Numerical methods for nonsmooth mechanical systems

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Objectives of the lecture

Formulation and numerical algorithms

Focus on discrete frictional contact 3D problem.

- Linear Complementarity Problem (LCP)
- Nonlinear Complementarity Problem (NCP)
- Generalized, Nonsmooth and semi–smooth equations,
- Variational Inequalities (VI) and Complementarity problems

Summary of the linear time discretized equations

Frictionless case

One step linear problem
$$\begin{cases} v_{k+1} = v_{free} + \widehat{M}^{-1} p_{k+1} \\ q_{k+1} = q_k + h\,[\theta v_{k+1} + (1-\theta) v_k] \end{cases}$$

Relations
$$\begin{cases} U^{\alpha}_{k+1} = H^{\alpha\,T}(q_k)\, v_{k+1} \\ p^{\alpha}_{k+1} = H^{\alpha}(q_k)\, P^{\alpha}_{k+1} \end{cases}$$

Nonsmooth Law
$$\begin{cases} \text{if } g^{\alpha}(\tilde{q}_{k+1}) \leqslant 0, \text{ then} \\ \qquad 0 \leqslant U^{\alpha}_{k+1} \perp P^{\alpha}_{k+1} \geqslant 0 \\ \\ \text{if } g^{\alpha}(\tilde{q}_{k+1}) > 0 \text{ then } P^{\alpha,}_{k+1} = 0 \end{cases}$$

Summary of the linear time discretized equations

Frictional case

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Relations
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Nonsmooth Law
$$\begin{cases} \widehat{U}^\alpha_{k+1} = \left[\begin{array}{c} U^\alpha_{\mathrm{N},k+1} + eU^\alpha_{\mathrm{N},k} + \mu^\alpha \,||U^\alpha_{\mathrm{T},k+1}|| \\ U^\alpha_{\mathrm{T},k+1} \end{array}\right] \\ \text{if } \; g^\alpha(\tilde{q}_{k+1}) \leqslant 0 \;\text{ then} \\ \qquad \mathbf{C}^{\alpha,*} \ni \widehat{U}^\alpha_{k+1} \perp P^\alpha_{k+1} \in \mathbf{C}^\alpha \\ \\ \text{if } \; g^\alpha(\tilde{q}_{k+1}) > 0 \;\text{ then } \; P^{\alpha,}_{k+1} = 0 \end{cases}$$

The time–discretized problems

The Index Set of Forecast Active Constraints

The index set I of all unilateral constraints in the system is denoted by:

$$I = \{1 \dots \nu\} \subset \mathbb{N} \tag{1}$$

The index-set I_a is the set of all *forecast* active constraints of the system and it is denoted by

$$I_a(\tilde{q}_{k+1}) = \{\alpha \in I \mid g^{\alpha}(\tilde{q}_{k+1}) \leqslant 0\} \subseteq I \tag{2}$$

where $\tilde{q}_{k+1}$ is an explicit forecast of the position.

The time–discretized problems

Cone complementarity reformulation for one contact α

$$\mathbf{C}^{\alpha,*} \ni \widehat{U}^{\alpha} \perp P^{\alpha}_{k+1} \in \mathbf{C}^{\alpha}, \;\; \forall\, \alpha \in I_a(\tilde{q}_{k+1}) \tag{3}$$

assuming implicitly that $P^{\alpha} = 0$ for all $\alpha \in I \setminus I_a(\tilde{q}_{k+1})$ and introducing the modified local velocity

$$\widehat{U}^{\alpha}_{k+1} = \left[U^{\alpha}_{\mathrm{N},k+1} + e^{\alpha}\, U^{\alpha}_{\mathrm{N},k} + \mu^{\alpha}\; ||U^{\alpha}_{\mathrm{T},k+1}||,\, U^{\alpha}_{\mathrm{T},k+1} \right]^{T} \tag{4}$$

The time–discretized problems

The time–discretized linear OSNSP $(\mathcal{P}_L)$

The time–discretized linear OSNSP, denoted by $(\mathcal{P}_L)$ is given by

$$(\mathcal{P}_L) \quad \begin{cases} U_{k+1} = \widehat{W} P_{k+1} + U_{\mathrm{free}} \\ \\ \forall \alpha \in I_a(\tilde{q}_{k+1}), \\ \begin{cases} \widehat{U}^\alpha_{k+1} = \left[U^\alpha_{\mathrm{N},k+1} + e^\alpha U^\alpha_{\mathrm{N},k} + \mu^\alpha \, || U^\alpha_{\mathrm{T},k+1} ||, U^\alpha_{\mathrm{T},k+1} \right]^T \\ \\ \mathbf{C}^{\alpha,*} \ni \widehat{U}^\alpha_{k+1} \perp P^\alpha_{k+1} \in \mathbf{C}^\alpha \end{cases} \end{cases}$$

The time–discretized problems

The time–discretized mixed linear OSNSP $(\mathcal{P}_{ML})$

$$
(\mathcal{P}_{ML}) \quad \begin{cases}
\widehat{M}(v_{k+1} - v_{\mathrm{free}}) = p_{k+1} = \displaystyle\sum_{\alpha \in I_a(\tilde{q}_{k+1})} p^{\alpha}_{k+1} \\[2ex]
\forall \alpha \in I_a(\tilde{q}_{k+1}), \\
\begin{cases}
U^{\alpha}_{k+1} = H^{\alpha,T} \; v_{k+1} \\[2ex]
p^{\alpha}_{k+1} = H^{\alpha} \; P^{\alpha}_{k+1} \\[2ex]
\widehat{U}^{\alpha}_{k+1} = \left[U^{\alpha}_{\mathrm{N},k+1} + e^{\alpha} U^{\alpha}_{\mathrm{N},k} + \mu^{\alpha} \, ||U^{\alpha}_{\mathrm{T},k+1}||, \, U^{\alpha}_{\mathrm{T},k+1} \right]^T \\[2ex]
\mathbf{C}^{\alpha,*} \ni \widehat{U}^{\alpha}_{k+1} \perp P^{\alpha}_{k+1} \in \mathbf{C}^{\alpha}
\end{cases}
\end{cases}
$$

The time–discretized problems

The time–discretized mixed nonlinear OSNSP ($\mathcal{P}_{MNL}$)

$$
(\mathcal{P}_{MNL}) \quad \begin{cases}
\mathcal{R}(v_{k+1}) = p_{k+1} = \sum\limits_{\alpha \in I_a(\tilde{q}_{k+1})} p^{\alpha}_{k+1} \\
\\
\forall \alpha \in I_a(\tilde{q}_{k+1}) \\
\begin{cases}
U^{\alpha}_{k+1} = H^{\alpha,T}(q_k+1)\; v_{k+1} \\
\\
p^{\alpha}_{k+1} = H^{\alpha}(q_k+1)\; P^{\alpha}_{k+1} \\
\\
\widehat{U}^{\alpha}_{k+1} = \left[U^{\alpha}_{\mathrm{N},k+1} + e^{\alpha}\, U^{\alpha}_{\mathrm{N},k} + \mu^{\alpha}\; ||U^{\alpha}_{\mathrm{T},k+1}||, U^{\alpha}_{\mathrm{T},k+1}\right]^T \\
\\
\mathbf{C}^{\alpha,*} \ni \widehat{U}^{\alpha}_{k+1} \perp P^{\alpha}_{k+1} \in \mathbf{C}^{\alpha}
\end{cases}
\end{cases}
$$

The time–discretized problems

Delassus' Matrix notation

$$\left\{\begin{array}{ll}\widehat{W} & = H^{a,T}\widehat{M}^{-1}H \\ \widehat{W}_{\mathrm{NT}} & = H_{\mathrm{N}}^{T}\widehat{M}^{-1}H_{\mathrm{T}} \\ \widehat{W}_{\mathrm{NN}} & = H_{\mathrm{N}}^{T}\widehat{M}^{-1}H_{\mathrm{N}} \\ \widehat{W}_{\mathrm{TT}} & = H_{\mathrm{T}}^{T}\widehat{M}^{-1}H_{\mathrm{T}}\end{array}\right. \tag{5}$$

Linear Complementarity Problem (LCP)

Definition (Linear Complementarity Problem (LCP))

Given $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$, the Linear Complementarity Problem, is to find a vector $z \in \mathbb{R}^n$, denoted by $\mathrm{LCP}(M, q)$ such that

$$0 \leq z \perp Mz + q \geq 0 \tag{6}$$

The inequalities have to be understood component-wise and the relation $x \perp y$ means $x^T y = 0$.

LCP . General properties

Definition (P–matrix)

A matrix, $M \in \mathbb{R}^{n\times n}$ is said to be a P–matrix if all its principal minors are positive.

Theorem

Let $M \in \mathbb{R}^{n\times n}$. The following statements are equivalent:

(a) *M is a P–matrix*

(b) *M reverses the sign of no nonzero vector[1], i.e. $x \circ Mx \leqslant 0, \quad \Longrightarrow \quad x = 0$ This property can be written equivalently,*

$$\forall x \neq 0, \exists i \text{ such that } x_i(Mx)_i > 0. \tag{7}$$

(c) *All real eigenvalues of M and its principal submatrices are positive.*

[1] A matrix $A \in \mathbb{R}^{n\times n}$ reverses the sign of a vector $x \in \mathbb{R}^n$ if $x_i(Ax)_i \leqslant 0,\ \forall i \in \{1, \dots, n\}$. The Hadamard product $x \circ y$ is the vector with coordinates $x_i y_i$.

LCP. Fundamental theorem

Theorem

A matrix $M \in \mathbb{R}^{n\times n}$ is a P–matrix if and only if $\mathrm{LCP}(M,q)$ has a unique solution for all vectors $q \in \mathbb{R}^n$.

Other properties

- In the worth case, the problem is N-P hard .i.e. there is no polynomial-time algorithm to solve it.
- In practice, this "P-matrix" assumption is difficult to ensure via numerical computation, but a definite positive matrix (not necessarily symmetric), which is a P-matrix is often encountered.

Mixed Linear Complementarity Problem (MLCP)

Definition (Mixed Linear Complementarity Problem (MLCP))

Given the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$, $C \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{m\times n}$, and the vectors $a \in \mathbb{R}^n, b \in \mathbb{R}^m$, the Mixed Linear Complementarity Problem denoted by $\mathrm{MLCP}(A, B, C, D, a, b)$ consists in finding two vectors $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ such that

$$\begin{cases} Au + Cv + a = 0 \\ 0 \leq v \perp Du + bv + b \geq 0 \end{cases} \tag{8}$$

Comments

The MLCP is a mixture between a LCP and a system of linear equations. Clearly, if the matrix A is non singular, we may solve the embedded linear system to obtain u and then reduced the MCLP to a LCP with $q = b - DA^{-1}a, M = b - DA^{-1}C$.

Linear Complementarity Problem (LCP)

Algorithms for LCP

- Projection/Splitting based methods
- Generalized Newton methods
- Interior point method
- Pivoting based method
- QP methods for a SDP matrix.

Projection/Splitting based methods

Principle

Decomposition of the matrix M as the sum of two matrices B and C

$$M = B + C \tag{9}$$

which define the splitting. Then $\mathrm{LCP}(M, q)$ is solved via a fixed–point iteration. For an arbitrary vector z^ν we consider LCP(B,q^ν) with

$$q^\nu = q + Cz^\nu \tag{10}$$

A vector $z = z^\nu$ solves $\mathrm{LCP}(M, q)$ if and only if z^ν is itself a solution of LCP(B,q^ν).

Choices of splitting

- The subproblem, $\mathrm{LCP}(B, q + Cz^\nu)$ needs to have at least one solution, i.e., B has to be a Q–matrix.
- The splitting must also lead to a subproblem which is relatively easier to solve.

