant 0$ has a unique solution λ^* for any $q \in \mathbb{R}^m$ if and only if M is a P-matrix.
In this case the solution λ^* is a piecewise linear function of q (with a finite number of pieces).

Remarks

- A P-matrix has all its principal minors positive. A positive definite matrix is a P-matrix.
- A symmetric P-matrix is a positive definite matrix.
- There exist non-symmetric P-matrices which are not positive definite. And there exist positive definite matrices which are not symmetric!

Solutions as continuously differentiable functions (C^1 solutions)

ODE with Lipschitz right-hand–side

The substitution of $\lambda(x)$ yields a Ordinary Differential Equation (ODE) with a Lipschitz right–hand–side.
➜ Solutions as continuously differentiable functions (C^1 solutions)

The LCS case

The solution $\lambda(x)$ of the following linear complementarity system

$$0 \leqslant \lambda \perp D\lambda + Cx + b \geqslant 0 \tag{21}$$

is unique for all $Cx + b$ if and only if D is a P-Matrix and moreover $\lambda(x)$ is a Lipschitz function of x.

see the example of the RLCD circuit

Example (The RLC circuit with a diode. A half wave rectifier)

A LC oscillator supplying a load resistor through a half-wave rectifier.

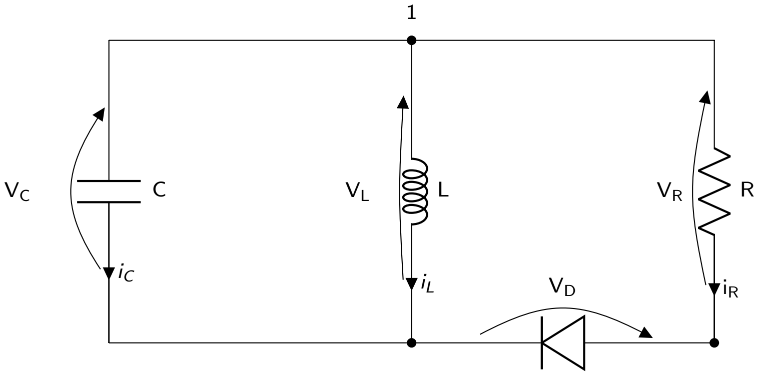


Figure: Electrical oscillator with half-wave rectifier

Example (The RLC circuit with a diode. A half wave rectifier)

The following linear complementarity system is obtained :

$$\begin{pmatrix} \dot{v}_L \\ \dot{i}_L \end{pmatrix} = \begin{pmatrix} 0 & \frac{-1}{C} \\ \frac{1}{L} & 0 \end{pmatrix} \cdot \begin{pmatrix} v_L \\ i_L \end{pmatrix} + \begin{pmatrix} \frac{-1}{C} \\ 0 \end{pmatrix} \cdot i_D$$

together with a state variable x and one of the complementary variables λ :

$$x = \begin{pmatrix} v_L \\ i_L \end{pmatrix}$$

and

$$y = \begin{pmatrix} -1 & 0 \end{pmatrix} x + \begin{pmatrix} R \end{pmatrix} \lambda, \quad \lambda = i_D.$$

Standard form for LCS

$$\begin{cases} \dot{x} = Ax + B\lambda \\ y = Cx + D\lambda \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases}$$

Example (Another RLC circuit with a diode. Circuit a) in [1])

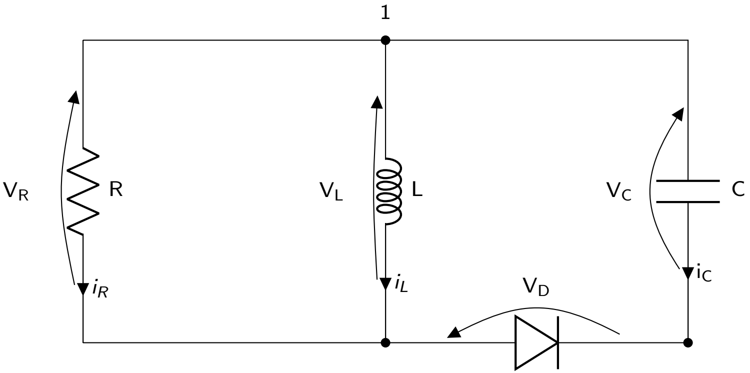


Figure: Electrical oscillator with half-wave rectifier

Example (Another RLC circuit with a diode. Circuit a) in [1])

The following linear complementarity system is obtained :

$$\begin{pmatrix} C\dot{V}_C \\ -\dot{i}_L \end{pmatrix} = \begin{pmatrix} \frac{-1}{RC} & 1 \\ \frac{-1}{LC} & 0 \end{pmatrix} \begin{pmatrix} Cv_C \\ -i_L \end{pmatrix} + \begin{pmatrix} \frac{-1}{R} \\ \frac{-1}{L} \end{pmatrix} (-v_D)$$

together with a state variable x and one of the complementary variables λ :

$$x = \begin{pmatrix} Cv_C \\ -i_L \end{pmatrix}$$

and

$$y = i_D = \begin{pmatrix} -\frac{1}{RC} & -1 \end{pmatrix} x + \begin{pmatrix} \frac{1}{R} \end{pmatrix} \lambda, \quad \lambda = -v_D.$$

Solutions as continuously differentiable functions (C^1 solutions)

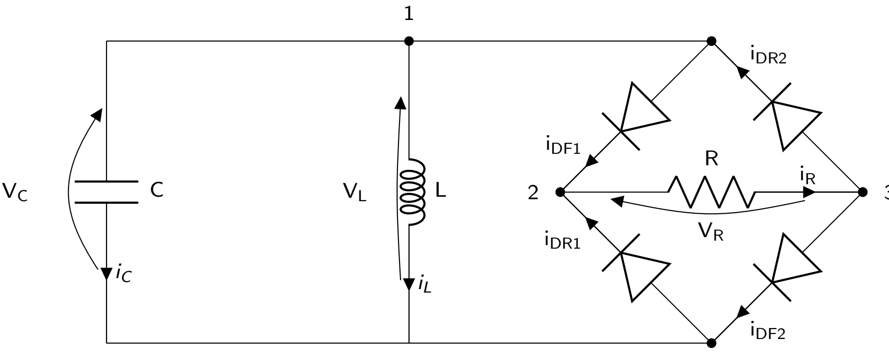


Figure: The 4-diode bridge rectifier. LC oscillator with a load resistor

Solutions as continuously differentiable functions (C^1 solutions)

The dynamical equations are stated choosing :

$$x = \begin{bmatrix} V_L \\ I_L \end{bmatrix}, \quad \text{and } y = \begin{bmatrix} I_{DR1} \\ I_{DF2} \\ V_2 - V_1 \\ V_1 - V_3 \end{bmatrix}, \quad \lambda = \begin{bmatrix} V_2 \\ -V_3 \\ I_{DF1} \\ I_{DR2} \end{bmatrix}, \tag{22}$$

and with

$$A = \begin{bmatrix} 0 & -1/C \\ 1/L & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & -1/C & 1/C \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad u = 0,$$

$$C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1/R & 1/R & -1 & 0 \\ 1/R & 1/R & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad a = 0, \quad K = K^* = \mathbb{R}^4_+. \tag{23}$$

Solutions as continuously differentiable functions (C^1 solutions)

$$D = \begin{bmatrix} 1/R & 1/R & -1 & 0 \\ 1/R & 1/R & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{22}$$

- D has full rank, but is only semi–definite positive then D is a P_0-matrix.
- The solution $x(t)$ is of class C^1 since $x \mapsto BSOL(\mathbb{R}^4_+, D\lambda + Cx + a)$ is a single valued Lipschitz function of x. (proof as an exercise)

Solutions as continuously differentiable functions (AC solutions)

Solutions as continuously differentiable functions (AC solutions)

Absolutely continuous functions

Definition

Let I be an interval in the real line $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is absolutely continuous on I if for every positive number ε, there exists a positive number δ such that whenever a finite sequence of pairwise disjoint sub-intervals (x_k, y_k) of I satisfies

$$\sum_k (y_k - x_k) < \delta \tag{23}$$

then

$$\sum_k |f(y_k) - f(x_k)| < \varepsilon \tag{24}$$

Absolutely continuous functions

Proposition

The following conditions on a real-valued function f on a compact interval $[a, b]$ are equivalent:

1. f is absolutely continuous
2. f has derivative almost everywhere, the derivative is Lebesque integrable, and
$$f(t) = f(a) + \int_a^t f'(t)dt \tag{23}$$
for all x on $[a, b]$.
3. there exists a Lebesgue integrable function g on $[a, b]$ such that
$$f(t) = f(a) + \int_a^t g(t)dt \tag{24}$$
for all x on $[a, b]$.

