# Lecture 2. Solvers for the time-discretized problems.

Vincent Acary

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# Quadratic Programming (QP) problem

## Definition (Quadratic Programming (QP) problem)

Let $Q \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Given the matrices $A \in \mathbb{R}^{m_i\times n}$, $C \in \mathbb{R}^{m_e\times n}$ and the vectors $p \in \mathbb{R}^n$, $b \in \mathbb{R}^{m_i}$, $d \in \mathbb{R}^{m_e}$, the Quadratic Programming (QP) problem is to find a vector $z \in \mathbb{R}^n$ denoted by $\mathrm{QP}(Q, p, A, b, C, d)$ such that

$$\begin{array}{ll} \text{minimize} & q(z) = \frac{1}{2} z^T Q z + p^T z \\ \text{subject to} & Az - b \geq 0 \\ & Cz - d = 0 \end{array} \tag{1}$$

## Associated Lagrangian function

With this constrained optimization problem, a Lagrangian function is usually associated

$$\mathcal{L}(z, \lambda, \mu) = \frac{1}{2} z^T Q z + p^T z - \lambda^T (Az - b) - \mu^T (Cz - d) \tag{2}$$

where $(\lambda, \mu) \in \mathbb{R}^{m_i} \times \mathbb{R}^{m_e}$ are the Lagrange multipliers.

# Quadratic Programming (QP) problem

## First order optimality conditions

The first order optimality conditions or Karush-Kuhn-Tucker (KKT) conditions of the QP problem(1) with a set of equality constraints lead to the following MLCP :

$$\begin{cases} \nabla_z \mathcal{L}(\bar{z}, \lambda, \mu) = Q\bar{z} + p - A^T\lambda - C^T\mu = 0 \\ C\bar{z} - d = 0 \\ 0 \leq \lambda \perp A\bar{z} - b \geq 0 \end{cases} . \qquad (3)$$

## Example

Let $a \in \mathbb{R}$,

$$\begin{array}{ll} \text{minimize} & q(z) = z^2 - 1 \\ \text{subject to} & z \geqslant a \end{array} \qquad \begin{array}{ll} \text{minimize} & q(z) = 1 - z^2 \\ \text{subject to} & z \geqslant a \end{array} \qquad (4)$$

# Quadratic Programming (QP) problem

## Basic properties

- The matrix $Q$ is usually assumed to be a symmetric positive definite (PD).
  ➜ the QP is then convex and the existence and the uniqueness of the minimum is ensured providing that the feasible set $C = \{z, Az - b \geq 0, Cz - d = 0\}$ is none empty.
- Degenerate case.
  - $Q$ is only Semi-Definite Positive (SDP) matrix. (Non existence problems).
  - $A$ (or $C$) is not full-rank. The constraints are not linearly independent. (Non uniqueness of the Lagrange Multipliers)
  - The strict complementarity does not hold. (we can have $0 = \bar{z} = \lambda = 0$ at the optimal point. )
- For positive definite $Q$, the ellipsoid method solves the problem in polynomial time. If, on the other hand, Q is indefinite, then the problem is NP-hard. In fact, even if Q has only one negative eigenvalue, the problem is NP-hard. If the objective function is purely quadratic, negative semi-definite and has fixed rank, then the problem can be solved in polynomial time.

# Quadratic Programming (QP) problem

## The dual problem and the Lagrangian relaxation

Due to the particular form of the Lagrangian function, the QP problem is equivalent to solving

$$\min_{z} \max_{\lambda \geq 0, \mu} \mathcal{L}(z, \lambda, \mu) \tag{5}$$

The idea of the Lagrangian relaxation is to invert the min and the max introducing the dual function

$$\theta(\lambda, \mu) = \min_{z} \mathcal{L}(z, \lambda, \mu) \tag{6}$$

and the dual problem

$$\max_{\lambda \geq 0, \mu} \theta(\lambda, \mu) \tag{7}$$

# Quadratic Programming (QP) problem

## The dual problem and the Lagrangian relaxation

In the particular case of a QP where the matrix $Q$ is non singular, the dual function is equal to :

$$\begin{aligned}\theta(\lambda,\mu) &= \min_z \mathcal{L}(z,\lambda,\mu) = \mathcal{L}(Q^{-1}(A^T\lambda + C^T\mu - p),\lambda,\mu) && (8)\\ &= -\frac{1}{2}(A^T\lambda + C^T\mu - p)^T Q^{-1}(A^T\lambda + C^T\mu - p) + b^T\lambda + d^T\mu && (9)\end{aligned}$$

and we obtain the following dual problem

$$\max_{\lambda\geq 0,\mu} -\frac{1}{2}(A^T\lambda + C^T\mu - p)^T Q^{-1}(A^T\lambda + C^T\mu - p) + b^T\lambda + d^T\mu \qquad (10)$$

which is a QP with only inequality constraints of positivity.

## Equivalences.

The strong duality theorem asserts that if the matrices $Q$ and $AQ^{-1}A^T$ are symmetric semi-definite positive, then if the primal problem (1) has an optimal solution then the dual has also an optimal solution.

# Quadratic Programming (QP) problem

## Algorithms for QP

For the standard case

- Active sets methods. see Fletcher book's [11]
- Interior point methods. see [3]
- Projection and splitting methods for large scale problems.