Projection/Splitting based methods

Require: $\mathsf{M}, \mathsf{q}, \mathsf{tol}$
Require: (B, C) a splitting of M
Ensure: z, w solution of $\mathrm{LCP}(\mathsf{M}, \mathsf{q})$.
  Compute a feasible initial point $\mathsf{z}_0 \geqslant 0$.
  $\nu \leftarrow 0$
  while error $>$ tol **do**
    Solve the $\mathrm{LCP}(\mathsf{B}, \mathsf{q} + \mathsf{C}\mathsf{z}^{\nu})$.
    Set $\mathsf{z}^{\nu+1}$ as an arbitrary solution.
    Evaluate error.
  end while

Projection/Splitting based methods

Projected Jacobi Method

B is the identity matrix or any positive diagonal matrix D.
➜ component–wise maximum:

$$z^{\nu+1} = \max\{0, z^{\nu} - D^{-1}(q + Mz^{\nu})\} \tag{11}$$

Comment

In particular, if the matrix D is chosen as the diagonal part of the matrix M, *i.e*, $D = \mathrm{diag}(m_{ii})$, we obtain the projected Jacobi method.

Projection/Splitting based methods

Projected Gauss–Seidel and Projected Successive Overrelaxation (PSOR)

The following splitting of M can be used

$$M = B + C, \text{ with } B = L + \omega^{-1} D, \quad C = U, \omega \in (0,2) \tag{12}$$

where

- L strictly lower part of the matrix M
- U strictly upper part of the matrix M

➜ Projected Successive OverRelaxation (PSOR) scheme

$$z_i^{k+1} = \max(0, z_i^k - \omega M_{ii}^{-1}(q_i + \sum_{j<i} M_{ij} z_i^{k+1} + \sum_{j \geq i} M_{ij} z_i^k)), \quad i = 1, \dots n \tag{13}$$

When $\omega = 1$ the PSOR method is called the Projected Gauss–Seidel (PGS) algorithm.

Projection/Splitting based methods

Convergence

- Chapter 5 in (Cottle et al., 1992) and Chapter 9 in (Murty, 1988)
- Basic assumptions :
  1.
  2. the symmetry and the positive definiteness of the matrix
  3. the contraction of the mapping defined in the fixed point algorithm
- Some results for PSD matrices
- few results concern the rate of convergence, but rather slow even extremely slow in practice.

Projection/Splitting based methods

Regularized PSOR

Diagonal matrix of M, $D = \mathrm{diag}\,(m_{ii})$ is not invertible.

$$\begin{cases} w = Mz + q + \rho(z - \tilde{z}) = (M + \rho I)z + q - \rho\tilde{z} \\ \\ 0 \leqslant w \perp z \geqslant 0 \end{cases} \tag{14}$$

Regularized Projected Successive OverRelaxation (RPSOR) scheme is given by

$$z_i^{k+1} = \max(0, z_i^k - \omega(M_{ii} + \rho)^{-1}(q_i + \sum_{j<i} M_{ij} z_i^{k+1} + \sum_{j\geq i} M_{ij} z_i^k - \rho z_i^k) \tag{15}$$

for $i = 1, \ldots n$. When $\omega = 1$ the RPSOR method is called the Regularized Projected Gauss–Seidel (RPGS) algorithm.

Projection/Splitting based methods

Line–search in the Symmetric Case

Sketch of the general splitting scheme with line–search for the $\mathrm{LCP}(M, q)$ with M symmetric.

Require: $\mathsf{M}, \mathsf{q}, \mathsf{tol}$ and (B, C) a splitting of a symmetric matrix M
Ensure: z, w solution of $\mathrm{LCP}(\mathsf{M}, \mathsf{q})$.
  Compute a feasible initial point $\mathsf{z}_0 \geqslant 0$.
  $\nu \leftarrow 0$
  while error $>$ tol **do**
    Solve the $\mathrm{LCP}(\mathsf{B}, \mathsf{q} + \mathsf{C}\mathsf{z}^\nu)$.
    Set $\mathsf{z}^\star$ as an arbitrary solution.
    Set $\mathsf{d}^\nu \leftarrow \mathsf{z}^\star - \mathsf{z}^\nu$ as the search direction.
    Determine the step size α^ν by a line-search procedure.
    Set $\mathsf{z}^{\nu+1} \leftarrow \mathsf{z}^\nu + \alpha^\nu \mathsf{d}^\nu$.
    Evaluate error.
  end while

Projection/Splitting based methods

Line–search in the Symmetric Case

If $z^\star$ is a solution of $\mathrm{LCP}(B, q + Cz^\nu)$, the line search determines the step–size in the direction d^ν defined by $d^\nu = z^\star - z^\nu$. If ${d^\nu}^T M d^\nu \leqslant 0$, then $\alpha^\nu = 1$; otherwise α^ν must be a nonnegative number satisfying

$$f(z^\nu + \alpha^\nu d^\nu) = \min\{f(z^\nu + \alpha^\nu d^\nu), z^\nu + \alpha^\nu d^\nu \geqslant 0, \alpha \geqslant 0\} \tag{16}$$

where $f(\cdot)$ is the objective function defined by

$$f(z) = z^T(Mz + q) \tag{17}$$

Analogy with QP

In the symmetric case, a clear analogy can be drawn with the following QP:

$$\begin{array}{ll} \text{minimize} & q(z) = \frac{1}{2} z^T M z + q^T z \\ \text{subject to} & z \geqslant 0 \end{array} \tag{18}$$

and d^ν is a descent direction of the objective function.

Pivoting based methods

Principle

Let us consider a $\mathrm{LCP}(M, q)$.

1. If $q \geqslant 0$, then $z = 0$ solves the problem.
2. If there exists an index r such that

$$q_r < 0 \qquad \text{and} \qquad m_{rj} \leqslant 0, \quad \forall j \in \{1 \dots n\} \tag{19}$$

   then there is no vector $z \geqslant 0$ such that $q_r + \sum_j m_{rj} z_i \geqslant 0$. Therefore the LCP is infeasible thus unsolvable.

The LCP rarely possesses these properties in its standard form. The goal of pivoting methods is to derive, by performing pivots, an equivalent system that has one of the previous properties.

Pivoting based methods

Pivotal Algebra

Let us consider the following linear system,

$$w = q + Mz \tag{20}$$

which is represented in a tableau as

	1	z_1	$\dots$	z_n
w_1	q_1	m_{11}	$\dots$	m_{1n}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
w_n	q_1	m_{n1}	$\dots$	m_{nn}

- w_i are called the basic variables
- the independent variables z_i are called the nonbasic variables

Pivoting based methods

Pivotal Algebra

Performing a pivot: exchanging a basic variable w_r with a non basic variable z_s
This operation is possible if and only if $m_{rs} \neq 0$ and yields a new definition of the tableau with (w', z', q', M') such that

$$
\begin{array}{rclcrcl}
w'_r & = & z_s, & & w'_i & = & w_i, \quad i \neq r \\
z'_s & = & w_s, & & z'_j & = & z_j, \quad j \neq s \\
\\
q'_r & = & -q_r/m_{rs}, & & q'_i & = & q_i - (m_{is}/m_{rs})q_r, \quad i \neq r \\
\\
m'_{rs} & = & 1/m_{rs}, & & m'_{is} & = & m_{is}/m_{rs}, \quad i \neq r \\
m'_{rj} & = & -m_{rj}/m_{rs}, & j \neq s, & m'_{ij} & = & m_{ij} - (m_{is}/m_{rs})m_{rj}, \quad i \neq r, j \neq s
\end{array} \tag{21}
$$

This pivot operation will be denoted by

$$(w', z', M', q') = \Pi_{rs}(w, z, M, q) \tag{22}$$

Pivoting based methods

Interest of pivoting operation

Conservation of the fundamental properties of the matrix M under principal pivoting and principal rearrangement

- the PD and PSD matrices are invariant under pivoting operations.
- the P–matrix property and the sufficiency are also conserved
- P_0 property is more tricky and needs some additional care (see (Cottle et al., 1992, Section 4.1)).

Pivoting based methods

Murty's Least Index Method

Simplest principal pivoting methods, also called "Bard–type" algorithm with convergence for a $P-$matrix

Require: M, q
Ensure: z, w solution of $\mathrm{LCP}(\mathsf{M}, \mathsf{q})$ with M a P–matrix.
$\nu \leftarrow 0$
$\mathsf{q}^\nu \leftarrow \mathsf{q}, \quad \mathsf{M}^\nu \leftarrow \mathsf{M}$
while $\mathsf{q}^\nu \ngeq 0$ **do**
Choose the pivot row of index r such that

$$r = \min\{i, q_i^\nu < 0\} \tag{23}$$

Pivoting w_r^ν and z_r^ν.

$$(\mathsf{w}^{\nu+1}, \mathsf{z}^{\nu+1}, \mathsf{M}^{\nu+1}, \mathsf{q}^{\nu+1}) \leftarrow \Pi_{rr}(\mathsf{w}^\nu, \mathsf{z}^\nu, \mathsf{M}^\nu, \mathsf{q}^\nu) \tag{24}$$

$\nu \leftarrow \nu + 1$
end while
$(\mathsf{z}^\nu = 0, \mathsf{w}^\nu = \mathsf{q}^\nu)$ solves $\mathrm{LCP}(\mathsf{M}^\nu, \mathsf{q}^\nu)$.
Recover the solution of $\mathrm{LCP}(\mathsf{M}, \mathsf{q})$.

Pivoting based methods

Lemke's Method

Lemke's Algorithm (Lemke and Howson, 1964 ; Lemke, 1965) belongs to the larger class of the complementary pivot algorithms.

- selection rule of the entering variable in each step, which is always the complementary variable of the dropping variable in the previous step.
- Augmented LCP $\mathrm{LCP}(\tilde{M}, \tilde{q})$

$$\begin{cases} w = Mz + q + dz_0 \\ w_0 = q_0 - d^T z \\ 0 \leqslant w \perp z \geqslant 0 \\ 0 \leqslant w_0 \perp z_0 \geqslant 0 \end{cases} \tag{25}$$

  for a sufficiently large scalar $q_0 \geqslant 0$ and a covering vector $d > 0$
- The $\mathrm{LCP}(\tilde{M}, \tilde{q})$ is known to always possess a solution
- The augmented LCP allows one to obtain a first feasible basic solution

$$\exists \bar{z}_0, \quad w = q + dz_0 \geqslant 0, \quad \forall\, z_0 \geqslant \bar{z}_0 \tag{26}$$

Pivoting based methods

Lemke's Method

- The first pivot row index α is chosen by the minimum ratio such that :

$$\alpha = \arg_i \min \left\{ -\frac{q_i}{d_i} \quad | \quad q_i < 0 \right\} \tag{27}$$

- This pivot index is chosen such that the basic variable component $w = w_\alpha$ equals zero for $z_0 = \bar{z}_0$. Lemke's method starts by pivoting z_0 and w_α.

Pivoting based methods

Lemke's Method. Algorithm Part I

Require: M, q LCP data and c the covering vector
Ensure: z, w solution of $\mathrm{LCP}(M, q)$
if $q \geqslant 0$ **then** $z = 0$, $w = q$ solves the $\mathrm{LCP}(M, q)$ **end if**.