If these equivalent conditions are satisfied then necessarily $g = f'$ almost everywhere. Equivalence between (1) and (3) is known as the fundamental theorem of Lebesgue integral calculus, due to Lebesgue.

Absolutely continuous functions

Properties

- The sum and difference of two absolutely continuous functions are also absolutely continuous.
- If the two functions are defined on a bounded closed interval, then their product is also absolutely continuous.
- If an absolutely continuous function is defined on a bounded closed interval and is nowhere zero then its reciprocal is absolutely continuous.
- Every absolutely continuous function is uniformly continuous and, therefore, continuous. Every Lipschitz-continuous function is absolutely continuous.
- If $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then it is of bounded variation on $[a, b]$.
- If $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then it can be written as the difference of two monotonic nondecreasing absolutely continuous functions on [a,b].
- If $f : [a, b] \to \mathbb{R}$ is absolutely continuous, then it has the Luzin N property (that is, for any $L \subseteq [a, b]$ such that $\lambda(L) = 0$, it holds that $\lambda(f(L)) = 0$, where λ stands for the Lebesgue measure on $\mathbb{R}$).
- $f : I \to \mathbb{R}$ is absolutely continuous if and only if it is continuous, is of bounded variation and has the Luzin N property.
- The composition of two absolutely continuous functions is **not** necessarily a absolutely continuous function

Absolutely continuous functions

Proposition

Let f be Lipschitz continuous on $\mathbb{R}$ and g be an absolutely continuous function on $[a, b]$. Then the composition $f \circ g$ is absolutely continuous on $[a, b]$.

Solutions as absolutely continuous functions (*AC* solutions)

General context

The mapping h is not an one–to–one mapping of λ.
For instance, if the Jacobian matrix $\nabla_{\lambda}^{T} h(t, x(t), \lambda(t))$ is singular or worse if the λ does not explicitly appear in the definition of h

Solutions as absolutely continuous functions (AC solutions)

The LCS case with $D = 0$ and $b = 0$

If we consider the LCS (17) with $D = 0$ and $b = 0$, we get

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + a, \quad x(0) = x_0 \\ y(t) = Cx(t) \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0. \end{cases} \tag{23}$$

Regularity: What should we expect ?

The time-derivative of the state $\dot{x}(t)$ and $\lambda(t)$ are expected to be, in this case, discontinuous functions of time.

Indeed, if the output $y(t)$ reaches the boundary of the feasible domain at time t_*, i.e., $y(t_*) = 0$, the time–derivative $\dot{y}(t)$ needs to jump if $\dot{y}(t_*) < 0$

Solutions as absolutely continuous functions (AC solutions)

Example (Scalar LCS with $D = 0$)

Let us search for a continuous solution $x(t)$ to

$$\begin{cases} x(0) = x_0 > 0 \\ \dot{x}(t) = -x(t) - 1 + \lambda(t) \\ 0 \leqslant x(t) \perp \lambda(t) \geqslant 0 \end{cases}$$

Two modes :

- free dynamics for $0 < t < t_*$ with $x(t) > 0$ and $x(t_*) = 0$:

$$\begin{cases} x(0) = x_0 > 0 \\ \dot{x}(t) = -x(t) - 1 \end{cases} \quad (24)$$

  Solution :

$$x(t) = \exp(-t)x_0 + \exp(-t) - 1 \quad (25)$$

  $x(t_*) = 0 \implies t_* = -\ln(\frac{1}{1+x_0}) > 0$

- dynamics for $t \geqslant t_*$

$$\begin{cases} x(t_*) = 0, \\ \dot{x}(t) + 1 = \lambda(t) \geqslant 0 \end{cases} \quad (26)$$

Solutions as absolutely continuous functions (AC solutions)

Example (Scalar LCS with $D = 0$)

Solving the dynamics for $t_* \leqslant t < T$:

$$\begin{cases} x(t_*) = 0 \\ \dot{x}(t) + 1 = \lambda(t) \geqslant 0 \end{cases} \tag{24}$$

if we are looking for an abs. continuous solution $x(t)$, the abs. continuity and $x(t_*) = 0$ implies that $\dot{x}(t) \geqslant 0, t \in [t_*, t_* + \varepsilon), \varepsilon > 0$, otherwise $x(t_* + \varepsilon) < 0$.

1. $\dot{x}(t) > 0, t \in [t_*, t_* + \varepsilon), \varepsilon > 0$.
   By continuity, $x(t + \epsilon) > 0, \lambda(t + \varepsilon) = 0$ then

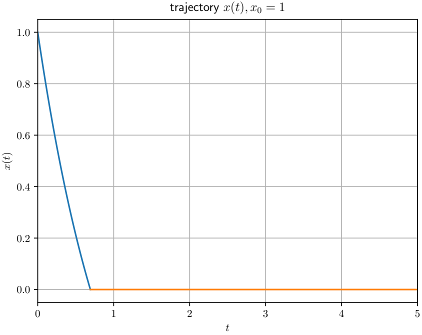
   $$\dot{x}(t + \varepsilon) = -x(t + \epsilon) - 1 < 0 \tag{25}$$

   No solution.
2. $\dot{x}(t) = 0, \lambda(t) = 1, x(t) = 0 \qquad \forall t \geqslant t_* \ (T = +\infty\ )$
   The only possible continuous solution.

Solutions as absolutely continuous functions (AC solutions)

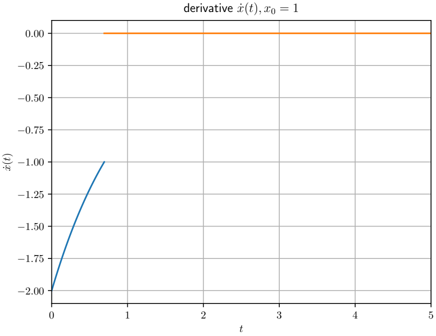
Example (Scalar LCS with $D = 0$)

Conclusion: A continuous $x(t)$ has been computed for all $t \in [0, +\infty)$. The time derivative of the solution $\dot{x}(t)$ jumps at from t_* from $x(t_*^-) = -1$ to $x(t_*^+) = 0$.



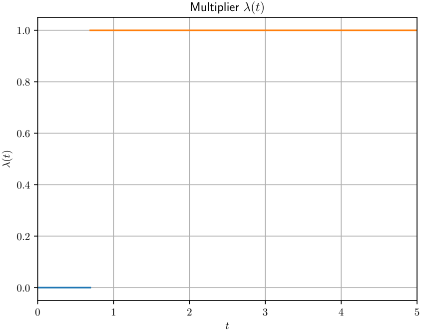
Solutions as absolutely continuous functions (*AC* solutions)

Example (Scalar LCS with $D = 0$)



Solutions as absolutely continuous functions (AC solutions)

Example (Scalar LCS with $D = 0$)



Solutions as absolutely continuous functions (*AC* solutions)

Idea of the general statement

If CB is a positive definite matrix (relative degree *one*) and $Cx_0 \geqslant 0$ (consistent initial condition), the unique solution of (59) is an absolutely continuous function.

Why the condition on CB ?