For the degenerate case,

- Lagrangian relaxation
- Active sets methods. see [12].
- Proximal point algorithm

## Interest of the QP problem

- Reliability with SDP matrix
- Minimization algorithms imply stability

# Equality constrained Quadratic Programming (QP) problem

## Definition

$$\begin{aligned} \text{minimize} \quad & q(z) = \frac{1}{2} z^T Q z + p^T z \\ \text{subject to} \quad & Cz - d = 0 \end{aligned} \tag{11}$$

## KKT conditions

$$\begin{bmatrix} Q & -C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} \bar{z} \\ \mu \end{bmatrix} = \begin{bmatrix} -p \\ d \end{bmatrix} \tag{12}$$

## Lemma (Existence and uniqueness)

*Let $C^T$ have full row rank and denote by $Z \in \mathbb{R}^{n \times (m_e)}$ whose columns form a basis of the null space of the matrix $C^T$, i.e. $\operatorname{Ker} C^T$, we have $C^T Z = 0$. Let us assume that the reduced Hessian matrix, $Z^T Q Z$ is PD. Then the KKT matrix*

$$K = \begin{bmatrix} Q & C^T \\ C & 0 \end{bmatrix} \tag{13}$$

*is non singular, and there exists a unique vector pair $(\bar{z}, \mu)$ satisfying (12).*

# Equality constrained Quadratic Programming (QP) problem

## Algorithms

Solving (12) amounts to solve a indefinite linear system.

- Direct Methods of the whole KKT Matrix.
  1. Dedicated symmetric indefinite factorizations for taking into account symmetry [5, 4, 6, 16].
  2. Iterative methods: QMR methods [13] and least-squares approaches such as LSQR method [21]
- Range-Space Method. Assume that $Q$ is positive definite. One obtains the Lagrange multiplier $\mu$

  $$(CQ^{-1}C^T)\mu = CQ^{-1}g - h \tag{14}$$

  Solve the system (14) for $\mu$ by standard methods (Cholesky, ...) and then

  $$Qr = C^T\mu - g \tag{15}$$

- Null-Space Method . Given a feasible vector $z_0$ (which is just a particular solution of the system $Cz = d$), any feasible vector can be expressed as

  $$z = z_0 + Zw, \quad w \in \mathbb{R}^{m_e} \tag{16}$$

  If the reduced Hessian $Z^TQZ$ is PD, then the unique solution $\bar{w}$ is given by the solution of the following linear system :

  $$Z^TQZw = -Z^T(Qz_0 + p) \tag{17}$$

# Inequality constrained Quadratic Programming (QP) problem

## Definition

$$
\begin{aligned}
\text{minimize} \quad & q(z) = \frac{1}{2} z^T Q z + p^T z \\
\text{subject to} \quad & h_i^T z - g_i \geqslant 0, \quad i \in \mathcal{I} \\
& h_i^T z - g_i = 0, \quad i \in \mathcal{E}
\end{aligned}
\tag{18}
$$

where $\mathcal{E}$ and $\mathcal{I}$ are finite sets of indices.

## Definition (Active set of constraints)

Let us define the active set, $\mathcal{A}(\bar{z})$ at an optimal point $\bar{z}$ in the following way

$$
\mathcal{A}(\bar{z}) = \{ i \in \mathcal{E} \cup \mathcal{I} \mid h_i^T z - g_i = 0 \}
\tag{19}
$$

## Associated Equality constrained QP

An active-set method starts by using a guess of the active set of constraints $\mathcal{A}(\bar{z})$, and solves the corresponding equality constrained QP

$$
\begin{aligned}
\text{minimize} \quad & q(z) = \frac{1}{2} z^T Q z + p^T z \\
\text{subject to} \quad & h_i^T z - g_i = 0, \quad i \in \mathcal{A}(\bar{z})
\end{aligned}
\tag{20}
$$

# Inequality constrained Quadratic Programming (QP) problem

## Reformulation

Given $z_k$ and $\mathcal{W}_k$ at the iteration $k$, we compute the step $r_k = z - z_k$ from the following QP

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2} r^T Q r + s_k^T r \\ \text{subject to} \quad & h_i^T r = 0, \quad i \in \mathcal{W}_k \end{aligned} \tag{21}$$

with $s_k = Q z_k + p$.

## How to add a new active constraints ?

If $r_k \neq 0$, we have to choose a step-length $\alpha_k$ as large as possible to maintain the feasibility with respect to all the constraints. We have the following property:

$$h_i^T (z_k + \alpha_k r_k) = h_i^T z_k = g_i \tag{22}$$

so the constraint value $h_i^T z$ is constant along the direction $r_k$. An explicit formula can be derived for $\alpha_k$ (see [11])

$$\alpha_k = \min \left\{ 1, \min_{i \in \mathcal{W}_k, h_i^T r_k < 0} \left\{ \frac{g_i - h_i^T z_k}{h_i^T r_k} \right\} \right\} \tag{23}$$

# Sketch of the active-set method for convex QP

**Require:** $Q, p, H, g$
**Ensure:** $\bar{z}, \lambda$
Compute a feasible initial point $z_0$.
Compute the working set $\mathcal{W}_0$ at $z_0$.
IsTheSolutionNotFound $\leftarrow$ true
**while** IsTheSolutionNotFound **do**
  Solve the equality constrained QP (21) for $r_k$ and $\hat{\lambda} = \lambda_k$.
  **if** $r_k = 0$ **then**
    **if** $\hat{\lambda}_i \geqslant 0, \forall i \in \mathcal{W}_k \cap \mathcal{I}$ **then**
      $\bar{z} \leftarrow z_k$
      IsTheSolutionNotFound $\leftarrow$ false
    **else**
      $j \leftarrow \mathrm{argmin}_{j \in W_k \cap \mathcal{I}} \{\hat{\lambda}_j\}$
      $z_{k+1} \leftarrow z_k$
      $\mathcal{W}_{k+1} \leftarrow \mathcal{W}_k \setminus \{j\}$
    **end if**
  **else**
    Compute $\alpha_k$ according to (23).
    $z_{k+1} \leftarrow z_k + \alpha_k r_k$.
    Update $\mathcal{W}_{k+1}$ by adding one of the blocking constraints if any.
  **end if**
**end while**

# How to Chose the Right Method ?

1. Active-set methods are the best suited:
   - for small to medium system sizes ($n < 5000$ ),
   - when a good initial point is known especially for the active-set identification point of view, for instance, in sequential quadratic programming or at each step of a dynamical process,
   - when an exact solution is searched. Active-set methods can be used as "purification" techniques of interior point methods.