$\nu \leftarrow 0, \quad \tilde{z}^{\nu} \leftarrow [\; z_0 \quad z \;]^T, \quad \tilde{w}^{\nu} \leftarrow [\; w_0 \quad w \;]^T, \quad \tilde{q}^{\nu} \leftarrow [\; q_0 \quad q \;]^T, \quad \tilde{M}^{\nu} \leftarrow \begin{bmatrix} 0 & -c^T \\ c & M \end{bmatrix}$

Find an index $\alpha > 1$ by using the minimum ratio test,

$$\alpha \leftarrow \arg_i \min \left\{ -\frac{q_i}{c_i} \quad | \quad q_i < 0 \right\} \tag{28}$$

Pivot $\tilde{z}_0^{\nu} = z_0$ and $\tilde{w}_{\alpha}^{\nu} = w_{\alpha}$.

$$(\tilde{w}^{\nu+1}, \tilde{z}^{\nu+1}, \tilde{M}^{\nu+1}, \tilde{q}^{\nu+1}) \leftarrow \Pi_{\alpha,0}(\tilde{w}^{\nu}, \tilde{z}^{\nu}, \tilde{M}^{\nu}, \tilde{q}^{\nu}) \tag{29}$$

Set the index of the driving variable $d \leftarrow \alpha$. The driving variable is z_{α}^{ν}.
IsFound $\leftarrow$ false, IsNotFound $\leftarrow$ false

Pivoting based methods

Lemke's Method. Algorithm Part II

while IsFound = false and IsNotFound = false **do**

Step 1. Determination of the blocking variable $\tilde{w}_b^\nu$

if $\exists i,\ m_{id}^\nu < 0$ **then**

Use the minimum ratio test,

$$b \leftarrow \arg_i \min \left\{ -\frac{q_i^\nu}{m_{id}^\nu} \quad | \quad m_{id}^\nu < 0 \right\} \tag{30}$$

else

The blocking variable is w_0. *IsNotFound = true*

end if

Step 2. Pivoting. The driving variable is blocked.

if $b = \alpha$ **then**

The blocking variable is z_0. Pivoting $\tilde{w}_b^\nu = z_0$ and z_d^ν.

$$(\tilde{w}^{\nu+1}, \tilde{z}^{\nu+1}, \tilde{M}^{\nu+1}, \tilde{q}^{\nu+1}) \leftarrow \Pi_{b+1d}(\tilde{w}^{\nu}, \tilde{z}^{\nu}, \tilde{M}^{\nu}, \tilde{q}^{\nu}) \tag{31}$$

The solution is found. $\tilde{z}^{\nu+1}$ solves $\mathrm{LCP}(M^{\nu+1}, q^{\nu+1})$ with $z_0 = 0$.

IsFound $\leftarrow$ true

else

Pivoting the blocking variable $\tilde{w}_b^\nu$ and the driving variable $z_d^\nu = z_0$.

$$(\tilde{w}^{\nu+1}, \tilde{z}^{\nu+1}, \tilde{M}^{\nu+1}, \tilde{q}^{\nu+1}) \leftarrow \Pi_{bd}(\tilde{w}^{\nu}, \tilde{z}^{\nu}, \tilde{M}^{\nu}, \tilde{q}^{\nu}) \tag{32}$$

end if

Set the index of the driving variable $d \leftarrow b$.

$\nu \leftarrow \nu + 1$

end while

Pivoting based methods

Lemke's Method. Algorithm Part III

if IsNotFound = true **then**
  Interpret the output in terms of infeasibility or unsolvability.
end if
if IsFound = true **then** Recover the solution of $\mathrm{LCP}(M, q)$ **end if**.

Pivoting based methods

Lemke's Method solves LCPs for a large class of matrices

- A result ensures that for a feasible LCP with a matrix M which is copositive plus

$$x^T Mx \geqslant 0 \text{ for all } x \geqslant 0 \text{ and } [x^T Mx = 0, x \geqslant 0] \Rightarrow (M + M^T)x = 0$$

  , the complementarity pivot algorithm terminates at a solution of the LCP. If it does not, the LCP is feasible.
- Other results can be found for other classes of matrices (semi–monotone, P_0–matrices,...) in the above cited books.

Interior points method

Principle

Let us start with the horizontal monotone LCP defined by

$$\begin{cases} Qx + Rs = q \\ 0 \leq x \perp s \geq 0 \end{cases} \tag{33}$$

together with the monotonicity property

$$Qx + Rs = 0 \Longrightarrow s^T x \geqslant 0 \tag{34}$$

Definition (central path)

The central path for the horizontal monotone LCP (33) is the set of points (x, s) defined by

$$\begin{cases} x \circ s = \mu \mathbb{1} \\ Qx + Rs = q \\ x \geqslant 0, \quad s \geq 0 \end{cases} \tag{35}$$

for μ describing the half–line, $\mathbb{R}_+$. Here, $\mathbb{1}$ is the vector whose components are all equal to 1.

Interior points methods

Principle

- for $\mu = 0$, the central path equation (35) is equivalent to the horizontal monotone LCP (33).
- for $\mu > 0$, any point of the central path lies in the strictly primal–dual feasible domain defined by

$$\mathcal{F}^{\circ} = \{x, s \in \mathbb{R}^n \mid Qx + Rs = q, x > 0, s > 0\} \tag{36}$$

- As with primal interior points and barrier methods, we see the link between the primal–dual interior point methods and the logarithmic penalty.

Interior points methods

Principle

Drive the iterates of μ to 0. Two methods :

- Approximate minimization of the mixed potential (or some other variants) for a sequence of μ that converges to 0 ➜ potential reduction methods.
- Central path is approximated for a sequence of μ that converges to 0.

In any case, the direction between two iterates is the Newton direction associated with

$$\begin{cases} x \circ s = \sigma\mu\mathbb{1} \\ Qx + Rs = q \end{cases} \tag{37}$$

The strict feasibility assumption is made, i.e, $(x, s) \in \mathcal{F}^\circ$ and $\sigma \in [0, 1]$ is the reduction parameter of μ.

Interior points methods

Implementation

Linearizing the problem (37) around the current point (x, s) results in the following linear system for the direction (u, v):

$$\begin{cases} s \circ u + x \circ v = \sigma\mu\mathbb{1} - x \circ s \\ Qu + Rv = 0 \end{cases} \tag{38}$$

We introduce a matrix notation of the previous system:

$$\begin{bmatrix} S & X \\ Q & R \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \sigma\mu\mathbb{1} - x \circ s \\ 0 \end{bmatrix} \tag{39}$$

where the matrix $S \in \mathbb{R}^{n\times n}$ and $X \in \mathbb{R}^{n\times n}$ are defined by $S = diag(s)$ and $X = diag(x)$.

Interior points methods

Implementation

Two extreme choices of σ are often encountered in practice,

(a) The value $\sigma = 1$ defines the so–called centering direction, (or a centralization displacement). Indeed a Newton step points toward (x^{μ}, s^{μ}) on the central path: $x_i^{\mu} s_i^{\mu} = \mu, i = 1 \ldots n$. A displacement along a centering direction makes little progress, if any, toward reducing the value of μ.

(b) The value $\sigma = 0$ defines the so–called affine–scaling direction, which the standard Newton step for the system

$$\begin{cases} x \circ s = 0 \\ Qx + Rs = q \end{cases} \tag{40}$$

This step therefore should ensure a decrease of μ.

Most of the algorithms choose an intermediate value for σ to have a good trade-off between reducing μ and improving centrality. Finally, once the direction is chosen through a value of σ, a step length α has to be chosen in the direction (u, v) to respect the strict feasibility.

Interior points methods

Algorihtm

Require: Q, R, q, tol
Require: $(x_0, s_0) \in \mathcal{F}^\circ$
Ensure: x, s solution of $\mathrm{hLCP}(Q, R, q)$

$\mu_0 \leftarrow \dfrac{x_0^T s_0}{n}$

$k \leftarrow 0$

while $\mu_k >$ tol **do**

Solve

$$\begin{bmatrix} S^k & X^k \\ Q & R \end{bmatrix} \begin{bmatrix} u^k \\ v^k \end{bmatrix} = \begin{bmatrix} \sigma^k \mu^k \mathbb{1} - x^k \circ s^k \\ 0 \end{bmatrix} \tag{41}$$

for some $\sigma_k \in (0, 1)$.

Choose α_k such that

$$(x^{k+1}, s^{k+1}) \leftarrow (x^k, s^k) + \alpha_k (u^k, v^k) \tag{42}$$

is strictly feasible i.e., $x^{k+1} > 0, s^{k+1} > 0$

$\mu_k \leftarrow \dfrac{x_k^T s_k}{n}$

end while

Interior points methods

A huge of family of methods

The path following primal–dual interior point methods generate a sequence of strictly feasible points satisfying *approximately* the central path equation (35) for a sequence of μ that converges to 0. The approximation may be measured by the centrality measure

$$\delta(s, x, \mu) = \left\| \frac{x \circ s}{\mu} - \mathbb{1} \right\|_2 \tag{43}$$

In most methods, the sequence of points is constrained to lie in one of the following two neighborhoods of the central path : the small neighborhood parametrized by θ

$$\mathcal{N}_2(\theta) \quad = \quad \{(x, s) \in \mathcal{F}^\circ \mid \ \|x \circ s - \mu\mathbb{1}\|_2 \geqslant \mu\theta\} \text{ for some } \theta \in (0, 1) \tag{44}$$

and the large neighborhood parametrized by ε

$$\mathcal{N}_{-\infty}(\varepsilon) \quad = \quad \{(x, s) \in \mathcal{F}^\circ \mid x_i s_i \geqslant \mu\varepsilon\} \text{ for some } \varepsilon \in (0, 1). \tag{45}$$

LCP solution methods

How to choose ?

1. The splitting methods are well suited
   - for very large and well conditioned LCP. Typically, the LCPs with symmetric PD matrix are solved very easily by a splitting method,
   - when a good initial solution is known in advance.
2. The pivoting techniques are well suited
   - for small to medium system sizes ($n < 5000$),
   - for "difficult problems" when the LCP has only a P–matrix, sufficient matrix or copositive plus matrix,
   - when one wants to test the solvability of the system.
3. Finally, interior–point methods can be used
   - for large scale–problems without the knowledge of a good starting point,
   - when the problem has a special structure that can be exploited directly in solving the Newton direction with an adequate linear solver.

The Frictionless Case with Newton's Impact

The frictionless contact problem $(\mathcal{P}_{LWF})$

Remind that the frictionless contact problem in the form $(\mathcal{P}_L)$ can be written as $(\mathcal{P}_{LWF})$:

$$
(\mathcal{P}_{LWF}) \quad \begin{cases} U_{\mathrm{N},k+1} = \widehat{W}_{\mathrm{NN}} P_{\mathrm{N},k+1} + U_{\mathrm{N},\mathrm{free}} \\ \left.\begin{array}{l} \widehat{U}^{\alpha}_{\mathrm{N},k+1} = U^{\alpha}_{\mathrm{N},k+1} + e^{\alpha} U^{\alpha}_{\mathrm{N},k} \\ 0 \leqslant \widehat{U}^{\alpha}_{\mathrm{N},k+1} \perp P^{\alpha}_{\mathrm{N},k+1} \geqslant 0 \end{array}\right\} \forall \alpha \in I_a(\tilde{q}_{k+1}) \end{cases}
$$

LCP formulation

The formulation in terms of LCP is straightforward,

$$
\begin{cases} U_{\mathrm{N},k+1} = \widehat{W}_{\mathrm{NN}} P_{\mathrm{N},k+1} + U_{\mathrm{N},\mathrm{free}} \\ \widehat{U}_{\mathrm{N},k+1} = U_{\mathrm{N},k+1} + e \circ U_{\mathrm{N},k} \\ 0 \leqslant P_{\mathrm{N},k+1} \perp \widehat{U}_{\mathrm{N},k+1} \geqslant 0 \end{cases} \tag{46}
$$

The Frictionless Case with Newton's Impact

LCP formulation

To obtain a proper LCP formulation it suffices to write

$$\begin{cases} \widehat{U}_{\mathrm{N},k+1} = \widehat{W}_{\mathrm{NN}} P_{\mathrm{N},k+1} + U_{\mathrm{N},\mathrm{free}} - e \circ U_{\mathrm{N},k} \\ 0 \leqslant \widehat{U}_{\mathrm{N},k+1} \perp P_{\mathrm{N},k+1} \geqslant 0 \end{cases} \tag{47}$$

and we can conclude that $(\widehat{U}_{\mathrm{N},k+1}, P_{\mathrm{N},k+1})$ solves the following LCP

$$\mathrm{LCP}(\widehat{W}_{\mathrm{NN}}, U_{\mathrm{N},\mathrm{free}} - e \circ U_{\mathrm{N},k}) \tag{48}$$

LCP Resolution

Almost all of the methods can be applied to solve (48).