Derivation of the output $y(t)$

$$\begin{array}{rcl} y(t) & = & Cx(t) \\ \dot{y}(t) & = & CAx(t) + CB\lambda(t) \text{ if } D = 0 \end{array} \tag{24}$$

If $CB > 0$, we have to solve the following LCP whenever $y(t) = 0$

$$\begin{cases} \dot{y}(t) = CAx(t) + CB\lambda(t) \\ 0 \leqslant \dot{y}(t) \perp \lambda(t) \geqslant 0 \end{cases} \tag{25}$$

The LCP (25) is a LCP for the time derivative $\dot{y}(t)$.

The good framework is the differential inclusion framework (see later)

Solutions as absolutely continuous functions (AC solutions)

- Link with Moreau's sweeping process with an assumption $R^2 = P > 0$ and $PB = C^T$.
- Include the case when D is not full rank. A non trivial linear combination of λ is continuous, but other are not.
- The system is also a piecewise linear (exercise) but the feasible domain is restricted by the constraints on x
- The assumption $CB > 0$ can be relaxed (P matrix, co-positive matrix)

Solutions as absolutely continuous functions (*AC* solutions)

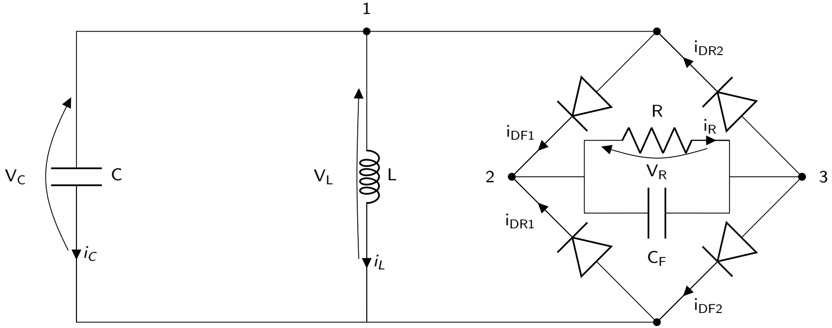


Figure: The 4-diode bridge rectifier. LC oscillator with a load resistor filtered by a capacitor

Solutions as absolutely continuous functions (AC solutions)

The second configuration of the 4-diode bridge is written in the LCS form choosing :

$$x = \begin{bmatrix} V_L \\ I_L \\ V_R \end{bmatrix}, \quad y = \begin{bmatrix} V_2 \\ I_{DF2} \\ V_2 - V_1 \\ V_L - V_3 \end{bmatrix}, \quad \text{and } \lambda = \begin{bmatrix} I_{DR1} \\ -V_3 \\ I_{DF1} \\ I_{DR2} \end{bmatrix}, \tag{26}$$

and with

$$A = \begin{bmatrix} 0 & -1/C & 0 \\ 1/L & 0 & 0 \\ 0 & 0 & -1/(RC_F) \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & -1/C & 1/C \\ 0 & 0 & 0 & 0 \\ 1/C_F & 0 & 1/C_F & 0 \end{bmatrix}, \quad u = 0,$$

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad a = 0. \tag{27}$$

For this second configuration, the matrix D does not have full rank ($\text{rank}(D) = 2$).

Existence and uniqueness results for LCS. Summary

Linear Complementarity Systems (LCS)

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + a, \quad x(0) = x_0 \\ y(t) = Cx(t) + D\lambda(t) + b \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0. \end{cases} \tag{28}$$

LCS with D a P-matrix

ODE with Lipschitz continuous right-hand side.
Cauchy–Lipschitz Theorem $\implies$ existence and uniqueness of solutions.

LCS with $D = 0$

Existence and uniqueness results based on

- Local (or nonzeno) solution based on the leading Markov parameters assumptions $(D, CB, CAB, CA^2B, ..)$
- or maximal monotone differential inclusion

Solutions as functions of Bounded Variations (BV solutions)

When discontinuities (jumps) are encountered in the solution $x(t)$, we often consider the solutions as functions of Bounded Variations (BV) [18].

Source of jumps

- inconsistency of the initial conditions.
- external input

Let us consider the previous example (59) with $Cx_0 + q < 0$. At the initial time, the solution have to jump to a consistent value with respect to the inequality.

Solutions as functions of Bounded Variations (BV solutions)

The dynamics in the problem (59) is written in terms of a measure differential equation as

$$dx = f(t, x(t))dt + Bdi, \tag{29}$$

where dx is the differential measure associated with the RCBV function $\dot{x}(t)$ and di is also a measure. The absolutely continuous function $\lambda(t)$ is the Radon-Nikodym derivative of di with respect to the Lebesgue measure, *i.e.* :

$$\frac{di}{dt} = \lambda(t). \tag{30}$$

If the singular part of the differential measure is neglected, a decomposition of the measure can be written as :

$$di = \lambda(t)dt + \sum_i \sigma_i \delta_{t_i} \tag{31}$$

where δ_{t_i} is the Dirac measure at times of discontinuities t_i and σ_i the magnitude. Thanks to (31), the differential measure equation (29) is decomposed in a smooth dynamics :

$$\dot{x}(t) = f(t, x(t)) + B\lambda(t), \quad dt - \text{almost everywhere}, \tag{32}$$

and in a jump dynamics at t_i :

$$x(t_i^+) - x(t_i^-) = B\sigma_i. \tag{33}$$

Solutions as functions of Bounded Variations (*BV* solutions)

Let us give an instance of a consistent state jump law.

Definition (State Jump Law)

Let us consider the LCS dynamics, and suppose that (A, B, C, D) is passive with storage function $V(x) = \frac{1}{2}x^T Px$, $P = P^T > 0$. For any $x(t^-)$, the state after the discontinuities, *i.e.* $x(t^+)$, is given by the solution of the generalized equation :

$$P(x(t^+) - x(t^-)) \in -\mathbb{N}_{\mathbf{K}}(x(t^+)). \tag{34}$$

The state jump law in (34) guarantees that $V(x(t^+)) - V(x(t^-)) \leqslant 0$ provided that $0 \in \mathbf{K}$.

The notion of relative degree. Well-posedness

Definition (Relative degree in the SISO case)

Let us consider a linear system in state representation given by the quadruplet $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times m}$:

$$\begin{cases} \dot{x} = Ax + B\lambda \\ y = Cx + D\lambda \end{cases} \tag{35}$$

- In the Single Input/ Single Output (SISO) case ($m = 1$), the relative degree is defined by the first non zero Markov parameters :

$$D, CB, CAB, CA^2B, \dots, CA^{r-1}B, \dots \tag{36}$$

- In the multiple input/multiple output (MIMO) case ($m > 1$), an *uniform* relative degree is defined as follows. If D is non singular, the relative degree is equal to 0. Otherwise, it is assumed to be the first positive integer r such that

$$CA^iB = 0, \quad i = 0 \dots q - 2 \tag{37}$$

while

$$CA^{r-1}B \text{ is non singular.} \tag{38}$$

The notion of relative degree. Well-posedness

Interpretation

The Markov parameters arise naturally when we derive with respect to time the output y,

$$
\begin{array}{rcl}
y & = & Cx + D\lambda \\
\dot{y} & = & CAx + CB\lambda, \text{ if } D = 0 \\
\ddot{y} & = & CA^2x + CAB\lambda, \text{ if } D = 0, CB = 0 \\
 & \dots & \\
y^{(r)} & = & CA^r x + CA^{r-1}B\lambda, \text{ if } D = 0, CB = 0, CA^{r-2}B = 0, r = 1 \dots r-2 \\
 & \dots &
\end{array}
$$

and the first non zero Markov parameter allows us to define the output y directly in terms of the input λ.

The notion of relative degree. Well-posedness

Example

Third relative degree LCS Let us consider the following LCS:

$$\begin{cases} \dddot{x}(t) = \lambda, x(0) = x_0 \geqslant 0 \\ y(t) = x(t) \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{35}$$

The function $x : [0, T] \to \mathbb{R}$ is usually assumed to be an absolutely continuous function of time.

