Nonsmooth dynamics for modeling natural gravity hazard.

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- Florian Vincent PhD: Thermodynamical admissible neural networks for granular materials
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Conclusions

Nonsmooth

- nonsmooth = lack of differentiability ($\notin \mathcal{C}^1$),
- graphs with peaks, kinks, jumps.

Dynamics

- systems that evolves with time,
- branch of mechanics concerned with the motion of objects.

Where is nonsmoothness?

- nonsmooth solutions in time and space:
  - continuous, functions of bounded variations, measures and distributions.
- nonsmooth modeling of constitutive laws:
  - set–valued mapping, inequality constraints, complementarity, impact laws,
  - ODE with discontinuous r.h.s, differential inclusion, measure equation.

TRIPOP INRIA project team (LJK)

Research object:

Modeling, Simulation and Control of Nonsmooth Dynamics.

Main application:

Natural environmental risks in mountains.

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  - Simon Le Berre (INRIA, 2024)

Scientific pairs: M. Jean, J.J. Moreau, M. Schatzman & C. Lemaréchal.

Our reference (bedside) books

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Variational inequalities

Differential inclusions

Switched systems

Nonsmooth Mechanical systems

Sliding mode control

Optimal control with state constraints

Hybrid systems and cyberphysical systems

Complementarity systems

Elasto-dynamics with plasticity, contact and impact.

A second order sweeping process

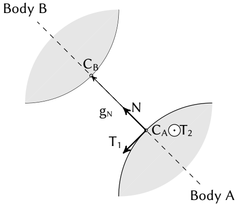
$$\begin{cases} v^+ = \dot{q}^+ & \text{(velocity of bounded variations)} \\ M(q)dv + F(q, v^+)dt + B^\top \sigma dt = \iota & \text{(differential measure)} \\ \dot{\sigma} = E(Bv - \dot{\varepsilon}^p) & \text{(elasticity)} \\ \dot{\varepsilon}^p \in N_C(\sigma) & \text{(plasticity)} \\ -\iota \in N_{T_M(q)}(v^+ + ev^-) & \text{(impact and contact)} \end{cases}$$

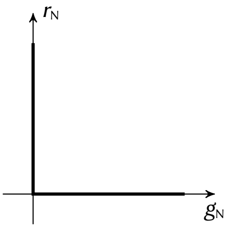
One–sided and Nonsmooth Mechanics. Pioneering work of J.J. Moreau

Irreversible processes in thermodynamics as convex subdifferentials

- Formulation of one-sided and threshold phenomena:
  - admissible (feasible) domains for the state, inequality constraints
- By duality (power), introduction of the force (multipliers):
  - set-valued laws derived from a convex potential thanks to subgradients,
  - potential with values in the completed real line $\mathbb{R}_+ \cup \{+\infty\}$,
  - variational inequalities (normal cone inclusion) $-F(z) \in N_C(z)$,
  - complementarity problems (C is a cone).
- Pseudo–potential of dissipation, $-A \in \partial\varphi(\dot{a})$, φ l.s.c. proper convex:
  - principle of maximum dissipation for friction
  - dual energy principles [Moreau, 1968, 1974].
- Gauss principle with unilateral constraints [Moreau, 1963, 1966]

Unilateral contact and Coulomb friction





- Signorini condition on the gap

$$0 \leq g_{\mathrm{N}} \perp r_{\mathrm{N}} \geq 0 \Leftrightarrow -r_{\mathrm{N}} \in N_{\mathbb{R}_+}(g_{\mathrm{N}})$$

- Signorini condition on the velocity

$$\begin{cases} 0 \leq u_{\mathrm{N}} \perp r_{\mathrm{N}} \geq 0 & \text{if } g_{\mathrm{N}} \leq 0 \\ r_{\mathrm{N}} = 0 & \text{otherwise} \end{cases} \Leftrightarrow -\mathbf{r}_{\mathrm{N}} \in N_{T_{\mathbb{R}_+}(g_{\mathrm{N}})}(u_{\mathrm{N}})$$

- Impact Law (Newton Impact law), e coefficient of restitution.

$$u_{\mathrm{N}}^+ = -e\, u_{\mathrm{N}}^-$$

- Moreau's Impact Law

$$\begin{cases} 0 \leq u_{\mathrm{N}}^+ + e u_{\mathrm{N}}^- \perp \iota_{\mathrm{N}} \geq 0 & \text{if } g_{\mathrm{N}} \leq 0 \\ \iota_{\mathrm{N}} = 0 & \text{otherwise} \end{cases}$$

$$\Updownarrow$$

$$-\iota_{\mathrm{N}} \in N_{T_{\mathbb{R}_+}(g_{\mathrm{N}})}(u_{\mathrm{N}}^+ + e u_{\mathrm{N}}^-)$$

Unilateral contact and Coulomb friction

Second order cone complementarity problem

De Saxcé [1992]; Acary and Brogliato [2008]; Acary et al. [2011, 2018].