- The matrix $\widehat{W}_{\mathrm{NN}}$ is a symmetric PSD matrix, provided that $\widehat{M}$ is PD
- The fact that $\widehat{W}_{\mathrm{NN}}$ is PSD and not necessarily PD can cause troubles in numerical applications. This is due to the rank deficiency of H and can be interpreted in terms of redundant constraints.
- In practice, it may happen that the splitting–based algorithms have difficulties to converge.

The Frictionless Case with Newton's Impact and Linear Perfect Bilateral Constraints

The mixed linear OSNSP

Remind that the mixed linear OSNSP with linear perfect bilateral constraints $G^T q + b = 0$ is given by

$$
(\mathcal{P}_{ML_b}) \quad \begin{cases}
\widehat{M}(v_{k+1} - v_{\mathrm{free}}) = p_{k+1} + GP_{\mu,k+1} \\[1em]
G^T v_{k+1} = 0 \\[1em]
U^{\alpha}_{\mathrm{N},k+1} = H^{\alpha,T}_{\mathrm{N}} \; v_{k+1} \\[1em]
p^{\alpha}_{k+1} = H^{\alpha}_{\mathrm{N}} \; P^{\alpha}_{\mathrm{N},k+1} \\[1em]
\left.\begin{array}{l}
\widehat{U}^{\alpha}_{\mathrm{N},k+1} = U^{\alpha}_{\mathrm{N},k+1} + e^{\alpha} U^{\alpha}_{\mathrm{N},k} \\[1em]
0 \leqslant \widehat{U}^{\alpha}_{\mathrm{N},k+1} \perp P^{\alpha}_{\mathrm{N},k+1} \geqslant 0
\end{array}\right\} \forall \alpha \in I_a(\tilde{q}_{k+1})
\end{cases}
$$

The Frictionless Case with Newton's Impact and Linear Perfect Bilateral Constraints

MCLP reformulation

Let us substitute the generalized velocities $\dot{q}_{k+1} = v_{k+1}$ thanks to

$$v_{k+1} = v_{\mathrm{free}} + \widehat{M}^{-1} \left(G P_{\mu,k+1} + H_{\mathrm{N}} P_{\mathrm{N},k+1} \right) = 0 \tag{49}$$

in order to obtain the following MLCP in the form,

$$\begin{cases} G^T v_{\mathrm{free}} + G^T \widehat{M}^{-1} G \, P_{\mu,k+1} + G^T \widehat{M}^{-1} H_{\mathrm{N}} \, P_{k+1} = 0 \\ \widehat{U}^{\alpha}_{\mathrm{N},k+1} = \left[H_{\mathrm{N}}^T v_{\mathrm{free}} + e \circ U_{\mathrm{N},k} + G^T \widehat{M}^{-1} G \, P_{\mu,k+1} + G^T \widehat{M}^{-1} H_{\mathrm{N}} \, P_{k+1} \right] \\ 0 \leqslant \widehat{U}^{\alpha}_{\mathrm{N},k+1} \perp P^{\alpha}_{\mathrm{N},k+1} \geqslant 0 \end{cases} \tag{50}$$

The 3D frictional contact problem as an LCP

The nonlinear nature of the friction cone

The second order cone **C** which cannot be written as a polyhedral cone due to the nonlinear nature of the section of the friction cone, i.e., the disk $\mathbf{D}(\mu R_{\mathrm{N}})$ defined by

$$\mathbf{D}(\mu R_{\mathrm{N}}) = \{R_{\mathrm{T}} \mid \sigma(R_{\mathrm{T}}) = \mu R_{\mathrm{N}} - \|R_{\mathrm{T}}\| \geq 0\} \tag{51}$$

Faceting of the Coulomb's cone.

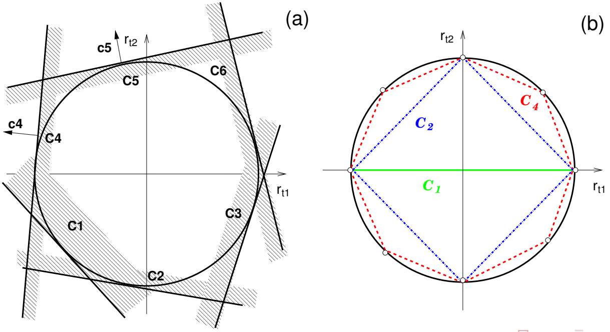


Figure: Approximation of the base of the Coulomb cone by an outer approximation (a) and by an

The 3D frictional contact problem as an LCP

Outer Faceting

The friction disk $\mathbf{D}$ can be approximated by an outer polygon (Klarbring, 1986 ; Klarbring and Björkman, 1988):

$$\mathbf{D}_{outer}(\mu R_{\mathrm{N}}) = \bigcap_{i=1}^{\omega} \mathbf{D}_i(\mu R_{\mathrm{N}})$$

$$\text{with} \tag{52}$$

$$\mathbf{D}_i(\mu R_{\mathrm{N}}) = \left\{ R_{\mathrm{T}}, \sigma_i(R_{\mathrm{T}}) = \mu R_{\mathrm{N}} - c_i^T \, R_{\mathrm{T}} \geq 0 \right\}$$

We now assume that the contact law is of the form

$$-U_{\mathrm{T}} \in N_{\mathbf{D}_{outer}(\mu R_{\mathrm{N}})}(R_{\mathrm{T}}) \tag{53}$$

From (Rockafellar, 1970), the normal cone to $\mathbf{D}_{outer}(\mu R_{\mathrm{N}})$ is given by :

$$N_{\mathbf{D}_{outer}(\mu R_{\mathrm{N}})}(R_{\mathrm{T}}) = \Sigma_{i=1}^{\omega} N_{\mathbf{D}_i(\mu R_{\mathrm{N}})}(R_{\mathrm{T}}) \tag{54}$$

and the inclusion can be stated as:

$$-U_{\mathrm{T}} = \Sigma_{i=1}^{\omega} \kappa_i \partial\sigma_i(R_{\mathrm{T}}), \quad 0 \leq \sigma_i(R_{\mathrm{T}}) \perp \kappa_i \geq 0 \tag{55}$$

Since $\sigma_i(R_{\mathrm{T}})$ is linear with respect to R_{T}, we obtain the following LCP :

The 3D frictional contact problem as an LCP

The time–discretized linear OSNSP, $(\mathcal{P}_L)$

$$
(\mathcal{P}_L)\begin{cases}
U_{k+1} = \widehat{W} P_{k+1} + U_{\mathrm{free}} \\[2ex]
\forall \alpha \in I_a(\tilde{q}_{k+1}), \\
\begin{cases}
-U^{\alpha}_{\mathrm{T},k+1} = \Sigma_{i=1}^{\omega} \kappa^{\alpha}_i c_i \\[2ex]
\sigma_i(P^{\alpha}_{\mathrm{T},k+1}) = \mu P^{\alpha}_{\mathrm{N},k+1} - c_i^T\, P^{\alpha}_{\mathrm{T},k+1} \\[2ex]
0 \leqslant U^{\alpha}_{\mathrm{N},k+1} + e^{\alpha} U^{\alpha}_{\mathrm{N},k} \perp P^{\alpha}_{\mathrm{N},k+1} \geqslant 0 \\[2ex]
0 \leq \sigma^{\alpha}_i(P^{\alpha}_{\mathrm{T},k+1}) \perp \kappa^{\alpha}_i \geq 0
\end{cases}
\end{cases}
\qquad (57)
$$

LCP Formulation

Generally, the MLCP (57) can be reduced in an LCP in standard form assuming that at least one pair of vectors c_i is linearly independent. (see Glocker (2001)).

The 3D frictional contact problem as an LCP

Inner Faceting

Another approach is based on an inner approximation as exposed in (Al-Fahed et al., 1991) and (Stewart and Trinkle, 1996),

$$\mathbf{D}_{inner}(\mu R_{\mathrm{N}}) = \left\{ R_{\mathrm{T}} = D\beta \mid \beta \geq 0, \mu R_{\mathrm{N}} \geq \mathbb{1}^T \beta \right\} \tag{58}$$

where

- $\beta \in \mathbb{R}^2$,
- $\mathbb{1} = [1, \ldots, 1]^T \in \mathbb{R}^\omega$,
- $D \in \mathbb{R}^{2\times\omega}$ whose columns are the directions vectors d_j which are the coordinates of the vertices of the polygon.

MLCP formulation

$$\begin{cases} R_{\mathrm{T}} = D\beta \\ \\ 0 \leq \beta \perp \lambda\mathbb{1} + D^T U_{\mathrm{T}} \geq 0 \\ 0 \leq \lambda \perp \mu R_{\mathrm{N}} - \mathbb{1}^T \beta \geq 0 \end{cases} \tag{59}$$

where $\lambda \in \mathbb{R}$.

The 3D frictional contact problem as an LCP

The time–discretized linear OSNSP, $(\mathcal{P}_L)$

$$\begin{cases} U_{k+1} = \widehat{W} P_{k+1} + U_{\text{free}} \\ \left.\begin{array}{l} P^\alpha_{T,k+1} = D^\alpha \beta^\alpha_{k+1} \\ \\ 0 \leqslant U^\alpha_{\mathrm{N},k+1} + e^\alpha U^\alpha_{\mathrm{N},k} \perp P^\alpha_{\mathrm{N},k+1} \geqslant 0 \\ 0 \leq \beta^\alpha_{k+1} \perp \lambda^\alpha_{k+1} \mathbb{1}^\alpha + D^{\alpha,T} U^\alpha_{\mathrm{T},k+1} \geq 0 \\ 0 \leq \lambda \perp \mu P^\alpha_{\mathrm{N},k+1} - \mathbb{1}^{T,alpha} \beta^\alpha_{k+1} \geq 0 \end{array}\right\} \forall \alpha \in I_a(\tilde{q}_{k+1}) \end{cases} \tag{60}$$

The 3D frictional contact problem as an LCP

LCP form for the inner faceting

$$\left\{\begin{array}{l} \left[\begin{array}{c} U_{\mathrm{N},k+1} + eU_{\mathrm{N},k} \\ \kappa_{k+1} \\ \sigma_{k+1} \end{array}\right] = M \left[\begin{array}{c} P_{\mathrm{N},k+1} \\ \beta_{k+1} \\ \lambda_{k+1} \end{array}\right] + q \\ \\ 0 \leqslant \left[\begin{array}{c} U_{\mathrm{N},k+1} + eU^{\alpha}_{\mathrm{N},k} \\ \kappa_{k+1} \\ \sigma_{k+1} \end{array}\right] \perp \left[\begin{array}{c} P_{\mathrm{N},k+1} \\ \beta_{k+1} \\ \lambda_{k+1} \end{array}\right] \geqslant 0 \end{array}\right. \tag{61}$$

where

$$M = \left[\begin{array}{ccc} \widehat{W}_{\mathrm{NN}} & \widehat{W}_{\mathrm{NT}} D & 0 \\ D^T \widehat{W}_{\mathrm{TN}} & D^T \widehat{W}_{\mathrm{TT}} D & \mathbb{1} \\ \mu & -\mathbb{1}^T & 0 \end{array}\right] \tag{62}$$

and

$$q = \left[\begin{array}{c} U_{\mathrm{N,free}} + eU_{\mathrm{N,free}} \\ D^T (U_{\mathrm{T,free}}) \\ 0 \end{array}\right] \tag{63}$$

The 3D frictional contact problem as an LCP

LCP form for the inner faceting

The variables $\kappa_{k+1} \in \mathbb{R}^{\omega}$ and $\sigma_{k+1} \in \mathbb{R}$ are given by the following equations:

$$\kappa_{k+1} = \lambda_{k+1} + D^T U_{\mathsf{T},k+1}, \qquad \sigma_{k+1} = \mu P_{\mathsf{N},k+1} - \mathbb{1}^T \beta_{k+1} \tag{61}$$

The 3D frictional contact problem as an LCP

Proposition ((Stewart and Trinkle, 1996))

Let $\widehat{W}$ be a PD matrix. The LCP defined by (61), (62) and (63) possesses solutions, which can be computed by Lemke's algorithm provided precautions are taken against cycling due to degeneracy.