- If $y = x \geqslant 0$ becomes active, i.e., $x = 0$,
  - If $\dot{x} > 0$, the system will instantaneously leaves the constraints.
  - If $\dot{x} < 0, \ddot{x} > 0$, the velocity needs to jump to respect the constraint in t^+. (B.V. function ?)
  - If $\dot{x} < 0, \ddot{x} < 0$, the velocity and the acceleration need to jump to respect the constraint in t^+. (Dirac + B.V. function)

➜ $\ddot{x} < 0$ and therefore λ may be derivative of Dirac distribution.

Problem: From the mathematical point of view, a constraint of the type $\lambda \geqslant 0$ has no mathematical meaning !!

Restrictions

➜ In this lecture, we will focus on LCS of relative degree $r \leqslant 1$.

The passive LCS.

Definition (Passivity properties and energy storage function. Continuous–time case)

The quadruple (A, B, C, D) is said to be *passive* if there exist matrices $L \in \mathbb{R}^{n\times m}$ and $W \in \mathbb{R}^{m\times m}$ and a symmetric positive semi-definite matrix $P \in \mathbb{R}^{n\times n}$, such that:

$$\begin{cases} A^T P + PA = -LL^T & (36) \\ B^T P - C = -W^T L^T & (37) \\ -D - D^T = -W^T W. & (38) \end{cases}$$

In this case, let $V(x) = \frac{1}{2}x^T Px$ denote the corresponding *energy storage function*.

The passive LCS.

The *dissipation equality*

$$V(x(T)) - V(x(0)) = -\frac{1}{2}\int_0^T (x^T(t), \lambda^T(t)) Q \begin{pmatrix} x(t) \\ \lambda(t) \end{pmatrix} dt, \quad \forall \quad T \geqslant 0 \tag{36}$$

in terms of the positive semi-definite matrix

$$Q \triangleq \begin{pmatrix} LL^T & W^T L^T \\ LW & W^T W \end{pmatrix}, \tag{37}$$

then implies that

$$V(x(T)) - V(x(0)) \leqslant 0. \tag{38}$$

The system is said to be *strictly passive* when Q is positive definite, and *lossless* when $Q = 0$. The system is said to be *state lossless* when $L = 0$ and *input lossless* when $W = 0$. The system is *dissipative*, *state dissipative*, and *input dissipative* when $Q \neq 0$, $L \neq 0$, or $W \neq 0$, respectively. In particular, we have

$$V(x(T)) - V(x(0)) \leqslant S(\lambda(t), w(t)), \tag{39}$$

where the supply rate $S(\lambda, w) \triangleq \lambda^T w$, since the LCS implies that $S(\lambda(t), w(t)) = 0$ for all $t \geqslant 0$.

The passive LCS.

Relative degree 0

Let us consider a LCS of relative degree 0 i.e. with D which is non singular.

$$\begin{cases} \dot{x} = Ax + B\lambda, \quad x(0) = x_0 \\ y = Cx + D\lambda \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{40}$$

Mathematical properties

- Existence and Uniqueness.
  - "$B.SOL(Cx, D)$ is a singleton":
    $B.SOL(Cx_0, D)$ is a singleton is equivalent to stating that the LCS (40) has a unique C^1 solution defined at all $t \geqslant 0$.
    Denoting by $\Lambda(x) = B.SOL(Cx, D)$, the LCS can be viewed as a standard ODE with a Lipschitz r.h.s :
    $$\dot{x} = Ax + \Lambda(x) = Ax + B.SOL(Cx, D) \tag{41}$$
  - Special important case: D is a P-matrix, ($LCP(q, M)$ has a unique solution for all $q \in \mathbb{R}^n$ if M is a P-matrix.) The Lipschitz property of the LCP solution with the respect to x is shown in [8].
- Stability theory [7] and for the numerical integration, the problem is a little more tricky because $\Lambda(x)$ is only B-differentiable.

The passive LCS.

Example

To complete this section, a example of non existence and non uniqueness of solutions is provided for a LCS of relative degree 0. This example is taken from [11]. Let us consider the following LCS

$$\begin{cases} \dot{x} = -x + \lambda \\ y = x - \lambda \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{42}$$

This system is strictly equivalent to

$$\dot{x} = \begin{cases} -x, & \text{if } x \geqslant 0 \\ 0, & \text{if } x \geqslant 0 \end{cases} \tag{43}$$

which leads to non existence of solutions for $x(0) < 0$ and to non uniqueness for for $x(0) > 0$.

The passive LCS.

Relative degree 1

Let us consider a LCS of relative degree 1 i.e. with CB which is non singular.

$$\begin{cases} \dot{x} = Ax + B\lambda, \quad x(0) = x_0 \\ y = Cx \\ 0 \leqslant y \perp \lambda \geqslant 0 \end{cases} \tag{44}$$

Mathematical properties

- The Rational Complementarity problem [10, 5, 6]. The P-matrix property plays henceforth a fundamental role and provides the existence of global solution of the LCS in the sense of Caratheodory.
- Special case $B = C^T$ uses some EVI results for the well-posedness and the stability of such a systems [9].

The passive LCS.

Comments

The passive linear systems are a class for which a "stored energy" in the system is only decreasing (see for more details, [5, 11]). The passive linear systems are of relative degree $\geqslant 1$.

Differential inclusion

Complementarity condition as a subdifferential inclusion

$$0 \leqslant y \perp \lambda \geqslant 0 \Longleftrightarrow -y \in \partial \Psi_{\mathbb{R}^m_+}(\lambda) \Longleftrightarrow -\lambda \in \partial \Psi_{\mathbb{R}^m_+}(y) \tag{45}$$

LCS as a differential inclusion with $D = 0$ and $b = 0$

$$\left\{\begin{array}{l} \dot{x}(t) = Ax(t) + B\lambda(t) + a \\ y(t) = Cx(t) \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0 \\ x(0) = x_0. \end{array}\right. \Longleftrightarrow \left\{\begin{array}{l} -(\dot{x}(t) - Ax(t) - a) \in B\partial\Psi_{\mathbb{R}^m_+}(Cx(t)), \\ \quad x(0) = x_0 \end{array}\right. \tag{46}$$

General differential inclusion

Concept of differential inclusions

Differential inclusions is a generalization of the concept of differential equations of the form

$$\dot{x}(t) \in A(x(t), t) \tag{47}$$

where $(x, t) \mapsto A(x, t)$ is a multi-valued map, *i.e.* $A(x, t)$ is a set rather than a single point.

A very general concept

Differential inclusions is a very general concept that contains Ordinary Differential Equations (ODE), Differential Algebraic Equations (DAE). There are many types if differential inclusions.

We will focus on Maximal Monotone Differential Inclusion

Maximal monotone operators

Let $2^{\mathbb{R}^n}$ be the set of the subsets of $\mathbb{R}^n$

Definition (Monotone multi-valued operator)

A multi–valued operator $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is monotone if

$$\forall y_1 \in T(x_1), \quad \forall y_2 \in T(x_2), \quad (y_2 - y_1)^T (x_2 - x_1) \geqslant 0 \tag{48}$$

Definition (Graph)

Let T multi–valued operator $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$. The graph of T is defined by

$$Gr(T) = \{(x, y) \mid y \in T(x)\} \tag{49}$$

Definition (Maximal Monotone multi-valued operator)

A operator T is maximal monotone if it is maximal for all the monotone operators for the inclusion of graphs.