   Recall that several methods are available to solve the equality constrained sub-problem depending on the structure of the original QP.
2. Gradient-projection methods are well suited for large QP with simple constraints (simple inequality, bound constrained, *etc*. . . )
3. Interior-point methods are well suited
   - for large systems without the knowledge of a good starting point,
   - when the problem has a special structure that can be exploited directly in solving the Newton iteration.

# Nonlinear Programming (NLP)

## Definition (Nonlinear Programming (NLP) Problem)

Given a differentiable function $\theta : \mathbb{R}^n \mapsto \mathbb{R}$, and two differentiable mappings $g : \mathbb{R}^n \mapsto \mathbb{R}^{m_i}$ $g : \mathbb{R}^n \mapsto \mathbb{R}^{m_e}$, the Nonlinear Programming (NLP) problem is to find a vector $z \in \mathbb{R}^n$ such that

$$\begin{array}{ll} \text{minimize} & f(z) \\ \text{subject to} & g(z) \geq 0 \\ & h(z) = 0 \end{array} \tag{24}$$

## Associated Lagrangian function

The Lagrangian of this NLP problem is introduced as follows

$$\mathcal{L}(z, \lambda, \mu) = f(z) - \lambda^T g(z) - \mu^T h(z) \tag{25}$$

where $(\lambda, \mu) \in \mathbb{R}^{m_i} \times \mathbb{R}^{m_e}$ are the Lagrange multipliers.

# Nonlinear Programming (NLP)

## First order optimality conditions

The Karush-Kuhn-Tucker (KKT) necessary conditions for the NLP problem are given the following NCP:

$$\begin{cases} \nabla_z \mathcal{L}(z,\lambda,\mu) = \nabla_z f(z) - \nabla_z^T g(z)\lambda - \nabla_z^T h(z)\mu = 0 \\ h(z) = 0 \\ 0 \leq \lambda \perp g(z) \geq 0 \end{cases} . \tag{26}$$

# Algorithms for NLP

- Penalty, Barrier and Augmented Lagrangian Approaches
  1. Exterior Penalty Approach
  $$\text{minimize} \quad f(z) + \frac{1}{2\varepsilon}\|h(z)\|^2 + \frac{1}{2\varepsilon}\|\max(0, -g(x))\|^2 \tag{27}$$
  2. Barrier Methods
  $$\text{minimize} \quad f(z) - \varepsilon \textstyle\sum_{i=1}^{m_i} \log g_i(x) \tag{28}$$
  3. Augmented Lagrangian Approach
  $$\mathcal{L}_\sigma(z,\lambda,\mu) = \mathcal{L}(z,\lambda,\mu) + \lambda^T \max(\frac{-\lambda}{\sigma}, g(x)) + \frac{\sigma}{2}\|\max(\frac{-\lambda}{\sigma}, g(x))\|^2 \tag{29}$$
- Successive Quadratic Programming (SQP)
- Gradient Projection Methods
  1. The Goldstein–Levitin–Polyak Gradient projection method [15, 18] consists of the iteration
  $$y_{j+1} = P_{\mathcal{D}}(y_j - \alpha_j \nabla f(y_j)), \tag{30}$$
  where $\alpha_j \geqslant 0$ denotes the step size and $\mathcal{D}$ the feasible domain.

# Linear Complementarity Problem (LCP)

## Definition (Linear Complementarity Problem (LCP))

Given $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$, the Linear Complementarity Problem, is to find a vector $z \in \mathbb{R}^n$, denoted by $\mathrm{LCP}(M, q)$ such that

$$0 \leq z \perp Mz + q \geq 0 \tag{31}$$

The inequalities have to be understood component-wise and the relation $x \perp y$ means $x^T y = 0$.

## Definition ($P$–matrix)

A matrix, $M \in \mathbb{R}^{n\times n}$ is said to be a $P$–matrix if all its principal minors are positive.

## Theorem

*Let $M \in \mathbb{R}^{n\times n}$. The following statements are equivalent:*

(a) *$M$ is a $P$–matrix*

(b) *$M$ reverses the sign of no nonzero vector*[1]*, i.e. $x \circ Mx \leqslant 0, \quad \Longrightarrow \quad x = 0$ This property can be written equivalently,*

$$\forall x \neq 0, \exists i \text{ such that } x_i(Mx)_i > 0. \tag{32}$$

(c) *All real eigenvalues of $M$ and its principal submatrices are positive.*

[1] A matrix $A \in \mathbb{R}^{n\times n}$ reverses the sign of a vector $x \in \mathbb{R}^n$ if $x_i(Ax)_i \leqslant 0, \forall i \in \{1, \dots, n\}$. The Hadamard product $x \circ y$ is the vector with coordinates $x_i y_i$.

# Linear Complementarity Problem (LCP)

## Theorem

*A matrix $M \in \mathbb{R}^{n\times n}$ is a P–matrix if and only if $\mathrm{LCP}(M, q)$ has a unique solution for all vectors $q \in \mathbb{R}^n$.*

## Other properties

- In the worth case, the problem is N-P hard .i.e. there is no polynomial-time algorithm to solve it.
- In practice, this "P-matrix" assumption is difficult to ensure via numerical computation, but a definite positive matrix (not necessarily symmetric), which is a P-matrix is often encountered.

# Linear Complementarity Problem (LCP)

## Definition (Mixed Linear Complementarity Problem (MLCP))

Given the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$, $C \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{m\times n}$, and the vectors $a \in \mathbb{R}^n, b \in \mathbb{R}^m$, the Mixed Linear Complementarity Problem denoted by $\mathrm{MLCP}(A, B, C, D, a, b)$ consists in finding two vectors $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ such that

$$\begin{cases} Au + Cv + a = 0 \\ 0 \leq v \perp Du + bv + b \geq 0 \end{cases} \tag{33}$$

## Comments

The MLCP is a mixture between a LCP and a system of linear equations. Clearly, if the matrix $A$ is non singular, we may solve the embedded linear system to obtain $u$ and then reduced the MCLP to a LCP with $q = b - DA^{-1}a, M = b - DA^{-1}C$.