Sketch of the proof

- M is co-positive
- $$z^T M z \geqslant 0 :, \quad \text{for all } z \geqslant 0, \tag{62}$$

  is satisfied
- Theorem 3.8.6 in (Cottle et al., 1992, page 179)

M only PSD

A condition has to be added on U_{free} to retrieve a similar result. This condition is related to the existence of a solution (**?**Cadoux, 2009)

The 3D frictional contact problem as an LCP

Advantages and weaknesses of LCP formulation

- ⊕ Existence results and convergence for pivoting algorithms
- ⊖ Anisotropy in the frictional behaviour
- ⊖ Scaling issues. Number of variables and pivoting algorithms.

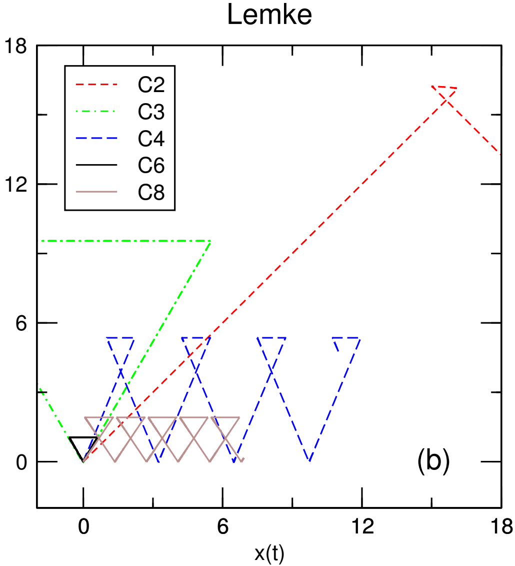
Faceting process and induced anisotropy

Let us consider a ball of mass m lying on a horizontal plane under gravity g. A cycling external force defined by

$$F(t) = \begin{cases} \mu mg(\cos\frac{\pi}{3}\ \mathbf{i} + \sin\frac{\pi}{3}\ \mathbf{j}), & t \in [15k, 5+15k) \\ -\mu mg\ \mathbf{i}, & t \in [5+15k, 10+15k) \\ \mu mg(\cos\frac{\pi}{3}\mathbf{i} + -\sin\frac{\pi}{3}\mathbf{j}), & t \in [10+15k, 15(k+1)) \end{cases}, \quad k \in \mathbb{N} \qquad (63)$$

is applied to the ball.

Faceting process and induced anisotropy



Nonlinear Complementarity Problem (NCP)

Definition (NCP)

Given a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$, the NCP denoted by $\mathrm{NCP}(F)$ is to find a vector $z \in \mathbb{R}^n$ such that

$$0 \leq z \perp F(z) \geq 0 \tag{63}$$

A vector z is called feasible (respectively strictly feasible) for the $\mathrm{NCP}(F)$ if $z \geqslant 0$ and $F(z) \geqslant 0$ (respectively $z > 0$ and $F(z) > 0$).

Definition

A given mapping $F : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be

(a) monotone on X if

$$(x-y)^T(F(x)-F(y)) \geqslant 0, \text{ for all } x, y \in X \tag{64}$$

(b) strictly monotone on X if

$$(x-y)^T(F(x)-F(y)) > 0, \text{ for all } x, y \in X, x \neq y \tag{65}$$

(c) strongly monotone on X if there exists $\mu > 0$ such that

$$(x-y)^T(F(x)-F(y)) \geqslant \mu\|x-y\|^2, \text{ for all } x, y \in X \tag{66}$$

Nonlinear Complementarity Problem (NCP)

Theorem (Characterization of monotone mapping)

Given a continuously differentiable mapping $F : \mathcal{D} \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ on the open convex set $\mathcal{D}$, the following statements are valid,

(a) *$F(\cdot)$ is monotone on $\mathcal{D}$ if and only if $\nabla^T F(x)$ is PSD for all $x \in \mathcal{D}$*

(b) *$F(\cdot)$ is strictly monotone on $\mathcal{D}$ if $\nabla^T F(x)$ is PD for all $x \in \mathcal{D}$*

(c) *$F(\cdot)$ is strongly monotone on $\mathcal{D}$ if and only if $\nabla^T F(x)$ is uniformly PD for all $x \in \mathcal{D}$, i.e.*

$$\exists\, \mu > 0, \qquad z^T \nabla^T F(x) z^T \geqslant \mu \|z\|^2, \quad \forall x \in \mathcal{D} \tag{67}$$

Theorem

Given a continuous mapping $F : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$, the following statements hold,

(a) *If $(\cdot)F$ is monotone on $X = \mathbb{R}^n_+$, the $\mathrm{NCP}(F)$ has a convex (possibly empty) solution set. Furthermore, if there exists a strictly feasible point, the $\mathrm{NCP}(F)$ has a non empty and compact solution set.*

(b) *If $F(\cdot)$ is strictly monotone on $X = \mathbb{R}^n_+$, the $\mathrm{NCP}(F)$ has at most one solution.*

(c) *If $F(\cdot)$ is strongly monotone on $X = \mathbb{R}^n_+$, the $\mathrm{NCP}(F)$ has a unique solution.*

Nonlinear Complementarity Problem (NCP)

Definition

A given mapping $F : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be

(a) a P-function on X if

$$\max_{i=1\ldots n} (x_i - y_i)(F_i(x) - F_i(y)) > 0, \qquad \forall x, y \in X, x \neq y \tag{68}$$

(b) a uniform P-function if

$$\exists\, \mu > 0, \quad \max_{i=1\ldots n} (x_i - y_i)(F_i(x) - F_i(y)) \geqslant \mu \|x - y\|^2, \qquad \forall x, y \in X, x \neq y \tag{69}$$

Theorem

Given a continuous mapping $F : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$, the following statements hold,

(a) *If $F(\cdot)$ is a P-function on X, then the $\mathrm{NCP}(F)$ has at most one solution*

(b) *If $F(\cdot)$ is a uniform P-function on X, then the $\mathrm{NCP}(F)$ has a unique solution.*

The proof can be found in (Moré, 1974).

Nonlinear Complementarity Problem (NCP)

Newton–Josephy's and Linearization Methods

The standard Newton method to linearize $F(\cdot)$ is used, the following LCP

$$0 \leqslant z \perp F(z_k) + \nabla F(z_k)(x - x_k) \geqslant 0 \tag{70}$$

has to be solved to obtain z_{k+1}.

Newton–Robinson's

For theoretical considerations, Robinson (1988, 1992) proposed to use a linearization of the so-called normal map

$$F^{nor}(y) = F(y^+) + (y - y^+) \tag{71}$$

where $y^+ = \max(0, y)$ stands for the positive part of y. Equivalence with the $\mathrm{NCP}(F)$, is as follows: y is a zero of the normal map if and only if y^+ solves $\mathrm{NCP}(F)$. The Newton–Robinson method uses a piecewise linear approximation of the normal map, namely

$$L_k(y) = F(y_k^+) + \nabla F(y_k^+)(y^+ - y_k^+) + y - y^+ \tag{72}$$

The Newton iterate y_{k+1} is a zero of $L_k(\cdot)$. The same y_{k+1} would be obtained by Newton–Josephy's method if z^k were set to y_k^+ in (70).

Nonlinear Complementarity Problem (NCP)

The PATH solver

- The PATH solver (Dirkse and Ferris, 1995) is an efficient implementation of Newton–Robinson's method together with the path-search scheme.
- A "path-search" (as opposed to line–search) is then performed using the merit function $\|F_+(y)\|$. Standard theory of damped Newton's method can be extended to prove standard local and global convergence results (Ralph, 1994 ; Dirkse and Ferris, 1995).
- The construction of the piecewise linear path p_k is based on the use of pivoting methods. Each pivot corresponds to a kink in the path. In (Dirkse and Ferris, 1995), a modification of Lemke's algorithm is proposed to construct the path.

Nonlinear Complementarity Problem (NCP)

Generalized or Semismooth Newton's Methods

The principle of the generalized or semismooth Newton's method for LCPs is based on a reformulation in terms of possibly nonsmooth equations using the so-called C-function also called NCP-function.

Definition

A function $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is called a C-function (for complementarity) if

$$0 \leqslant w \perp z \geqslant 0 \Longleftrightarrow \phi(w,z) = 0 \tag{73}$$

Well-known examples of C-function are

$$\begin{aligned}
\phi(w,z) &= \min(w,z) && \text{(74a)}\\
\phi(w,z) &= \max(0, w - \rho z) - w, \rho > 0 && \text{(74b)}\\
\phi(w,z) &= \max(0, z - \rho w) - z, \rho > 0 && \text{(74c)}\\
\phi(w,z) &= \sqrt{w^2 + z^2} - z - w && \text{(74d)}\\
\phi(w,z) &= \lambda(\sqrt{w^2 + z^2} - z - w) - (1-\lambda) w_+ z_+), \ \lambda \in (0,1) && \text{(74e)}\\
\phi(w,z) &= -wz + \frac{1}{2}\min{}^2(0, w + z) && \text{(74f)}
\end{aligned}$$

Nonlinear Complementarity Problem (NCP)

Then defining the following function associated with $\mathrm{NCP}(F)$:

$$\Phi(z) = \begin{bmatrix} \phi(F_1(z), z_1) \\ \vdots \\ \phi(F_i(z), z_i) \\ \vdots \\ \phi(F_n(z), z_n) \end{bmatrix} \tag{75}$$

we obtain as an immediate consequence of the definitions of $\phi(\cdot)$ and $\Phi(\cdot)$ the following equivalence.

Lemma

Let $\phi(\cdot)$ be a C-function and the corresponding operator $\Phi(\cdot)$ defined by (75). A vector $\bar{z}$ is a solution of $\mathrm{NCP}(F)$ if and only if $\bar{z}$ solves the nonlinear system of equations $\Phi(z) = 0$.

Numerical method

The standard Newton method is generalized to the nonsmooth case by the following scheme

$$z_{k+1} = z_k - {H_k}^{-1}\Phi(z_k), \quad H_k \in \partial\Phi(z_k) \tag{76}$$

Because the set $\partial\Phi(z_k)$ may not be a singleton (if z_k is a point of discontinuity of $\Phi(\cdot)$), we have to select an arbitrary element for H_k.