In other words, T is monotone and for all other monotone operator S then $Gr(T) \subset Gr(S) \implies T = S$

Maximal monotone operators

Definition (Domain)

The domain of an operator T is defined by $D(T) = \{x \mid T(x) \neq \emptyset\}$

Definition (Range of T)

Let $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be an operator. The range of T is defined by

$$R(T) = \cup_{x \in \mathbb{R}^n} \{y \mid y \in T(x)\} \tag{50}$$

Definition (Inverse of T)

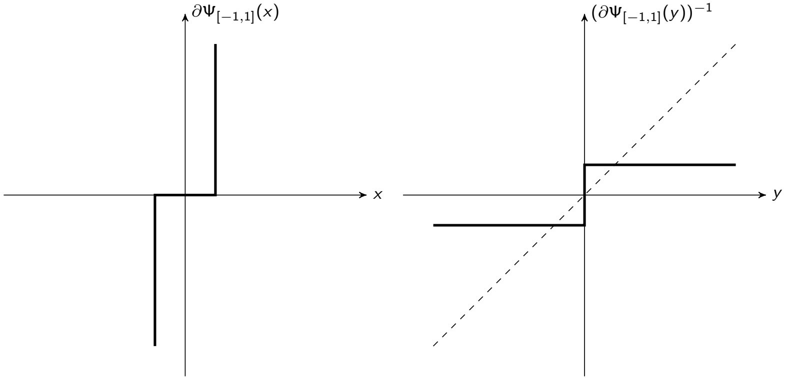
Let $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a maximal monotone operator. Its inverse T^{-1} is defined by

$$y \in T(x) \Longleftrightarrow x \in T^{-1}(y) \tag{51}$$

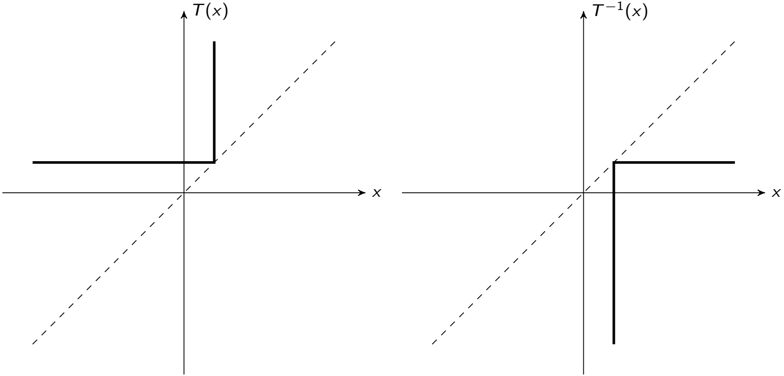
and we have $D(T^{-1}) = R(T)$ and $R(T^{-1}) = D(T)$

Its inverse is defined by the symmetry of its graph with respect to $y = x$

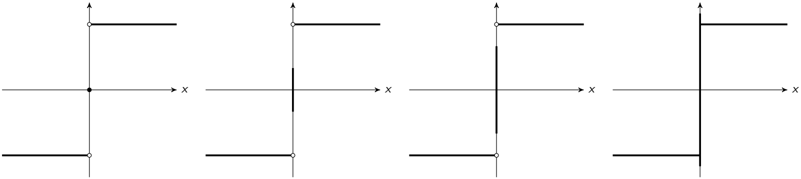
Maximal monotone operators



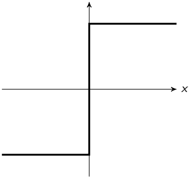
Maximal monotone operators



Maximal monotone operators



$\mathrm{sgn}(x) = \partial|x|$



Maximal monotone differential inclusion

Definition (Maximal monotone differential inclusion)

Let T multi–valued operator $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$. A maximal monotone differential inclusion is defined by

$$-\dot{x}(t) \in T(x(t)) \tag{50}$$

Definition (Perturbed maximal monotone differential inclusion)

Let T multi–valued operator $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$. A maximal monotone differential inclusion is defined by

$$-(\dot{x}(t) + f(x,t)) \in T(x(t)) \tag{51}$$

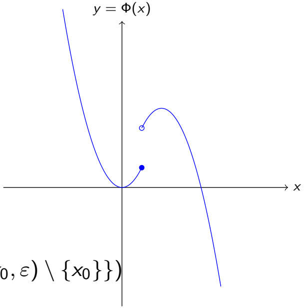
where f is a Lipschitz continuous map w.r.t x.

Maximal monotone differential inclusion

Definition (lower semi-continuity)

A function $\Phi : \mathbb{R}^n \to \mathbb{R} \cup +\infty$ is lower semi-continuous if one of the following equivalent assertions is satisfied:

- $$\lim\inf_{x \to x_0} \Phi(x) \geqslant \Phi(x_0)$$
- Its epigraph is closed



Remarks

- $\lim\inf_{x\to x_0} \Phi(x) = \lim_{\varepsilon \to 0}(\inf\{\Phi(x), x \in B(x_0, \varepsilon) \setminus \{x_0\}\})$
- Continuity implies semi-continuity.

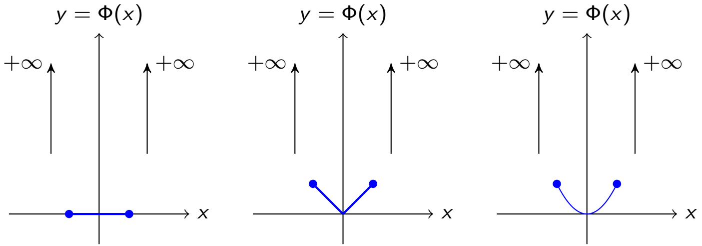
Maximal monotone differential inclusion

For a convex proper function Φ, the semi–continuity property has only to be checked on the boundary of the domain of definition

$$\partial D(\Phi) = \overline{D(\Phi)} \setminus \mathring{D}(\Phi)$$

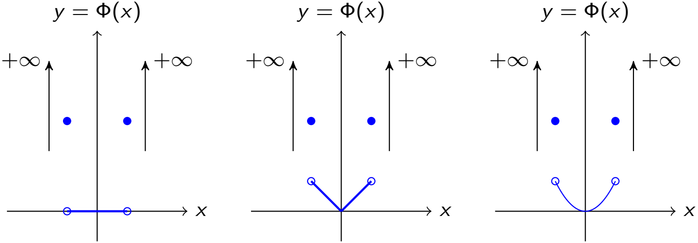
.

Examples



Maximal monotone differential inclusion

Counter-examples



Maximal monotone differential inclusion

Theorem

For a lower semi–continuous convex proper function Φ, the subdifferential $\partial\Phi(x)$ is a maximal monotone operator

Remarks

- Obvious in the regular case: $\phi(x) : \mathbb{R} \to \mathbb{R}$ a convex potential C^2 $\phi''(x) \geqslant 0$ and $\phi'(x)$ is monotone (increasing single–valued function)
- For a maximal monotone operator in $\mathbb{R}$, i.e. $T : \mathbb{R} \to 2^{\mathbb{R}}$ it exists a lower semi–continuous convex proper function Φ such that $T = \partial\Phi$

Maximal monotone differential inclusion

Examples

- $\Phi(x) = 0 = \Psi_{\mathbb{R}}$, $T(x) = 0$
$$-\dot{x} + f(x,t) = 0 \tag{52}$$
- $\Phi(x) = \Psi_c(x)$, $T(x) = \partial\Psi_C(x)$
$$-\dot{x} + f(x,t) \in \partial\Psi_C(x) \tag{53}$$
- relay or sign function $\Phi(x) = |x|$, $T(x) = \partial|x|$
$$-\dot{x} \in \partial|x| \iff -\dot{x} \in \mathrm{sgn}(x) \tag{54}$$
- 2-norm $\Phi(x) = \|x\|,\ T(x) = \partial\|x\| = \begin{cases} \frac{x}{\|x\|} & \text{if} \quad x \neq 0 \\ \{s \mid \|s\| \leqslant 1\} & \text{if} \quad x = 0 \end{cases}$

Maximal monotone differential inclusion

Examples

- relay with dead zone

$$\Phi(x) = \begin{cases} -x+1, \text{ if } x \leqslant -1 \\ 0, \text{ if } -1 \leqslant x \leqslant 1 \\ x-1, \text{ if } x \geqslant 1 \end{cases} \tag{52}$$