# Linear Complementarity Problem (LCP)

## Link with the QP

If the matrix $M$ of $\mathrm{LCP}(M, q)$ is symmetric PD, a QP formulation of (31) is direct into $\mathrm{QP}(M, q, I_{n\times n}, 0_n, \emptyset, \emptyset)$, $m_i = n, m_e = 0$. For a non symmetric PD matrix $M$, the inner product may be chosen as an objective function:

$$\begin{array}{ll} \text{minimize} & q(z) = z^T(q + Mz) \\ \text{subject to} & q + Mz \geq 0 \\ & z \geq 0 \end{array} \tag{34}$$

and to identify (34) with (1), we set
$Q = M + M^T, Az = (Mz, z)^T, b = (-q, 0)^T, m_i = 2n, m_e = 0.$ Moreover, the first order optimality condition may be written as

$$\begin{cases} (M + M^T)\bar{z} + p - A^T\lambda - M^T\mu \geqslant 0 \\ z^T((M + M^T)\bar{z} + p - A^T\lambda - M^T\mu) = 0 \\ \mu \geqslant 0 \\ u^T(q + M\bar{z}) = 0 \end{cases} . \tag{35}$$

Let us recall that a non symmetric matrix $M$ is PD if and only if its symmetric part, $(M + M^T)$ is PD.

# Linear Complementarity Problem (LCP)

## Algorithms for LCP

- Splitting based methods
- Generalized Newton methods
- Interior point method
- Pivoting based method
- QP methods for a SDP matrix.

# Linear Complementarity Problem (LCP)

## Principle

1. Decomposition the matrix $M$ as the sum of two matrices $B$ and $C$

$$M = B + C \tag{36}$$

   which define the splitting.

2. Then $\mathrm{LCP}(M, q)$ is solved via a fixed–point iteration.

$$q^{\nu} = q + Cz^{\nu} \tag{37}$$

   A vector $z = z^{\nu}$ solves $\mathrm{LCP}(M, q)$ if and only if $z^{\nu}$ is itself a solution of LCP($B$,$q^{\nu}$).

# Linear Complementarity Problem (LCP)

**Require:** $\mathsf{M}, \mathsf{q}, \mathsf{tol}$
**Require:** $(\mathsf{B}, \mathsf{C})$ a splitting of $\mathsf{M}$
**Ensure:** $\mathsf{z}, \mathsf{w}$ solution of $\mathrm{LCP}(\mathsf{M}, \mathsf{q})$.
  Compute a feasible initial point $\mathsf{z}_0 \geqslant 0$.
  $\nu \leftarrow 0$
  **while** error $>$ tol **do**
    Solve the $\mathrm{LCP}(\mathsf{B}, \mathsf{q} + \mathsf{C}\mathsf{z}^\nu)$.
    Set $\mathsf{z}^{\nu+1}$ as an arbitrary solution.
    Evaluate error.
  **end while**

## Projected Jacobi Method

$B$ is to choose the identity matrix or any positive diagonal matrix $D$

$$z^{\nu+1} = \max\{0, z^\nu - D^{-1}(q + Mz^\nu)\} \tag{38}$$

if the matrix $D$ is chosen as the diagonal part of the matrix $M$, *i.e*, $D = \mathrm{diag}(m_{ii})$, we obtain the projected Jacobi method.

# Linear Complementarity Problem (LCP)

## Projected Gauss–Seidel and Projected Successive Overrelaxation (PSOR) Methods

he following splitting of $M$ can be used

$$M = B + C, \text{ with } B = L + \omega^{-1} D, \quad C = U \tag{39}$$

where the matrices $L$ and $U$ are respectively the strictly lower part and upper part of the matrix $M$ and $\omega \in (0, 2)$ is an arbitrary relaxation parameter.

$$z_i^{k+1} = \max(0, z_i^k - \omega M_{ii}^{-1}(q_i + \sum_{j<i} M_{ij} z_i^{k+1} + \sum_{j\geq i} M_{ij} z_i^k)), \quad i = 1, \ldots n \tag{40}$$

# Linear Complementarity Problem (LCP)

## Principle

If $q \geqslant 0$, then $z = 0$ solves the problem.
If there exists an index $r$ such that

$$q_r < 0 \qquad \text{and} \qquad m_{rj} \leqslant 0, \quad \forall j \in \{1 \dots n\} \tag{41}$$

then there is no vector $z \geqslant 0$ such that $q_r + \sum_j mrjz_i \geqslant 0$. Therefore the LCP is infeasible thus unsolvable.
The goal of pivoting methods is to derive, by performing pivots, an equivalent system that has one of the previous properties.