Nonlinear Complementarity Problem (NCP)

Comparison of the following implementation of solvers

- MILES (Rutherford, 1993) which is an implementation of the classical Newton–Josephy method,
- PATH
- NE/SQP (Gabriel and Pang, 1992 ; Pang and Gabriel, 1993) which is a generalized Newton's method based on the minimum function (74a); the search direction is computed by solving a convex QP at each iteration,
- QPCOMP (Billups and Ferris, 1995) which is an enhancement of the NE/SQP algorithm to allow iterates to escape from local minima,
- SMOOTH (Chen and Mangasarian, 1996) which is based on solving a sequence of smooth approximations of the NCP,
- PROXI (Billups, 1995) which is a variant of the QPCOMP algorithm using a nonsmooth Newton solver rather than a QP solver,
- SEMISMOOTH (DeLuca et al., 1996) which is an implementation of a semismooth Newton method using the Fischer–Bursmeister function,
- SEMICOMP (Billups, 1995) which is an enhancement of SEMISMOOTH based on the same strategy as QPCOMP.

All of these comparisons, which have been made in the framework of the MCP show that the PROXI, PATH and SMOOTH are superior on a large sample of test problems.

The Frictionless Case with Newton's Impact as an NCP

This problem can stated in the following NCP for v_{k+1} and $P^{\alpha}_{\mathrm{N},k+1}$, $\alpha \in I_a(\tilde{q}_{k+1})$

$$\begin{cases} \mathcal{R}(v_{k+1}) = \displaystyle\sum_{\alpha \in I_a(\tilde{q}_{k+1})} H^{\alpha}_{\mathrm{N}}(q_k+1)\, P^{\alpha}_{\mathrm{N},k+1} \\ 0 \leqslant \widehat{H}^{\alpha,T}_{\mathrm{N}}(q_k+1)\, v_{k+1} + e^{\alpha} U^{\alpha}_{\mathrm{N},k} \perp P^{\alpha}_{\mathrm{N},k+1} \geqslant 0, \qquad \forall \alpha \in I_a(\tilde{q}_{k+1}). \end{cases} \tag{77}$$

A first NCP formulation

The Coulomb friction model can be easily reformulated into the following CP:

$$\begin{cases} R_{\mathsf{T}}\, \|U_{\mathsf{T}}\| + \|R_{\mathsf{T}}\| U_{\mathsf{T}} = 0 \\ \\ 0 \leqslant \|U_{\mathsf{T}}\| \perp \mu R_{\mathsf{N}} - \|R_{\mathsf{T}}\| \geq 0 \end{cases} \tag{78}$$

Let us denote $\kappa = \|U_{\mathsf{T}}\|$ the norm of U_{T}. The following CP can be stated

$$\begin{cases} \kappa = \|U_{\mathsf{T}}\| \\ \\ R_{\mathsf{T}}\, \kappa + \|R_{\mathsf{T}}\| U_{\mathsf{T}} = 0 \\ \\ 0 \leqslant \kappa \perp \mu R_{\mathsf{N}} - \|R_{\mathsf{T}}\| \geq 0 \end{cases} \tag{79}$$

A first NCP formulation

$$\left\{\begin{array}{l} U_{k+1} = \widehat{W} P_{k+1} + U_{\mathrm{free}} \\ \left.\begin{array}{l} \kappa^{\alpha}_{k+1} = \|U^{\alpha}_{\mathrm{T},k+1}\| \\ \tilde{U}^{\alpha}_{\mathrm{N},k+1} = U^{\alpha}_{\mathrm{N},k+1} + e^{\alpha} U^{\alpha}_{\mathrm{N},k} \\ \\ P^{\alpha}_{\mathrm{T},k+1}\,\kappa^{\alpha}_{k+1} + \|P^{\alpha}_{\mathrm{N},k+1}\| U^{\alpha}_{\mathrm{T},k+1} = 0 \\ \\ 0 \leqslant \kappa^{\alpha}_{k+1} \perp \mu^{\alpha} P^{\alpha}_{\mathrm{N},k+1} - \|P^{\alpha}_{\mathrm{T},k+1}\| \geq 0 \\ 0 \leqslant \tilde{U}^{\alpha}_{\mathrm{N},k+1} \perp P^{\alpha}_{\mathrm{N},k+1} \geqslant 0 \end{array}\right\} \forall \alpha \in I_a(\tilde{q}_{k+1}) \end{array}\right. \tag{80}$$

which can be casted into the MCP form with $u = [U^{\alpha}_{\mathrm{T},k+1}, P^{\alpha}_{\mathrm{N},k+1}, P^{\alpha}_{\mathrm{T},k+1}]^T$ and $v = [\kappa^{\alpha}_{k+1}, U_{\mathrm{N}}]^T$.

Weaknesses of the formulation

Besides the difficulty to directly deal with a MCP in general form, the main drawback of this formulation is the lack of differentiability in the mapping involved in the CP.

A clever NCP formulation due to Glocker (1999)

Glocker (1999) adds three inequalities

$$\sigma_i(R_{\mathrm{T}}) = \mu R_{\mathrm{N}} - e_i^T\, R_{\mathrm{T}} \geq 0, \quad i = 1, 2, 3 \tag{81}$$

where e_1, e_2, e_3 are three unit outwards vector defined by

$$e_i = [\cos\alpha_i, \sin\alpha_i], \quad \alpha_i = \frac{(4i-3)\pi}{6} \tag{82}$$

We can remark that

$$\mathbf{D}(\mu R_{\mathrm{N}}) = \cap_{i=1}^{3} \mathbf{D}_i \cap \mathbf{D}(\mu r_{\mathrm{N}}) \tag{83}$$

thus the Coulomb's frictional law remains identical. This normal cone condition leads to

$$-U_{\mathrm{T}} \in \Sigma_{i=1}^{3} e_i \kappa_i + \partial\sigma_D(R_{\mathrm{T}})\,\kappa_D \tag{84}$$

where $\sigma_D(R_{\mathrm{T}}) = \mu^2 R_{\mathrm{N}}^2 - \|R_{\mathrm{T}}\|^2$ is a nonlinear friction saturation associated with the second–order cone.

The trick introduced by Glocker lies into the reformulation of this inclusion into an equation of the form

$$-U_{\mathrm{T}} = \Sigma_{i=1}^{3} e_i \kappa_i + 2R_{\mathrm{T}}\,\kappa_D, \qquad 0 \leq \kappa_j \perp \sigma_j \geq 0, \quad j = 1, 2, 3, D \tag{85}$$

A clever NCP formulation due to Glocker (1999)

The previous CP formulation yields

$$\left\{\begin{array}{l} U_{k+1} = \widehat{W} P_{k+1} + U_{\mathrm{free}} \\ \\ \left.\begin{array}{l} -U^{\alpha}_{\mathrm{T},k+1} = \displaystyle\sum_{i=1}^{3} e_i \kappa^{\alpha}_{i,k+1} + 2P^{\alpha}_{\mathrm{T},k+1}\, \kappa^{\alpha}_{C,k+1} \\ \sigma^{\alpha}_{i}(P^{\alpha}_{\mathrm{T},k+1}) = \mu^{\alpha} P^{\alpha}_{\mathrm{N},k+1} - e_i^T\, P^{\alpha}_{\mathrm{T},k+1}, i = 1,2,3 \\ \sigma^{\alpha}_{D}(P^{\alpha}_{\mathrm{T},k+1}) = (\mu^{\alpha} P^{\alpha}_{\mathrm{N},k+1})^2 - \|P^{\alpha}_{\mathrm{T},k+1}\|^2 \\ \\ 0 \leqslant U^{\alpha}_{\mathrm{N},k+1} + e^{\alpha} U^{\alpha}_{\mathrm{N},k} \perp P^{\alpha}_{\mathrm{N},k+1} \geqslant 0 \\ 0 \leq \kappa^{\alpha}_{j,k+1} \perp \sigma^{\alpha}_{j,k+1} \geq 0, j = 1,2,3,D \end{array}\right\} \forall \alpha \in I_a(\tilde{q}_{k+1}) \end{array}\right. \tag{86}$$

Advantages

The mappings involved in the NCP formulation are differentiable, so standard NCP solvers should work

NCP summary

- It is difficult to say something on the abilities of the NCP formulation to provide good numerical solvers, mainly due to the fact there is quite no attempt to use it in the literature.
- Note that NCP methods based on generalized newton methods provide new solvers for LCP.
- If the problem is over-constrained (hyper-staticity), standard NCP solvers are in troubles.

The Frictional contact 3D problem as a nonsmooth equation

Principle

$$
(\mathcal{P}_L)\quad \begin{cases} U_{k+1} = \widehat{W} P_{k+1} + U_{\mathrm{free}} \\ \\ \forall \alpha \in I_a(\tilde{q}_{k+1}), \\ \begin{cases} \widehat{U}^{\alpha}_{k+1} = \left[U^{\alpha}_{\mathrm{N},k+1} + e^{\alpha} U^{\alpha}_{\mathrm{N},k} + \mu^{\alpha}\, ||U^{\alpha}_{\mathrm{T},k+1}||, U^{\alpha}_{\mathrm{T},k+1} \right]^T \\ \\ \mathbf{C}^{\alpha,*} \ni \widehat{U}^{\alpha}_{k+1} \perp P^{\alpha}_{k+1} \in \mathbf{C}^{\alpha} \end{cases} \end{cases}
$$

is equivalent to

$$
\Phi(U, R) = 0 \tag{87}
$$

The Frictional contact 3D problem as a nonsmooth equation

Alart& Curnier's formulation

Curnier and Alart (1988) ; Alart and Curnier (1991) The previous linear problem can be written as

$$\begin{cases} U_{k+1} = \widehat{W} P_{k+1} + U_{\mathrm{free}} \\ \left.\begin{array}{l} P^{\alpha}_{\mathrm{N},k+1} = \mathrm{proj}_{\mathbb{R}_+}(P^{\alpha}_{\mathrm{N},k+1} - \rho^{\alpha}_{\mathrm{N}}\,(U^{\alpha}_{\mathrm{N},k+1} + e^{\alpha} U^{\alpha}_{\mathrm{N},k})) \\ P^{\alpha}_{\mathrm{T},k+1} = \mathrm{proj}_{\widehat{\mathbf{D}}^{\alpha}(P^{\alpha}_{\mathrm{N},k+1}, U^{\alpha}_{\mathrm{N},k+1})}(P^{\alpha}_{\mathrm{T},k+1} - \rho^{\alpha}_{\mathrm{T}} \circ\, U^{\alpha}_{\mathrm{T},k+1}) \end{array}\right\} \forall \alpha \in I_a(\tilde{q}_{k+1}) \end{cases} \tag{88}$$

where $\rho^{\alpha}_{\mathrm{N}} > 0$, $\rho^{\alpha}_{\mathrm{T}} \in \mathbb{R}^2_+ \setminus \{0\}$ for all $\alpha \in I_a(\tilde{q}_{k+1})$ and the modified friction disk is

$$\widehat{\mathbf{D}}^{\alpha}(P^{\alpha}_{\mathrm{N},k+1}, U^{\alpha}_{\mathrm{N},k+1}) = \mathbf{D}(\mu(\mathrm{proj}_{\mathbb{R}_+}(P^{\alpha}_{\mathrm{N},k+1} - \rho^{\alpha}_{\mathrm{N}}\,(U^{\alpha}_{\mathrm{N},k+1} + e^{\alpha} U^{\alpha}_{\mathrm{N},k}))) \tag{89}$$

for all $\alpha \in I_a(\tilde{q}_{k+1})$.