Maximal monotone differential inclusion

Examples

- Sum of (proper) convex functions $\Phi_1 + \Phi_2$ is convex. Moreover, if the relative interior $\mathrm{ri}(D(\partial\Phi_1))$ and $\mathrm{ri}(D(\partial\Phi_2)$ have a common point then

$$\partial(\Phi_1(x) + \Phi_2(x)) = \partial\Phi_1(x) + \partial\Phi_2(x) \tag{52}$$

  Relative interior : $\mathrm{ri}(X) = \{x \in X \mid \exists\varepsilon > 0, B_\varepsilon \cap \mathrm{Aff}(X) \subset X\}$ where $\mathrm{Aff}(\mathrm{X})$ is the affine hull of X, the smallest affine set containing X:

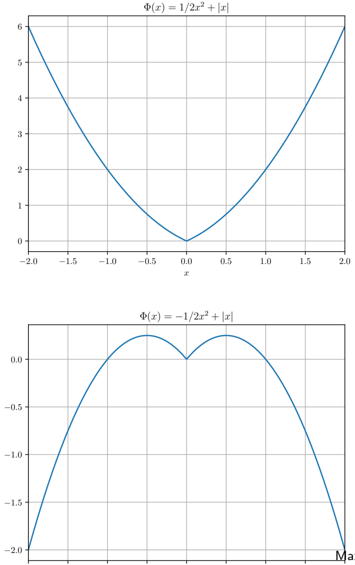
$$\mathrm{Aff}(X) = \{\sum_{i=0}^{k} \alpha_i x_i \mid k > 0, x_i \in X, \alpha_i \in \mathbb{R}, \sum_{i=0}^{k} \alpha_i = 1\} \tag{53}$$

  Ex: $C = \{x \in \mathbb{R}^2 \mid x_1 \in [-1, 1], x_2 = 0\}$ $\mathrm{Aff}(C) = \mathbb{R} \times \{0\}$
- $\Phi(x) = 1/2 * ax^2 + |x|$, $T(x) = ax + \mathrm{sgn}(x)$

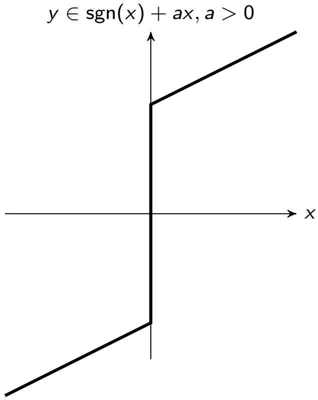
$$-\dot{x} \in ax + \partial|x| \iff -\dot{x} - ax \in \mathrm{sgn}(x) \tag{54}$$

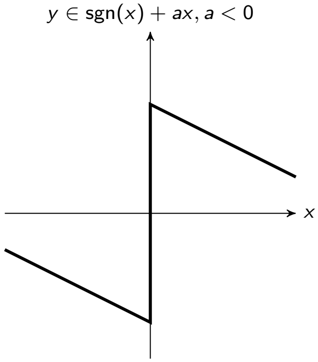
  1. $a > 0$. $\Phi(x)$ is convex and $T(x)$ is maximal monotone.
  2. $a < 0$. $\Phi(x)$ is not convex and $T(x)$ is not monotone.

Maximal monotone differential inclusion



Maximal monotone differential inclusion





Maximal monotone differential inclusion

Link with gradient systems with convex potentials

- $\phi(x) : \mathbb{R} \to \mathbb{R}$ a convex potential C^2
  $\phi''(x) \geqslant 0$ and $\phi'(x)$ is monotone (increasing function)

$$-\dot{x} = \phi'(x) \tag{52}$$

- $\Phi(x) : \mathbb{R} \to \mathbb{R}$ a convex potential not necessarily differentiable, but proper and lower semi–continuous $\partial\Phi(x)$ is a maximal monotone operator.

$$-\dot{x} = \partial\Phi(x) \tag{53}$$

Existence and uniqueness results

Theorem (Brézis 1973)

Let $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a maximal monotone operator such that $D(\mathring{T}) \neq \emptyset$. Let a function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ such that

1. *the function $f(x, \cdot)$ is Lipschitz continuous on $D(T)$ that is*

$$\exists L \geqslant 0, \forall t \in [0, t_{\max}], \forall x_1, x_2 \in \overline{D(T)}, \quad \|f(t, x_1) - f(t, x_2)\| \leqslant L\|x_1 - x_2\| \quad (54)$$

2. *$\forall x \in \overline{D(T)}$, the mapping $t \mapsto f(x, t)$ belongs to $\mathcal{L}^{\infty}(0, t_{\max}; \mathbb{R}^n)$*

Then, for all $x_0 \in \overline{D(T)}$, it exists a unique solution $x(t)$ which is absolutely continuous such that

$$\begin{cases} -(\dot{x}(t) + f(x(t), t)) \in T(x(t)), \ \textit{almost everywhere on } [0, t_{\max}] \\ x(0) = x_0 \end{cases} \quad (55)$$

Existence and uniqueness results

Existence

- By using the Moreau-Yosida regularization of T

$$T_\lambda(x) = \frac{1}{\lambda}(I - J_\lambda(x)), \lambda > 0, \tag{56}$$

with $J_\lambda(x)$ the resolvent of $T(x)$ given by

$$J_\lambda(x) = (I + \lambda T(x))^{-1}. \tag{57}$$

For a maximal monotone operator T or $\mathbb{R}$, J_λ is defined over $\mathbb{R}$ and is contracting. The mapping T_λ is a maximal monotone operator and Lipschitz continuous with a Lipschitz constant of $\frac{1}{\lambda}$. We consider that ODE with Lipschitz r.h.s.

$$-(\dot{x}_\lambda(t) + f(x_\lambda(t), t)) = T_\lambda(x_\lambda(t)) \tag{58}$$

and then the limit $\lambda \to 0$ of the sequence of solutions x_λ.

- By approximation using a discretization scheme

Existence and uniqueness results

Uniqueness

Simple case $-\dot{x}(t) \in T(x(t))$. $x \in \mathbb{R}$
Let us consider two solution x_1 and x_2
Since $T(x)$ is monotone, we have

$$(\dot{x}_1(s) - \dot{x}_2(s))^T (x_1(s) - x_2(s)) \leqslant 0 \text{ almost everywhere on } [0, T] \tag{56}$$

By integrating over $[0, t]$, we get

$$\frac{1}{2}(x_2(t) - x_1(t))^2 - \frac{1}{2}(x_2(0) - x_1(0))^2 \leqslant 0 \tag{57}$$

If $x_1(0) = x_2(0)$, we have

$$\frac{1}{2}(x_2(t) - x_1(t))^2 \leqslant 0 \implies x_2 = x_1 \tag{58}$$

Existence and uniqueness results

Uniqueness

$-(\dot{x}(t) + f(x,t)) \in T(x(t))$
Let us consider two solution x_1 and x_2
Since $T(x)$ is monotone, we have

$$(\dot{x}_1(s) + f(x_1(s),s) - \dot{x}_2(s) - f(x_2(s),s))^T(x_1(s) - x_2(s)) \leqslant 0 \tag{56}$$

almost everywhere on $[0, T]$.
By integrating over $[0, t]$, we get

$$\frac{1}{2}(x_2(t) - x_1(t))^2 \leqslant \int_0^t (f(x_2(s),s) - f(x_1(s),s))^T(x_1(s) - x_2(s))ds \tag{57}$$

Since f is lipschitz, we have

$$(x_2(t) - x_1(t))^2 \leqslant 2L \int_0^t \|x_1(s) - x_2(s)\|^2 ds \tag{58}$$

Existence and uniqueness results

Gronwall Lemma

Let a a positive constant and m a integrable function, nonnegative almost everywhere on $(0, t_{\max})$ and a function ϕ a continuous function on $[0, t_{\max}]$. If

$$\forall t \in [0, t_{\max}], \phi(t) \leqslant a + \int_0^t m(s)\phi(s)\, ds \tag{56}$$

then

$$\forall t \in [0, t_{\max}], \phi(t) \leqslant a \exp(\int_0^t m(s)\, ds) \tag{57}$$

Applying the Gronwall Lemma, for $a = 0$ and $m(s) = 2L$ and $\phi(s) = \|x_1(s) - x_2(s)\|^2$, we get

$$\|x_2(t) - x_1(t))\|^2 \leqslant 0 \implies x_2 = x_1 \tag{58}$$

Come back to LCS with $D = 0$ but $B \neq I_d \neq C$

Theorem (LCS as maximal monotone differential inclusion)

Let us consider the following LCS

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + a(t), \quad x(0) = x_0 \\ y(t) = Cx(t) \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0. \end{cases} \tag{59}$$

If there exists P a symmetric definite positive matrix such that

$$PB = C^T \tag{60}$$

then we can perform a change of variable $z = Rx$ with $R^2 = P, R \geqslant 0, R = R^T$

$$-(\dot{z}(t) - RAR^{-1}z(t) - Ra(t)) \in RB\,\partial\Psi_{\mathbb{R}^m_+}(CR^{-1}z(t)) \tag{61}$$

such that (61) is a maximal monotone differential inclusion.