# Linear Complementarity Problem (LCP)

## Pivotal Algebra

Canonical Tableau defined as

| | $1$ | $z_1$ | $\dots$ | $z_n$ |
|---|---|---|---|---|
| $w_1$ | $q_1$ | $m_{11}$ | $\dots$ | $m_{1n}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |
| $w_n$ | $q_1$ | $m_{n1}$ | $\dots$ | $m_{nn}$ |

This pivot operation will be denoted by

$$(w', z', M', q') = \Pi_{rs}(w, z, M, q) \tag{42}$$

and corresponds to

$$\begin{aligned}
w'_r &= z_s, & w'_i &= w_i, \quad i \neq r \\
z'_s &= w_s, & z'_j &= z_j, \quad j \neq s \\
q'_r &= -q_r/m_{rs}, & q'_i &= q_i - (m_{is}/m_{rs})q_r, \quad i \neq r \\
m'_{rs} &= 1/m_{rs}, & m'_{is} &= m_{is}/m_{rs}, \quad i \neq r
\end{aligned}$$

# Linear Complementarity Problem (LCP)

## Murty's Least Index Method

**Require:** $\mathsf{M}, \mathsf{q}$
**Ensure:** $\mathsf{z}, \mathsf{w}$ solution of $\mathrm{LCP}(\mathsf{M}, \mathsf{q})$ with M a P–matrix.
$\nu \leftarrow 0$
$\mathsf{q}^\nu \leftarrow \mathsf{q}, \quad \mathsf{M}^\nu \leftarrow \mathsf{M}$
**while** $\mathsf{q}^\nu \not\geq 0$ **do**
Choose the pivot row of index r such that

$$r = \min\{i, q_i^\nu < 0\} \tag{44}$$

Pivoting $\mathsf{w}_r^\nu$ and $\mathsf{z}_r^\nu$.

$$(\mathsf{w}^{\nu+1}, \mathsf{z}^{\nu+1}, \mathsf{M}^{\nu+1}, \mathsf{q}^{\nu+1}) \leftarrow \Pi_{rr}(\mathsf{w}^\nu, \mathsf{z}^\nu, \mathsf{M}^\nu, \mathsf{q}^\nu) \tag{45}$$

$\nu \leftarrow \nu + 1$
**end while**
$(\mathsf{z}^\nu = 0, \mathsf{w}^\nu = \mathsf{q}^\nu)$ solves $\mathrm{LCP}(\mathsf{M}^\nu, \mathsf{q}^\nu)$.
Recover the solution of $\mathrm{LCP}(\mathsf{M}, \mathsf{q})$.

# Linear Complementarity Problem (LCP)

Other well–known pivoting based methods:

1. Lemke's Algorithm
2. Cottle–Dantzig method
3. Van de Panne method

# Linear Complementarity Problem (LCP)

## Horizontal Monotone LCP

Let us start with the horizontal monotone LCP defined by

$$\begin{cases} Qx + Rs = q \\ 0 \leq x \perp s \geq 0 \end{cases} \tag{46}$$

together with the monotonicity property $Qx + Rs = 0 \Longrightarrow s^T x \geqslant 0$.

## Definition (central path)

The central path for the horizontal monotone LCP (46) is the set of points $(x, s)$ defined by

$$\begin{cases} x \circ s = \mu \mathbb{1} \\ Qx + Rs = q \\ x \geqslant 0, \quad s \geq 0 \end{cases} \tag{47}$$

for $\mu$ describing the half–line, $\mathbb{R}_+$.Here, $\mathbb{1}$ is the vector whose components are all equal to 1.

Obviously, for $\mu = 0$, the central path equation (47) is equivalent to the horizontal monotone LCP (46).

# Linear Complementarity Problem (LCP)

## Principle

In any case, the direction between two iterates is the Newton direction associated with

$$\begin{cases} x \circ s = \sigma \mu \mathbb{1} \\ Qx + Rs = q \end{cases} \tag{48}$$

The strict feasibility assumption is made, i.e, $(x, s) \in \mathcal{F}^\circ$ and $\sigma \in [0, 1]$ is the reduction parameter of $\mu$. Linearizing the problem (48) around the current point $(x, s)$ results in the following linear system for the direction $(u, v)$:

$$\begin{cases} s \circ u + x \circ v = \sigma \mu \mathbb{1} - x \circ s \\ Qu + Rv = 0 \end{cases} \tag{49}$$

We introduce a matrix notation of the previous system:

$$\begin{bmatrix} S & X \\ Q & R \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \sigma \mu \mathbb{1} - x \circ s \\ 0 \end{bmatrix} \tag{50}$$

where the matrix $S \in \mathbb{R}^{n \times n}$ and $X \in \mathbb{R}^{n \times n}$ are defined by $S = diag(s)$ and $X = diag(x)$.

# Linear Complementarity Problem (LCP)

## General scheme of the primal–dual interior–point methods

**Require:** $Q, R, q, tol$
**Require:** $(x_0, s_0) \in \mathcal{F}^\circ$
**Ensure:** $x, s$ solution of $\mathrm{hLCP}(Q, R, q)$

$\mu_0 \leftarrow \dfrac{x_0^T s_0}{n}$
$k \leftarrow 0$
**while** $\mu_k > tol$ **do**
Solve

$$\begin{bmatrix} S^k & X^k \\ Q & R \end{bmatrix} \begin{bmatrix} u^k \\ v^k \end{bmatrix} = \begin{bmatrix} \sigma^k \mu^k \mathbb{1} - x^k \circ s^k \\ 0 \end{bmatrix} \tag{51}$$

for some $\sigma_k \in (0, 1)$.
Choose $\alpha_k$ such that

$$(x^{k+1}, s^{k+1}) \leftarrow (x^k, s^k) + \alpha_k (u^k, v^k) \tag{52}$$

is strictly feasible i.e., $x^{k+1} > 0, s^{k+1} > 0$
$\mu_k \leftarrow \dfrac{x_k^T s_k}{n}$
**end while**

# Linear Complementarity Problem (LCP)

## How to choose an LCP solver ?

1. The splitting methods are well suited
   - for very large and well conditioned LCP. Typically, the LCP s with symmetric PD matrix are solved very easily by a splitting method,
   - when a good initial solution is known in advance.
2. The pivoting techniques are well suited
   - for small to medium system sizes ($n < 5000$ ),
   - for "difficult problems" when the LCP has only a $P$–matrix, sufficient matrix or copositive plus matrix,
   - when one wants to test the solvability of the system.
3. Finally, interior–point methods can be used
   - for large scale–problems without the knowledge of a good starting point,
   - when the problem has a special structure that can be exploited directly in solving the Newton direction with an adequate linear solver.