The Frictional contact 3D problem as a nonsmooth equation

Alart& Curnier's formulation

The use of the projection operators proj(·), or more generally the natural and normal map, allows one to restate a CP or a VI into a system of nonlinear nonsmooth equations,

$$\Phi(U_{k+1}, P_{k+1}) = \left[\begin{array}{l} -U_{k+1} + \widehat{W} P_{k+1} + U_{\mathrm{free}} \\ \\ P_{\mathrm{N},k+1} - \mathrm{proj}_{\mathbb{R}^a_+}(P_{\mathrm{N},k+1} - \rho_{\mathrm{N}} \circ (U_{\mathrm{N},k+1} + e \circ U_{\mathrm{N},k})) \\ \\ P_{\mathrm{T},k+1} - \mathrm{proj}_{\widehat{\mathbf{D}}(P_{\mathrm{N},k+1}, U_{\mathrm{N},k+1})}(P_{\mathrm{T},k+1} - \rho_{\mathrm{T}} \circ U_{\mathrm{T},k+1}) \end{array}\right] = 0 \tag{90}$$

The Frictional contact 3D problem as a nonsmooth equation

Nonsmooth Newton's method

The standard Newton method is generalized to the nonsmooth case by the following scheme

$$z_{k+1} = z_k - {H_k}^{-1}\Phi(z_k), \quad H_k \in \partial\Phi(z_k) \tag{91}$$

Because the set $\partial\Phi(z_k)$ may not be a singleton (if z_k is a point of discontinuity of $\Phi(\cdot)$), we have to select an arbitrary element for H_k.

The Frictional contact 3D problem as a nonsmooth equation

Computation of a subgradient for the Alart–Curnier formulation

$H(U,P) \in \partial\Phi(U,P)$ which has the structure

$$H(U,P) = \begin{bmatrix} -I & 0 & \widehat{W}_{\mathrm{NN}} & \widehat{W}_{\mathrm{NT}} \\ 0 & -I & \widehat{W}_{\mathrm{TN}} & \widehat{W}_{\mathrm{TT}} \\ \partial_{U_{\mathrm{N}}}\Phi_2(U,P) & 0 & \partial_{P_{\mathrm{N}}}\Phi_2(U,P) & 0 \\ 0 & \partial_{U_{\mathrm{T}}}\Phi_3(U,P) & \partial_{P_{\mathrm{N}}}\Phi_3(U,P) & \partial_{P_{\mathrm{T}}}\Phi_3(U,P) \end{bmatrix} \tag{92}$$

where the components of Φ are defined by

$$\begin{aligned} \Phi_1(U,P) &= -U_{k+1} + \widehat{W}P_{k+1} + U_{\mathrm{free}} \\ \Phi_2(U,P) &= P_{\mathrm{N}} - \mathrm{proj}_{\mathbb{R}^a_+}(P_{\mathrm{N}} - \rho_{\mathrm{N}} \circ (U_{\mathrm{N}} + e \circ U_{\mathrm{N},k})) \\ \Phi_3(U,P) &= P_{\mathrm{T}} - \mathrm{proj}_{\widehat{\mathbf{D}}(P_{\mathrm{N}},U_{\mathrm{N}})}(P_{\mathrm{T},k+1} - \rho_{\mathrm{T}} \circ\ U_{\mathrm{T}}) \end{aligned} \tag{93}$$

The Frictional contact 3D problem as a nonsmooth equation

Computation of a subgradient for the Alart–Curnier formulation

$$
\partial_{U_\mathrm{N}} \Phi_2(U,P) = \begin{cases} \rho_\mathrm{N} & \text{if } P_\mathrm{N} - \rho_\mathrm{N} U_\mathrm{N} > 0 \\ 0 & \text{otherwise} \end{cases}
$$

$$
\partial_{P_\mathrm{N}} \Phi_2(U,P) = \begin{cases} 0 & \text{if } P_\mathrm{N} - \rho_\mathrm{N} U_\mathrm{N} > 0 \\ 1 & \text{otherwise} \end{cases}
$$

$$
\partial_{U_\mathrm{T}} \Phi_3(U,P) = \begin{cases} \rho_\mathrm{T} I_{2\times 2} & \text{if } P_\mathrm{T} - \rho_\mathrm{T} U_\mathrm{T} \in \widehat{\mathbf{D}}(P_\mathrm{N}, U_\mathrm{N}) \\ \rho_\mathrm{T} \mu P_\mathrm{N}\; \Gamma(P_\mathrm{T} - \rho_\mathrm{T} U_\mathrm{T}) & \text{otherwise} \end{cases}
$$

$$
\partial_{P_\mathrm{N}} \Phi_3(U,P) = \begin{cases} 0 & \text{if } P_\mathrm{T} - \rho_\mathrm{T} U_\mathrm{T} \in \widehat{\mathbf{D}}(P_\mathrm{N}, U_\mathrm{N}) \\ 0 & \text{if } P_\mathrm{T} - \rho_\mathrm{T} U_\mathrm{T} \notin \widehat{\mathbf{D}}(P_\mathrm{N}, U_\mathrm{N}) \text{ and } P_\mathrm{N} - \rho_\mathrm{N} U_\mathrm{N} \leqslant 0 \\ -\mu \dfrac{P_\mathrm{T} - \rho_\mathrm{T} U_\mathrm{T}}{\|P_\mathrm{T} - \rho_\mathrm{T} U_\mathrm{T}\|} & \text{otherwise} \end{cases}
$$

$$
\partial_{P_\mathrm{T}} \Phi_3(U,P) = \begin{cases} 0 & \text{if } P_\mathrm{T} - \rho_\mathrm{T} U_\mathrm{T} \in \widehat{\mathbf{D}}(P_\mathrm{N}, U_\mathrm{N}) \\ I_2 & \text{if } P_\mathrm{T} - \rho_\mathrm{T} U_\mathrm{T} \notin \widehat{\mathbf{D}}(P_\mathrm{N}, U_\mathrm{N}) \text{ and } P_\mathrm{N} - \\ I_2 - \mu(P_\mathrm{N} - \rho_\mathrm{N} U_\mathrm{N})\; \Gamma(P_\mathrm{T} - \rho_\mathrm{T} U_\mathrm{T}) & \text{otherwise} \end{cases} \quad (94)
$$

The Frictional contact 3D problem as a nonsmooth equation

Line–search procedure.

- (Christensen et al., 1998 ; Christensen and Pang, 1998 ; Christensen, 2000), the semi–smoothness is shown and a line–search procedure is based on a merit function such as
$$\Psi(U,P) = \frac{1}{2}\Phi(U,P)^T\Phi(U,P). \tag{95}$$
Let $\alpha^i = \rho^{m^i}$, where $\rho \in (0,1)$ and m^i is the smallest non-negative integer m for which the following decrease criterion holds:
$$\Psi(z^i + \rho^m d^i) \geqslant (1 - 2\sigma\rho^m)\Psi(z^i) \tag{96}$$
where $\sigma \in (0, \frac{1}{2})$ is a given parameter.
- There is no clear study of the influence of the line–search procedure.

The Frictional contact 3D problem as a nonsmooth equation

Variants

- (Park and Kwak, 1994 ; Leung et al., 1998) write the 2D problem as

$$\text{If } g(q) \leqslant 0, \quad \Psi_1(U_{\mathrm{N}}, R_{\mathrm{N}}) = \min(U_{\mathrm{N}}, R_{\mathrm{N}}) = 0 \tag{97}$$

$$\begin{aligned}\Psi_2(U_{\mathrm{T}}, R_{\mathrm{T}}) = U_{\mathrm{T}} &+ \min(0, \mu \max(0, R_{\mathrm{N}} - U_{\mathrm{N}}) + R_{\mathrm{T}} - U_{\mathrm{T}}) \\ &+ \max(0, -\mu \max(0, R_{\mathrm{N}} - U_{\mathrm{N}}) + R_{\mathrm{T}} - U_{\mathrm{T}}) = 0\end{aligned} \tag{98}$$

  and the 3D case, they introduce the sliding angle as variable.
- citepXuewen.ea2000, another direct equation–based reformulation is presented

$$\begin{cases} \Psi_1(U, R) = \min(U_{\mathrm{N}}, R_{\mathrm{N}}) = 0 \\ \Psi_2(U, R) = \min(\|U_{\mathrm{T}}\|, \mu R_{\mathrm{N}} - \|R_{\mathrm{T}}\|) \\ \Psi_3(U, R) = |U_{\mathrm{T}1} R_{\mathrm{T}2} - U_{\mathrm{T}2} R_{\mathrm{T}1}| + \max(0, U_{\mathrm{T}1} R_{\mathrm{T}1}) \end{cases} \tag{99}$$

The Frictional contact 3D problem as a VI/CP problem

With the following definitions for the Cartesian product of Coulomb cones,

$$\mathbf{C} = \prod_{\alpha \in I_a(\tilde{q}_{k+1})} \mathbf{C}^{\alpha}, \qquad \mathbf{C}^* = \prod_{\alpha \in I_a(\tilde{q}_{k+1})} \mathbf{C}^{\alpha,*}, \tag{100}$$

the following CP over cones can be written

$$\begin{cases} \widehat{U}_{k+1} = \widehat{W} P_{k+1} + U_{\mathrm{free}} - G(P_{k+1}) \\ \mathbf{C}^* \ni \widehat{U}_{k+1} \perp P_{k+1} \in \mathbf{C}. \end{cases} \tag{101}$$

The function $G : \mathbb{R}^{3a} \to \mathbb{R}^{3a}$ defined by

$$G(P) = \left[\left[\mu^{\alpha} \| [WP + U_{\mathrm{free}}]^{\alpha}_{\mathrm{T}} \| + e^{\alpha} U^{\alpha}_{\mathrm{N},k}, 0 \right], \alpha \in I_a(\tilde{q}_{k+1}) \right]^T \tag{102}$$

is a nonlinear and nonsmooth function of P.

The Frictional contact 3D problem as a VI/CP problem

The formulation in terms of VI is straightforward due to the equivalence between VIs and CPs. The linear problem $(\mathcal{P}_L)$ is equivalent to the following VI

$$(\widehat{W} P_{k+1} + U_{\mathrm{free}} - G(P_{k+1}))^T (P^* - P_{k+1}) \geqslant 0, \quad \text{for all } P^* \in \mathbf{C}. \tag{103}$$

The Frictional contact 3D problem as a VI/CP problem

Projection–type methods for VI

We recall that the most basic projection–type method for the VI

$$F(P)^T(P^* - P) \geqslant 0, \quad \text{for all } P^* \in \mathbf{C} \tag{104}$$

is a fixed point method such that

$$P^{i+1} = \mathrm{proj}_{\mathbf{C}}(P^i - \rho F(P^i)), \quad \rho > 0. \tag{105}$$

Variants: Extra–gradient, hyperplane method

Projection–type methods for VI

For the three-dimensional frictional contact, i.e. $F(P) = \widehat{W}P + U_{\mathrm{free}} - G(P)$, it yields

$$P^{i+1} = \mathrm{proj}_{\mathbf{C}}\left[(I - \rho\widehat{W})P^i - \rho G(P^i) + \rho U_{\mathrm{free}}\right] \quad \rho > 0. \tag{106}$$

This method has been initiated by De Saxcé and Feng (1991) and extensively tested in (Feng, 1991, 1995 ; De Saxcé and Feng, 1998). The authors term this method an Uzawa's method for solving the variational inequality.

The Frictional contact 3D problem as a VI/CP problem

Projection/splitting methods. The NSGS method (Jean and Moreau, 1992)

1. The Delassus operator W is usually sparse block structured in multibody dynamics (see the formulation. The splitting is chosen to take advantage of this structure.
2. Each subproblem of frictionless/frictional contact for a single contact α can be either analytically solved or easily approximated.

Principle

$$\begin{cases} U^{\alpha,i+1}_{k+1} - \widehat{W}^{\alpha\alpha} P^{\alpha,i+1}_{k+1} = U^{\alpha}_{\mathrm{free}} + \sum\limits_{\beta<\alpha} \widehat{W}^{\alpha\beta} P^{\beta,i+1}_{k+1} + \sum\limits_{\beta>\alpha} \widehat{W}^{\alpha\beta} P^{\beta,i}_{k+1} \\[2ex] \widehat{U}^{\alpha,i+1}_{k+1} = \left[U^{\alpha,i+1}_{\mathrm{N}} + e^{\alpha} U^{\alpha}_{\mathrm{N},k} + \mu^{\alpha}\, ||U^{\alpha,i+1}_{\mathrm{T},k+1}||, U^{\alpha,i+1}_{\mathrm{T},k+1} \right]^T \\[2ex] \mathbf{C}^{\alpha,*} \ni \widehat{U}^{\alpha,i+1}_{k+1} \perp P^{\alpha,i+1}_{k+1} \in \mathbf{C}^{\alpha} \end{cases} \tag{107}$$

for all $\alpha, \beta \in I_a(\tilde{q}_{k+1})$.