Come back to LCS with $D = 0$ but $B \neq I_d \neq C$

We have the following equivalence

$$\begin{cases} \dot{x}(t) = Ax(t) + B\lambda(t) + a(t) \\ y(t) = Cx(t) \\ 0 \leqslant y(t) \perp \lambda(t) \geqslant 0, \\ x(0) = x_0 \end{cases} \iff \begin{cases} -(\dot{x}(t) - Ax(t) - a(t)) \in B\partial\Psi_{\mathbb{R}^m_+}(Cx(t)), \\ x(0) = x_0 \end{cases} \tag{59}$$

We can perform a change of variable $z = Rx$ with $R^2 = P, R \geqslant 0, R = R^T$

$$-(\dot{z}(t) - RAR^{-1}z(t) - Ra(t)) \in RB\,\partial\Psi_{\mathbb{R}^m_+}(CR^{-1}z(t)) \tag{60}$$

Come back to LCS with $D = 0$ but $B \neq I_d \neq C$

For a matrix E, the function $\phi(x) = \Psi_{\mathbb{R}^m_+}(Ex)$ is a proper convex function and its subdifferential is given by

$$\partial\phi(x) = E^T \partial\Psi_{\mathbb{R}^m_+}(Ex) \tag{59}$$

$(\mathrm{Im}(E)$ contains a point of $\mathrm{ri}(D(\partial\Psi_{\mathbb{R}^m_+})))$ (Chain rule)
In our application, we set $E = CR^{-1}$ and we have

$$E^T = R^{-T}C^T = R^{-1}R^2B = RB \tag{60}$$

The obtained inclusion

$$-(\dot{z}(t) - RAR^{-1}z(t) - Ra) \in \partial\Phi(z(t)) = E^T\partial\Psi_{\mathbb{R}^m_+}(Ez(t)), \tag{61}$$

is a maximal monotone differential inclusion

The Moreau's sweeping process of first order

The Moreau's sweeping process of first order

Definition (The Moreau's sweeping process (of first order))

The Moreau's sweeping process (of first order) is defined by the following Differential inclusion (DI)

$$\begin{cases} -\dot{x}(t) \in N_{K(t)}(x(t)) \quad t \in [0,T], \\ x(0) = x_0 \in K(0). \end{cases} \tag{62}$$

where

- $K(t)$ is a moving closed and nonempty convex set.
- $N_K(x)$ is the normal cone to K at x

$$N_K(x) := \{s \in \mathbb{R}^n : \langle s, y - x \rangle \leqslant 0, \text{ for all } y \in K\},$$

Comment

This terminology is explained by the fact that $x(t)$ can be viewed as a point which is swept by a moving convex set.

References

[15, 16, 17, 14, 13]

The Moreau's sweeping process of first order

Basic mathematical properties [14].

- A solution $x(.)$ for such type of DI is assumed to be differentiable almost everywhere satisfying the inclusion $x(t) \in K(t), t \in [0, T]$.
- If the set-valued application $t \mapsto K(t)$ is supposed to be Lipschitz continuous, i.e.

$$\exists l \leqslant 0, \quad d_H(K(t), K(s)) \leqslant l|t - s| \tag{63}$$

  where d_H is the Hausdorff distance between two closed sets, then
  - existence of a solution which is l-Lipschitz continuous
  - uniqueness in the class of absolutely continuous functions.

  [14].

Definition (State dependent sweeping process [12])

The state dependent sweeping process is defined

$$\begin{cases} -\dot{x}(t) \in N_{K(t,x(t))}(x(t)) \;\; t \in [0, T], \\ x(0) = x_0 \in K(0). \end{cases} \tag{64}$$

Variants of the Moreau's sweeping process

Definition (RCBV sweeping process [12])

The RCBV sweeping process of the type is defined

$$\begin{cases} -du \in N_{K(t)}(u(t)) \ \ (t \geqslant 0), \\ \ \ u(0) = u_0. \end{cases} \tag{65}$$

where the convex set is RCBV i.e

$$d_H(K(t), K(s)) \leqslant r(t) - r(s) \tag{66}$$

for some right-continuous non-decreasing function $r : [0, T] \to \mathbb{R}$ is made.

Mathematical properties

- the solution $u(.)$ is searched as a function of bounded variations (B.V.)
- the measure du associated with the B.V. function u is a differential measure or a Stieltjes measure.
- Inclusion of measure into cone

Unbounded DI and Maximal monotone operator

Definition (Unbounded Differential Inclusion (UDI))

The following UDI can be defined (together with the initial condition $x(0) = x_0 \in C$)

$$-(\dot{x}(t) + f(x(t)) + g(t)) \in \mathbb{N}_K(x(t)) \tag{67}$$

where K is the feasible set and $g : \mathbb{R}_+ \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$.

Basic properties

- A solution of such a UDI is understood as an absolutely continuous $t \mapsto x(t)$ lying in the convex set C.

Comment

The Terminology is explained by the fact that $\mathbb{N}_K(x(t))$ is neither compact nor bounded. Standard DI analysis no longer apply.

Unbounded DI and Maximal monotone operator

Link with Maximal monotone operator

- In [2], a existence and uniqueness theorem for

$$\dot{x}(t) + A(x(t)) + g(t) \ni 0 \tag{68}$$

where A is a maximal monotone operator, and g a absolutely continuous function of time.

- If f which is monotone and Lipschitz continuous, then

$$A(x(t)) = f(x(t)) + \mathbb{N}_K(x(t)) \tag{69}$$

is then a maximal monotone operator.

- Equivalence [4]

$$-\left(\dot{x}(t) + f(x(t)) + g(t)\right) \in \mathbb{N}_{\mathrm{T}_{K(x(t))}}(\dot{x}(t))\,, \tag{70}$$

providing that the UDI (67) has the so-called slow solution, that is $\dot{x}(t)$ is of minimal norm in $\mathbb{N}_{K(x(t))}(x(t)) + f(x,t) + g(t)$.

Special case when K is finitely represented.

Assumptions

$$K = \{x \in \mathbb{R}^n, h(x) \leqslant 0\} \tag{71}$$

For $x \in K$, we denote by

$$I(x) = \{i \in \{i \dots m\}, h_i(x) = 0\} \tag{72}$$

the set of active constraints at x. The tangent cone can be defined by

$$T^h(x) = \{s \in \mathbb{R}^n, \langle \nabla h_i(x), s \rangle \leqslant 0, i \in I(x)\} \tag{73}$$

and the normal cone by

$$N^h(x) := [T^h(x)]^\circ = \{ \sum_{i \in I(x)} \lambda_i \nabla h_i(x), \lambda_i \geqslant 0,\ i \in I(x)\} \tag{74}$$

- $N_K(x) \supset N^h(x)$ and $T_K(x) \subset T^h(x)$ always hold.
- $N_K = N^h$ and equivalently $T_K = T^h$ holds if a constraints qualification condition is satisfied

Special case when K is finitely represented.