# Nonlinear Complementarity Problem (NCP)

## Definition (NCP)

Given a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$, the NCP denoted by $\mathrm{NCP}(F)$ is to find a vector $z \in \mathbb{R}^n$ such that

$$0 \leq z \perp F(z) \geq 0 \tag{53}$$

A vector $z$ is called feasible (respectively strictly feasible) for the $\mathrm{NCP}(F)$ if $z \geqslant 0$ and $F(z) \geqslant 0$ (respectively $z > 0$ and $F(z) > 0$).

## Definition

A given mapping $F : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be

(a) monotone on $X$ if

$$(x-y)^T(F(x)-F(y)) \geqslant 0, \text{ for all } x, y \in X \tag{54}$$

(b) strictly monotone on $X$ if

$$(x-y)^T(F(x)-F(y)) > 0, \text{ for all } x, y \in X, x \neq y \tag{55}$$

(c) strongly monotone on $X$ if there exists $\mu > 0$ such that

$$(x-y)^T(F(x)-F(y)) \geqslant \mu \|x-y\|^2, \text{ for all } x, y \in X \tag{56}$$

# Nonlinear Complementarity Problem (NCP)

## Theorem

*Given a continuously differentiable mapping $F : \mathcal{D} \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ on the open convex set $\mathcal{D}$, the following statements are valid,*

(a) *$F(\cdot)$ is monotone on $\mathcal{D}$ if and only if $\nabla^T F(x)$ is PSD for all $x \in \mathcal{D}$*

(b) *$F(\cdot)$ is strictly monotone on $\mathcal{D}$ if $\nabla^T F(x)$ is PD for all $x \in \mathcal{D}$*

(c) *$F(\cdot)$ is strongly monotone on $\mathcal{D}$ if and only if $\nabla^T F(x)$ is uniformly PD for all $x \in \mathcal{D}$ , i.e.*

$$\exists\, \mu > 0, \qquad z^T \nabla^T F(x) z^T \geqslant \mu \|z\|^2, \quad \forall x \in \mathcal{D} \tag{57}$$

# Nonlinear Complementarity Problem (NCP)

## Definition

A given mapping $F : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be

(a) a $P$-function on $X$ if

$$\max_{i=1\dots n}(x_i - y_i)(F_i(x) - F_i(y)) > 0, \qquad \forall x, y \in X, x \neq y \tag{58}$$

(b) a uniform $P$-function if

$$\exists\, \mu > 0, \quad \max_{i=1\dots n}(x_i - y_i)(F_i(x) - F_i(y)) \geqslant \mu\|x - y\|^2, \qquad \forall x, y \in X, x \neq y \tag{59}$$

## Theorem

*Given a continuous mapping $F : X \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$, the following statements hold,*

(a) *If $F(\cdot)$ is a $P$-function on $X$, then the $\mathrm{NCP}(F)$ has at most one solution*

(b) *If $F(\cdot)$ is a uniform $P$-function on $X$, then the $\mathrm{NCP}(F)$ has a unique solution.*

The proof can be found in [20].

# Nonlinear Complementarity Problem (NCP)

## Newton–Josephy's and Linearization Methods

The standard Newton method to linearize $F(\cdot)$ is used, the following LCP

$$0 \leqslant z \perp F(z_k) + \nabla F(z_k)(x - x_k) \geqslant 0 \tag{60}$$

has to be solved to obtain $z_{k+1}$.

## Newton–Robinson's

For theoretical considerations, Robinson [24, 25] proposed to use a linearization of the so-called normal map (see Section 6 for a general definition for VIs)

$$F^{nor}(y) = F(y^+) + (y - y^+) \tag{61}$$

where $y^+ = \max(0, y)$ stands for the positive part of $y$. Equivalence with the $\mathrm{NCP}(F)$, is as follows: $y$ is a zero of the normal map if and only if $y^+$ solves $\mathrm{NCP}(F)$. The Newton–Robinson method uses a piecewise linear approximation of the normal map, namely

$$L_k(y) = F(y_k^+) + \nabla F(y_k^+)(y^+ - y_k^+) + y - y^+ \tag{62}$$

The Newton iterate $y_{k+1}$ is a zero of $L_k(\cdot)$. The same $y_{k+1}$ would be obtained by Newton–Josephy's method if $z^k$ were set to $y_k^+$ in (60).

# Nonlinear Complementarity Problem (NCP)

## The PATH solver

- The PATH solver [9] is an efficient implementation of Newton–Robinson's method together with the path-search scheme.
- A "path-search" (as opposed to line–search) is then performed using the merit function $\|F_+(y)\|$. Standard theory of damped Newton's method can be extended to prove standard local and global convergence results [23, 9].
- The construction of the piecewise linear path $p_k$ is based on the use of pivoting methods. Each pivot corresponds to a kink in the path. In [9], a modification of Lemke's algorithm is proposed to construct the path.

# Nonlinear Complementarity Problem (NCP)

## Generalized or Semismooth Newton's Methods

The principle of the generalized or semismooth Newton's method for LCPs is based on a reformulation in terms of possibly nonsmooth equations using the so-called C-function also called NCP-function.

## Definition

A function $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ is called a C-function (for complementarity) if

$$0 \leqslant w \perp z \geqslant 0 \Longleftrightarrow \phi(w,z) = 0 \tag{63}$$

Well-known examples of C-function are

$$\begin{aligned}
\phi(w,z) &= \min(w,z) && \text{(64a)}\\
\phi(w,z) &= \max(0, w-\rho z) - w, \rho > 0 && \text{(64b)}\\
\phi(w,z) &= \max(0, z-\rho w) - z, \rho > 0 && \text{(64c)}\\
\phi(w,z) &= \sqrt{w^2+z^2} - z - w && \text{(64d)}\\
\phi(w,z) &= \lambda(\sqrt{w^2+z^2} - z - w) - (1-\lambda) w_+ z_+), \ \lambda \in (0,1) && \text{(64e)}\\
\phi(w,z) &= -wz + \frac{1}{2}\min^2(0, w+z) && \text{(64f)}
\end{aligned}$$

# Nonlinear Complementarity Problem (NCP)

Then defining the following function associated with $\mathrm{NCP}(F)$:

$$\Phi(z) = \begin{bmatrix} \phi(F_1(z), z_1) \\ \vdots \\ \phi(F_i(z), z_i) \\ \vdots \\ \phi(F_n(z), z_n) \end{bmatrix} \tag{65}$$

we obtain as an immediate consequence of the definitions of $\phi(\cdot)$ and $\Phi(\cdot)$ the following equivalence.