The Frictional contact 3D problem as a VI/CP problem

Possible local solvers :

- Analytical solutions for the frictionless and the 2D frictional case
- Projection onto the friction cone or the friction disk
- Local newton solver on one contact.

The Frictional contact 3D problem as a VI/CP problem

Newton's method for VI

the following nonsmooth equations holds

$$\mathbf{F}^{nat}_{C}(P) = P - \mathrm{proj}_{\mathbf{C}}((I - \widehat{W})P + U_{\mathrm{free}} - G(P)) = 0 \tag{108}$$

or equivalently

$$\Phi(P) = P - \mathrm{proj}_{\mathbf{C}}((I - \rho\widehat{W})P + \rho U_{\mathrm{free}} - \rho G(P)) = 0, \quad \rho > 0. \tag{109}$$

Use the Clarke generalized subgradient to generate iterates such that

$$P^{i+1} = P^{i} - H^{i-1}\Phi(P^{i}), \text{ with } H^{i-1} \in \partial\Phi(P^{i}). \tag{110}$$

The Frictional contact 3D problem as a VI/CP problem

C-function for CP over second order cone. Second Order Cone Complementarity (SOCC) function.

A SOCC-function ϕ is defined by

$$K^\star \ni x \perp y \in K \Longleftrightarrow \phi(x,y) = 0. \tag{111}$$

Clearly, the nonsmooth equations of the previous sections provides several examples of SOCC-functions and the natural map offers the most simplest SOCC-function.

Jordan Algebra

- In Fukushima et al. (2001), Extension of standard C-function (min and Fischer-Burmeister) to the SOCCP by means of Jordan algebra.
- Smoothing functions are also given with theirs Jacobians and they studied their properties in view of the application of Newton's method.

The Frictional contact 3D problem as a VI/CP problem

Jordan Algebra

For the second order cone K, the Jordan algebra can be defined with the following non-associative Jordan product,

$$x \cdot y = \begin{bmatrix} x^\top y \\ \mu y_{\mathrm{N}} x_{\mathrm{T}} + \mu x_{\mathrm{N}} y_{\mathrm{T}} \end{bmatrix} \tag{112}$$

and the usual componentwise addition $x + y$. The vector x^2 denotes $x \cdot x$ and their exists an unique vector $x^{1/2} \in K$ for all $x \in K$ such that

$$(x^{1/2})^2 = x^{1/2} \cdot x^{1/2} = x. \tag{113}$$

A direct calculation yields

$$x^{1/2} = \begin{bmatrix} s \\ \dfrac{x_{\mathrm{T}}}{2s} \end{bmatrix}, \quad \text{where } s = \sqrt{(x_{\mathrm{N}} + \sqrt{x_{\mathrm{N}}^2 - \|x_{\mathrm{T}}\|^2})/2} \tag{114}$$

If $x = 0$, we can remark that $x_{\mathrm{T}} = 0$ and then $s = 0$. In this case, $x/2s$ is defined to be zero, that is $x^{1/2} = 0$. The vector $|x| \in K$ denotes $(x^2)^{1/2}$.

The Frictional contact 3D problem as a VI/CP problem

Jordan Algebra

Thanks to this algebra and its associated operator, the projection onto K can be written as

$$P_K(x) = \frac{x + |x|}{2}. \tag{115}$$

This formula provides a new expression for the natural map and its associated nonsmooth equations. More interesting is the fact that more complicated C-functions can be extended and smoothed version of this function can be also developed. Let us start with the Fischer-Burmeister function for NCP that can be written

$$\phi_{\mathrm{FB}}(x, y) = x + y - (x^2 + y^2)^{1/2} \tag{116}$$

and its smoothed version with a regularization parameter $\mu > 0$ as

$$\phi_{\mathrm{FB},\mu}(x, y) = x + y - (x^2 + y^2 + 2\mu^2 e)^{1/2} \tag{117}$$

where e is the identity element of the Jordan algebra, that is $e \cdot x = x$. In the same vein, the class of smoothing function of the natural map for NCP developed in Chen and Mangasarian (1996) is extended to SOCCP in Fukushima et al. (2001).

Optimization–based algorithms.

Principle

Try to rely the solvers for the 3D frictional contact problem on (convex) optimization problems.
Why ?

- Reliable existing solvers may be available
- Convergence may be shown on stability arguments (decreasing an cost function)

Main approaches

Sequential convex QP on

- the Tresca's cylinder
- the second order cone

Optimization–based algorithms.

Tresca's Friction

the so-called Tresca friction model can be invoked

$$\begin{cases} \text{If } U_{\mathrm{T}} = 0 \quad \text{then } ||R_{\mathrm{T}}|| \leqslant \theta \\ \\ \text{If } U_{\mathrm{T}} \neq 0 \quad \text{then } ||R_{\mathrm{T}}(t)|| = \theta, \quad \text{and } \exists\, a \geqslant 0 \text{ such that } R_{\mathrm{T}}(t) = -aU_{\mathrm{T}}(t) \end{cases} \tag{118}$$

where θ is the friction threshold.

Equivalent formulation

$$-U \in \partial \psi_{\mathbf{T}_\theta}(R), \qquad \text{or } R \in \partial \psi^*_{\mathbf{T}_\theta}(-U) \tag{119}$$

with $\mathbf{T}_\theta = \mathbb{R}_+ \times \mathbf{D}_\theta$ is the friction cylinder or the Tresca cylinder and $\mathbf{D}_\theta$ is a disk with radius θ.

Optimization–based algorithms.

Inclusion into the normal cone for Tresca's friction.

Introducing the Cartesian product of the Tresca cylinders such that

$$\mathbf{T} = \prod_{\alpha \in I_a(\tilde{q}_{k+1})} \mathbf{T}^{\alpha}_{\theta}, \qquad \mathbf{T}^* = \prod_{\alpha \in I_a(\tilde{q}_{k+1})} \mathbf{T}^{\alpha,*}_{\theta}, \tag{120}$$

the following inclusion can be written for the linear problem $(\mathcal{P}_L)$ with Tresca's friction

$$-\left(\widehat{W} P_{k+1} + U^e_{\mathrm{free}}\right) \in \partial\psi_{\mathbf{T}}(P_{k+1}) \tag{121}$$

where $U^e_{\mathrm{free}} = U_{\mathrm{free}} + \left[\left[e^{\alpha} \circ U^{\alpha}_{\mathrm{N},k}, 0\right], \quad \alpha \in I_a(\tilde{q}_{k+1})\right]^T$.

Equivalent minimization problem

Under the assumption that $\widehat{W}$ is a symmetric PSD matrix, we may consider the following minimization problem

$$\begin{array}{ll} \text{minimize} & \dfrac{1}{2} P^T_{k+1} \widehat{W} P_{k+1} + P^T_{k+1} U^e_{\mathrm{free}} \\ \text{subject to} & P_{k+1} \in \mathbf{T} \end{array} \tag{122}$$

Optimization–based algorithms.

Require: $W, U^{e}_{\mathrm{free}}, U_{\mathrm{free}}, \mu$
Require: P^{0}_{k+1}
Ensure: U_{k+1}, P_{k+1} solution of the problem $(\mathcal{P}_L)$
$i \leftarrow 0$
while error $>$ tol **do**
$\theta^{i} \leftarrow \mu \circ P^{i}_{N,k+1}$
Solve (possibly inexactly) the minimization problem (122)that is

$$P^{i+1}_{k+1} \leftarrow \underset{P \in \mathbf{T}_{\theta^{i}}}{\mathrm{argmin}} \frac{1}{2} P^{T} \widehat{W} P + P^{T} U^{e}_{\mathrm{free}}$$

$i \leftarrow i+1$
Evaluate error.
end while
$U_{k+1} \leftarrow \widehat{W} P_{k+1} + U_{\mathrm{free}}$

Fixed–point algorithm on the friction threshold

Optimization–based algorithms.

PhD Thesis of Florent Cadoux. (co-supervised with C. Lemaréchal)

- Introduce extra variable s^i at each contact

$$s^i := \|U^i_T\| \tag{123}$$

- perform the change of variables (cf De Saxcé)

$$U \longrightarrow \hat{U} := U + \mu s n$$

Incremental problem

$$\left\{ \begin{array}{lcl} Mv + f & = & H^\top P \\ \hat{U} & = & Hv + w + \mu s n \\ C^* \ni \hat{U} \perp P \in C & & \end{array} \right. \tag{124}$$

Why should we do that ?

(124) are KKT conditions of two convex optimization problems (SOCP: second order cone programs)

primal problem

$$\left\{\begin{array}{l} \min \quad J(v) := \frac{1}{2} v^\top M v + f^\top v \\ Hv + w + \alpha s \in C^* \end{array}\right. \tag{D_s}$$

dual problem

$$\left\{\begin{array}{l} \min \quad J_s(P) := \frac{1}{2} P^\top W P - b_s^\top P \\ P \in C \end{array}\right. \tag{P_s}$$

with $W = HM^{-1}H^\top$ and $b_s = \alpha s + \beta$

Side note: when $\mu = 0$, incremental problem is a QP (Moreau)

Final reformulation

- Introducing
$$U(s) := \underset{U}{\operatorname{argmin}}(P_s) = \underset{U}{\operatorname{argmin}}(D_s)$$
practically computable by optimization software, and
$$F^i(s) := \|U_T^i(s)\|,$$
- the incremental problem becomes

fixed point problem

$$F(s) = s$$

Assumption

Assumption

$$\exists v \in \mathbb{R}^m \ : \ Hv + w \in C^*$$

- Interpretation: it is kinematically possible to enforce

$$u \in C^*$$

  at each contact
- not only the intuitive $u_N \geqslant 0$!

Consequence

Using the assumption,

- the application $F : \mathbb{R}^n_+ \rightarrow \mathbb{R}^n_+$ is well-defined
- it is continuous
- it is bounded
- apply Brouwer's theorem

Theorem

A fixed point exists

Can this be used in practice ?

The fixed point equation $F(s) = s$ can be tackled by

- fixed-point iterations
$$s \leftarrow F(s)$$
- Newton iterations
$$s \leftarrow \mathrm{Jac}[F](s) \backslash F(s)$$
- Variants possible (truncated resolution of inner problem...)

Does it work ?

- fixed-point iterations:
  - expensive
  - not very robust
- Newton:
  - usually very few iterations. . .
  - . . . but they are expensive
- bottleneck: SOCP solver
- practical interest is unclear yet
  - more robust ?
  - faster ?

Comparisons between solvers

There is no thorough and systematic comparisons of the various methods ➜ We need a standard library of benchmarks (as for MCPLib).
but

- For the 3D second–order cone, there is no results of convergence of solvers if a solution exists.
- When $\widehat{W}$ has full rank, damped newton methods based on the Alart–Curnier formulation or the SOCP formulation provides quite good results.
- When $\widehat{W}$ has not full rank, the projection–splitting based (projected gradient–like) methods provide a robust alternative but very slow convergence is achieved. Cycling can also be noted.
- The methods based on some optimization sub–problems can provide a good alternative to the previous methods, but their convergence are not shown. Moreover, they need specific developments of optimization solver that take into account the specific nature of the mechanical problem.

Thank you for your attention.

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