Link with Differential Complementarity Systems (DCS)

Equivalence with the following DCS of Gradient Type (GTCS)

$$\begin{cases} -\dot{x}(t) = f(x(t)) + g(t) + \nabla h(x(t))\lambda(t) \\ 0 \leqslant -h(x(t)) \perp \lambda(t) \geqslant 0 \end{cases} \tag{71}$$

Link with Evolution Variational Inequalities (EVI)

Equivalence with the following EVI

$$\langle \dot{x}(t) + f(x(t)) + g(t), y - x \rangle \geqslant 0 \tag{72}$$

- existence and uniqueness theorem for maximal monotone operators
- existence result is given for this last EVI under the assumption that f is continuous and hypo-monotone [4].

Applications

- Quasi-static analysis (first order) of viscoelastic mechanical systems
  - with perfect (associated) plasticity
  - with associated friction
- Quasi static analysis (first order) of quasi-brittle mechanical systems
  - cohesion, damage and fracture mechanics
  - geomaterials
- Dynamic analysis of mechanical systems with Coulomb's friction with permanent contact
- Many other applications, in economy and in control.

Differential Variational Inequalities (DVI)

Differential Variational Inequalities (DVI)

Definition (Differential Variational inequalities (DVI) [19])

A Differential Variational inequality can be defined as follows:

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)) & (73) \\ u(t) &= \mathrm{SOL}(K, F(t, x(t), \cdot)) & (74) \\ 0 &= \Gamma(x(0), x(T)) & (75) \end{aligned}$$

where :

- $x : [0, T] \to \mathbb{R}^n$ is the differential trajectory (state variable),
- $u : [0, T] \to \mathbb{R}^m$ is the algebraic trajectory
- $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is the ODE right-hand side
- $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is the VI function
- K is nonempty closed convex subset of $\mathbb{R}^m$
- $\Gamma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is the boundary conditions function.
  - Initial Value Problem (IVP), $\Gamma(x, y) = x - x_0$
  - linear Boundary Value Problem (BVP), $\Gamma(x, y) = Mx + Ny - b$

The notation $u(t) = \mathrm{SOL}(K, \Phi)$ means that $u(t) \in K$ is the solution of the following VI

$$(v - u)^T \Phi(u) \geqslant 0, \quad \forall v \in K \qquad (76)$$

Differential Variational Inequalities (DVI)

The DVI is a slightly more general framework in the sense that it includes at the same time:

- Differential Algebraic equations(DAE)

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)) && (77)\\ u(t) &= F(t, x(t), u(t)) && (78) \end{aligned}$$

- Differential Complementarity systems (DCS)

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)) && (79)\\ C &\ni u(t) \perp F(t, x(t), u(t)) \in C^* && (80) \end{aligned}$$

  where C and C^* are a pair of dual closed convex cones ($C^* = -C^\circ$). The Linear Complementarity systems are also special case of DVI (see the section 1).

Differential Variational Inequalities (DVI)

The DVI is a slightly more general framework in the sense that it includes at the same time:

- Evolution variational inequalities (EVI)

$$-\left(\dot{x} + f(x)\right) \in \mathbb{N}_K(x) \tag{77}$$

  - When K is a cone, the preceding EVI is equivalent to a DCS of the type :

$$\begin{aligned} \dot{x}(t) + f(x(t)) &= u(t) && (78)\\ K \ni x(t) \perp u(t) &\in K^* && (79) \end{aligned}$$

  - When K is finitely represented i.e. $K = \{x \in \mathbb{R}^n, g(x) \leqslant 0\}$ then under some appropriate constraints qualifications, we obtain another DCS which is often called a Gradient type Complementarity Problem (GTCS) (see 1) :

$$\begin{aligned} \dot{x}(t) + f(x(t)) &= -\nabla_x^T g(x)u(t) && (80)\\ 0 \leqslant -g(x(t)) \perp u(t) &\geqslant 0 && (81) \end{aligned}$$

  - Finally, if K is a closed convex and nonempty set then the EVI is equivalent to the following DVI :

$$\begin{aligned} \dot{x}(t) + f(x(t)) &= w(t) && (82)\\ 0 &= x(t) - y(t) && (83)\\ y(t) \in K,\ (v - y(t))^T w(t) &\geqslant 0, \forall v \in K && (84) \end{aligned}$$

Thank you for your attention.

[1] V. Acary and B. Brogliato. *Numerical methods for nonsmooth dynamical systems. Applications in mechanics and electronics.* Lecture Notes in Applied and Computational Mechanics 35. Berlin: Springer. xxi, 525 p. , 2008.

[2] H. Brezis. *Opérateurs Maximaux Monotones et Semi-groupe de Contraction dans les Espaces de Hilbert*. North Holland, Amsterdam, 1973.

[3] B. Brogliato, editor. *Impact in Mechanical systems: Analysis and Modelling*, volume 551 of *Lecture Notes in Physics*. Springer, 2000.

[4] B. Brogliato, A. Daniilidis, C. Lemaréchal, and V. Acary. On the equivalence between complementarity systems, projected systems and differential inclusions. *Systems and Control Letters*, 55:45–51, 2006.

[5] K. Camlibel. *Complementarity Methods in the Analysis of Piecewise Linear Dynamical Systems*. PhD thesis, Katholieke Universiteit Brabant, 2001. ISBN: 90 5668 073X.

[6] K. Camlibel, W.P.M.H. Heemels, and J.M. Schumacher. Consistency of a time-stepping method for a class of piecewise-linear networks. *IEEE Transactions on Circuits and Systems I*, 49:349–357, 2002.

[7] K. Camlibel, J.S. Pang, and J. Shen. Lyapunov stability of complementarity and extended systems. *SIAM Journal on Optimization*, 2006. in revision.

[8] R. W. Cottle, J. Pang, and R. E. Stone. *The Linear Complementarity Problem*. Academic Press, Inc., Boston, MA, 1992.

[9] D. Goeleven and B. Brogliato. Stability and instability matrices for linear evolution variational inequalities. *IEEE Transactions on Automatic Control*, 49 (4):521–534, 2004.

[10] W.P.M.H. Heemels. *Linear Complementarity Systems. A Study in Hybrid Dynamics*. PhD thesis, Technical University of Eindhoven, 1999. ISBN 90-386-1690-2.

[11] W.P.M.H. Heemels and B. Brogliato. The complementarity class of hybrid dynamical systems. *European Journal of Control*, 9:311–349, 2003.

[12] M. Kunze and M.D.P. Monteiro Marquès. On parabolic quasi-variational inequalities and state-dependent sweeping processes. *Topol. Methods Non Linear Analysis*, 12:179–191, 1998.

[13] M. Kunze and M.D.P. Monteiro Marquès. An introduction to Moreau's sweeping process. In Brogliato [3], pages 1–60.

[14] M.D.P. Monteiro Marques. *Differential Inclusions in Nonsmooth Mechanical Problems. Shocks and Dry Friction*. Progress in Nonlinear Differential Equations and their Applications, vol.9. Birkhauser, Basel, 1993.

[15] J.J. Moreau. Rafle par un convexe variable (première partie), exposé no 15. *Séminaire d'analyse convexe, University of Montpellier*, page 43 pages, 1971.

[16] J.J. Moreau. Rafle par un convexe variable (deuxième partie) exposé no 3. *Séminaire d'analyse convexe, University of Montpellier*, page 36 pages, 1972.

[17] J.J. Moreau. Evolution problem associated with a moving convex set in a Hilbert space. *Journal of Differential Equations*, 26:347–374, 1977.

[18] J.J. Moreau. Bounded variation in time. In J.J Moreau, P.D. Panagiotopoulos, and G. Strang, editors, *Topics in Nonsmooth Mechanics*, pages 1–74, Basel, 1988. Birkhäuser.

[19] D. Pang, J.-S. an Stewart. Differential variational inequalities. *Mathematical Programming A.*, 2006. submitted, preprint available at http://www.cis.upenn.edu/davinci/publications/pang-stewart03.pdf.