## Lemma

*Let $\phi(\cdot)$ be a C-function and the corresponding operator $\Phi(\cdot)$ defined by (65). A vector $\bar{z}$ is a solution of $\mathrm{NCP}(F)$ if and only if $\bar{z}$ solves the nonlinear system of equations $\Phi(z) = 0$.*

## Numerical method

The standard Newton method is generalized to the nonsmooth case by the following scheme

$$z_{k+1} = z_k - {H_k}^{-1}\Phi(z_k), \quad H_k \in \partial\Phi(z_k) \tag{66}$$

Because the set $\partial\Phi(z_k)$ may not be a singleton (if $z_k$ is a point of discontinuity of $\Phi(\cdot)$), we have to select an arbitrary element for $H_k$.

# Nonlinear Complementarity Problem (NCP)

## Comparison of the following implementation of solvers

- MILES [26] which is an implementation of the classical Newton–Josephy method,
- PATH
- NE/SQP [14, 22] which is a generalized Newton's method based on the minimum function (64a); the search direction is computed by solving a convex QP at each iteration,
- QPCOMP [2] which is an enhancement of the NE/SQP algorithm to allow iterates to escape from local minima,
- SMOOTH [7] which is based on solving a sequence of smooth approximations of the NCP,
- PROXI [1] which is a variant of the QPCOMP algorithm using a nonsmooth Newton solver rather than a QP solver,
- SEMISMOOTH [8] which is an implementation of a semismooth Newton method using the Fischer–Bursmeister function,
- SEMICOMP [1] which is an enhancement of SEMISMOOTH based on the same strategy as QPCOMP.

All of these comparisons, which have been made in the framework of the MCP show that the PROXI, PATH and SMOOTH are superior on a large sample of test problems.

# The Variational Inequalities and Complementarity Problem(VI/CP)

## Definition (Variational Inequality (VI) problem)

Let $X$ be a nonempty subset of $\mathbb{R}^n$ and let $F$ be a mapping form $\mathbb{R}^n$ into itself. The Variational Inequality problem, denoted by $\mathrm{VI}(X, F)$ is to find a vector $z \in \mathbb{R}^n$ such that

$$F(z)^T(y - z) \geq 0, \forall y \in X \tag{67}$$

# The Variational Inequalities (VI)

## Basic properties

- The set $X$ is often assumed to closed and convex. In most of the applications, $X$ is polyhedral. The function is also assumed to continuous, nevertheless some VI are defined for set-valued mappings and nonconvex sets.
- If $X$ is a closed set and $F$ continuous, the solution set of $\mathrm{VI}(X, F)$ denoted by $\mathrm{SOL}(X, F)$ is always a closed set.
- A geometrical interpretation if the $\mathrm{VI}(X, F)$ leads to the equivalent formulation in terms of inclusion into a normal cone of $X$, i.e.,

$$-F(x) \in N_X(x) \tag{68}$$

or equivalently, in terms of Generalized Equation(GE)

$$0 \in F(x) + N_X(x) \tag{69}$$

# The Variational Inequalities and Complementarity Problem(VI/CP)

## Basic properties

- It is noteworthy that the $\mathrm{VI}(X, F)$ extends the problem of solving non linear equations, $F(x) = 0$ taking $X = \mathbb{R}^n$.
- If $F$ is affine function, $F(x) = Mz + q$, the $\mathrm{VI}(X, F)$ is called Affine VI denoted by, $\mathrm{AVI}(X, F)$.
- If $X$ is polyhedral, we say that the $\mathrm{VI}(X, F)$ is linearly constrained, or that is a linearly constrained VI. A important case is the box constrained VI where the set $X$ is a closed rectangle (possibly unbounded) of $\mathbb{R}^n$, i.e

$$K = \{x \in \mathbb{R}^n, -\infty \leqslant a_i \leqslant x \leqslant b_i \leqslant +\infty\} \tag{70}$$

# The Variational Inequalities and Complementarity Problem(VI/CP)

## Definition (Complementarity Problem (CP))

Given a cone $K \subset \mathbb{R}^n$ and a mapping $F : \mathbb{R}^n \mapsto \mathbb{R}^n$,the Complementarity Problem is to find a vector $x \in \mathbb{R}^n$ denoted by $\mathrm{CP}(K, F)$ such that

$$K \ni x \perp F(z) \in K^\star \tag{71}$$

where $K^\star$ is the dual (negative polar) cone of $K$ defined by

$$K^\star = \{d \in \mathbb{R}^n, v^T d \geqslant 0, \forall v \in K\} \tag{72}$$

# The Variational Inequalities and Complementarity Problem(VI/CP)

## Links between VI and CP, NCP, MCP, LCP, ...

- Let $X = K \subset \mathbb{R}^n$ be a cone. A vector $x$ solves the $\mathrm{VI}(X, F)$ if and only if $x$ solves the $\mathrm{CP}(K, F)$. If $K$ is equal to the non negative orthant of $\mathbb{R}^n_+$, a vector $x$ solves the $\mathrm{VI}(X, F)$ if and only if $x$ solves the $\mathrm{NCP}(F)$.
- A box-constrained VI is equivalent to a NCP or a MCP choosing the bounds $a_i$ and $b_i$ in the right way. If, in $\mathrm{CP}(K, F)$, $K$ is polyhedral and $F$ is affine, we get an LCP.
- An interesting non polyhedral example is when

$$K = \{z \in \mathbb{R}^{n+1} \mid z_0 \geqslant \|(z_1, \dots z_n)\|\} \tag{73}$$

is the so–called *second–order cone* or *ice–cream cone*.

# The Variational Inequalities and Complementarity Problem(VI/CP)

## Links between VI and NLP

Let us consider the following NLP

$$\begin{array}{ll} \text{minimize} & G(z) \\ \text{subject to} & z \in K \end{array} \tag{74}$$

where $G$ is supposed continuously differentiable. If the set $K$ is convex, any local minimizer $\bar{z}$ of (74) must satisfy the following first order optimality conditions:

$$(y - \bar{z})^T \nabla G(\bar{z}) \geqslant 0, \forall y \in K \tag{75}$$

## Theorem

*Let $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable mapping on a convex set $\Omega$; then the following statements are equivalent:*

(a) *there exists a real-valued function $G$ such that $F(x) = \nabla G^T(x)$ on $\Omega$,*

(b) *the Jacobian matrix, $\nabla F^T(x)$ is symmetric on $\Omega$,*

(c) *the integral of $F$ along any closed curve in $\Omega$ is zero.*

# The Variational Inequalities and Complementarity Problem(VI/CP)

## Problem (Generalized Equation (GE) problem)

*Let $\Omega \subset \mathbb{R}^n$ be an open set. Given a continuously Fréchet differentiable mapping $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ and a maximal monotone operator $T : \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$, find a vector $z \in \mathbb{R}^n$ such that*

$$0 \in F(z) + T(z) \tag{76}$$

## Link between VI and Generalized Equation (GE)

The GE problem is closely related to CP problems and to the NLP. For instance, the NCP (53) can represented into a GE by

$$0 \in F(z) + N_{\mathbb{R}^n_+}(z) \tag{77}$$

and the MCP (26), which provides the KKT conditions for the NLP can be recast into a GE of the form

$$0 \in F(z) + N_K(z), \;\; z \in \mathbb{R}^{n+m_e+m_i} \tag{78}$$

with

$$F(z) = \begin{bmatrix} \mathcal{L}(z,u,v) \\ -g(z) \\ -h(z) \end{bmatrix} \quad K = \mathbb{R}^n \times \mathbb{R}^{m_i}_+ \times \mathbb{R}^{m_e} \tag{79}$$

# The Variational Inequalities and Complementarity Problem(VI/CP)

## Lemma

*Let $X \subset \mathbb{R}^n$ be a closed convex set and a mapping $F : X \to \mathbb{R}^n$ an let $P_X$ denote the projection operator onto $X$. The following statement holds*

$$x \text{ solves } \mathrm{VI}(X, F) \Longleftrightarrow \mathbf{F}_X^{nat}(x) = 0 \tag{80}$$

*where $F_X^{nat}$ is the so-called natural map, defined by*

$$\mathbf{F}_X^{nat}(y) = y - P_X(y - F(y)) \tag{81}$$

## Lemma

*Let $X \subset \mathbb{R}^n$ be a closed convex set and a mapping $F : X \to \mathbb{R}^n$. The following statement holds*

$$x \text{ solves } \mathrm{VI}(X, F) \Longleftrightarrow x = P_X(z) \text{ for some } z \text{ such that } \mathbf{F}^{nor}(z) = 0 \tag{82}$$

*where $F_X^{nor}$ is the so-called normal map, defined by*

$$\mathbf{F}_X^{nor}(y) = F(P_X(y)) + y - P_X(y) \tag{83}$$

# The Variational Inequalities and Complementarity Problem(VI/CP)

## Algorithms for VI

- General VI (unstructured closed convex set $K$).
  Reformulation with the normal map associated the $\mathrm{VI}(K, F)$

$$\mathbf{F}_K^{nor}(z) = F(\Pi_K(z)) + z - \Pi_K(z) \tag{84}$$

  A solution $x$ of the $\mathrm{VI}(K, F)$ is given by $\mathbf{F}_K^{nor}(z) = 0$ with $x = \Pi_K(z)$
  - General projection algorithm for VI/CP. (Fixed point). Need at least the definition of the projection onto the cone.
    ➜ Slow and inefficient algorithm.
  - Newton Methods for VI/CP. Need the definition of the projection and the Jacobian of $\mathbf{F}_K^{nor}(z)$
    ➜ Difficult computation for a unstructured closed convex set $K$
- If the problem has a better structure, the problem is then reformulated into a specific complementarity problem through a nonsmooth equation.

# The Variational Inequalities and Complementarity Problem(VI/CP)

## Main Types of Algorithms for the VI

(a) Projection–type and splitting methods.

(b) Minimization of merit functions.

(b) Generalized Newton Methods.

(c) Interior and smoothing methods.

# The Variational Inequalities and Complementarity Problem(VI/CP)

## Projection–type and Splitting Methods

1. Basic Fixed–Point Scheme

$$z_{k+1} = P_X(z_k - \gamma F(z_k)), \gamma > 0 \tag{85}$$

2. The Extragradient Method.

$$z_{k+1} = P_X(z_k - \gamma F(P_X(z_k - \gamma F(z_k)))) \tag{86}$$

3. Splitting Methods
   In the case of the $\mathrm{AVI}(X, q, M)$, most of the previous projection methods have been extended by splitting the matrix $M$ as for the LCP case in [27, 28, 19, 10].
4. The Hyperplane Projection Method [17]

# The Variational Inequalities and Complementarity Problem(VI/CP)

# The Variational Inequalities and Complementarity Problem(VI/CP)

# The Variational Inequalities and Complementarity Problem(VI/CP)

Thank you for your attention.

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