- Coulomb friction $K = \{r \in \mathbb{R}^3 \mid \|r_{\mathsf{T}}\| \leq \mu r_n\}$
  - nonassociated character (loss of monotony) [De Saxcé, 1992]

$$-\hat{u} := -(u + \mu \|u_{\mathsf{T}}\| \mathsf{N}) \in N_K(r)$$

  - Second order cone complementarity condition[1].

$$K^\star \ni \hat{u} \perp r \in K$$

- Existence Theorem [Acary V., F. Cadoux, C. Lemaréchal, J. Malick, 2011].

$$\begin{cases} u = Wr + q \\ -(u + \mu \|u_{\mathsf{T}}\| \mathsf{N}) \in N_K(r) \end{cases}$$

with a simple Slater assumption (constraints qualification).

[1] The set $K^\star$ is the dual cone to K defined by $K^\star = \{u \in \mathbb{R}^3 \mid r^\top u \geq 0, \quad \text{for all } r \in K\}$.

Unilateral contact and Coulomb friction

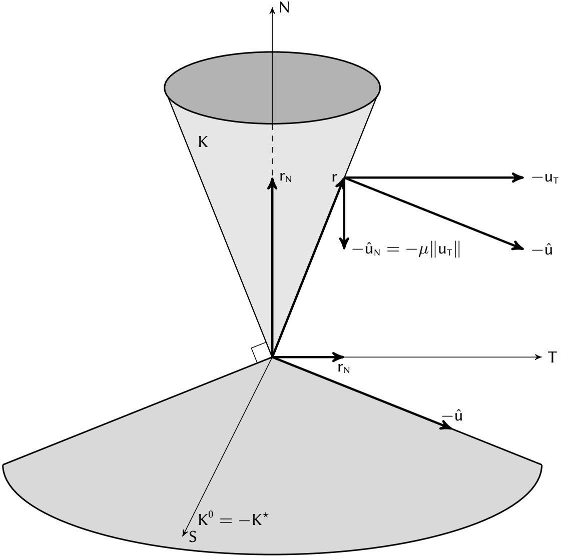


Figure: Coulomb friction and modified relative velocity $\hat{u}$. Sliding case.

Other applications of unilateral and nonsmooth mechanics

Non exhaustive list of applications

- Cavitation in fluids [Moreau, 1964]
  The pressure must be positive or higher than the vaporization pressure
- Plasticity and generalized standard materials [Moreau, 1974, 1976; Halphen and Nguyen, 1975]
  The stresses and strain hardening variables belong to a convex set
- Granular materials [Moreau, 1997, 2001]
- No tension materials and tension field modeling
- Fracture and damage (cohesive zone models)
- Non Newtonian Fluids
  - Quasi-brittle and visco-plastic fluids (Bingham, damage, . . .)
  - Multiphase fluid flows,

Modeling for the environment and natural hazards

- Debris flows, avalanches, block falls, threshold fluids, complex rheology
- Coastal swell protection, ice pack modeling, . . .

Nonsmooth mechanics

Motivated historically by theoretical mechanics, Convex analysis
is the appropriate tools for modeling and mathematical analysis

With mathematical programming and optimisation,
it paves the way to numerical efficient methods

Nonsmooth Dynamics

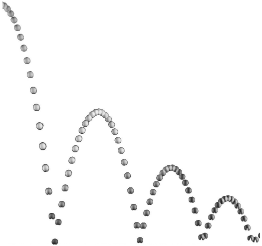
nonsmooth? Késako?

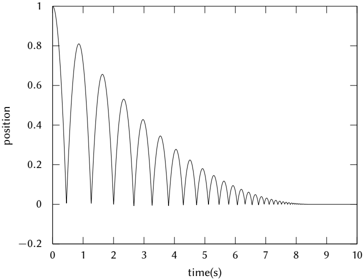
Lack of mathematical regularity of functions.
Everything that is not everywhere differentiable

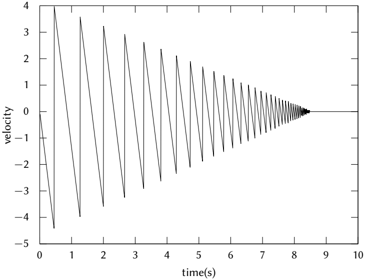
in Mechanics:

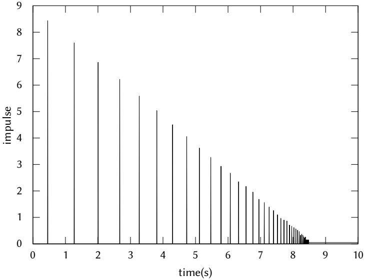
A non-smooth formulation of the laws of constitutive laws
(multi-valued function, inequalities, complementarity)
which can imply non-smooth solutions in time
(angular points, jumps, measures, distributions)

Nonsmooth Dynamics









Nonsmooth Dynamics

Nonsmooth formulation based on differential inclusion

- Writing quasi-static or dynamic evolutions in the form of differential inclusion (parallel research of J.J. Moreau, H. Brézis, M. Schatzman):
  - Second order Moreau's sweeping process
  - Measure differential inclusion
  - The state lies in the space of functions of bounded variations,,
    and its derivatives are differential measures
  - Impact laws as variational inequalities on differential measures

Nonsmooth Dynamics

Efficient numerical methods

- Numerical time integration schemes of these formulations
  - “Event-capturing time–stepping schemes”
  - The discrete variables are the velocities and impulses
  - Iterative solution methods at each time step of the non-smooth and non-convex variational problem based on optimization and mathematical programming techniques

One sided constraint as an inclusion

Definition (Dynamics with perfect one-sided constraints)

[Moreau, 1988]]

$$
\begin{cases}
\dot{q} = v \\
M(q)\dfrac{dv}{dt} + F(q, v) = r \\
-r \in N_{\mathcal{C}(t)}(q)
\end{cases}
\tag{1}
$$

where r is the generalized reaction force.

- Extension of Lagrange equations with one-sided constraints
- Second order differential inclusion (relatice degree 2)
- The constraints are said to be perfect since their work is vanishes (Normality law in coordinates.)

Nonsmooth Lagrangian Dynamics

Fundamental assumptions

- The velocity $v = \dot{q}$ is a function of bounded variations. The unknown of the equation of motion is its right limit.

$$v^+ = \dot{q}^+ \tag{2}$$

- The coordinate q is an absolutely continuous function by the Lebesgue fundamental Theorem of integration:

$$q(t) = q(t_0) + \int_{t_0}^{t} v^+(t)\, dt \tag{3}$$

- The acceleration ($\dot{v} = \ddot{q}$ in the usual sense) is a differential measure associated with v such that

$$dv(]a,b]) = \int_{]a,b]} dv = v^+(b) - v^+(a) \tag{4}$$

Nonsmooth Lagrangian Dynamics

Definition (Nonsmooth Lagrangian Dynamics [Moreau, 1988])

$$\begin{cases} M(q)dv + F(q, v^+)dt = \iota \\ \\ v^+ = \dot{q}^+ \end{cases} \tag{5}$$

where ι is the generalized reaction measure

Advantages

- The formulation allows to take into account complex behaviors such as finite accumulations in time (Zenon phenomenon)
- The formulation is useful for mathematical analysis [Schatzman, 1973, 1978; Monteiro Marques, 1993; Ballard, 2000]
- The non-smooth dynamics contains both the impact equations and the equations of continuous motion

Impact Equations and equations of motion

Using the densities of the differential measures, with respect to the Lebesgue measure and the discrete measures, we obtain

Définition (Impact equations at any time)

$$M(q)(v^+ - v^-)d\nu = pd\nu, \qquad avec\ p = \frac{d\iota}{d\nu} \tag{6}$$

or, equivalently,

$$M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \tag{7}$$

Définition (Continuous dynamics almost-everywhere)

$$M(q)\dot{v}dt + F(q,v)dt = fdt \qquad avec\ f = \frac{d\iota}{dt} \tag{8}$$

or, equivalently,

$$M(q)\dot{v}^+ + F(q,v^+) \quad = \quad f^+ \quad [dt - a.e.] \tag{9}$$

Second order Moreau's sweeping process

Définition (Moreau [1983, 1988])

The keystone of the formulation is the inclusion of measurements in terms of speed:

$$\begin{cases} M(q)dv + F(t,q,v^+)dt = \iota \\ v^+ = \dot{q}^+ \\ -\iota \in N_{T_C(q)}(v^+) \end{cases} \tag{10}$$

Comments

An inclusion that involves measures

A single framework for non-smooth dynamics with inelastic impacts.

➜ Foundations of the numerical scheme of Moreau-Jean

Second order Moreau's sweeping process

Newton-Moreau impact law

$$-\iota \in N_{T_C(q(t))}(v^+ + ev^-) \tag{11}$$

where e is the coefficient of restitution ($v^+ = -ev^-$)

Moreau's viability lemma

$$\begin{array}{c} 0 \leq g_{\mathrm{N}} \perp r_{\mathrm{N}} \geq 0 \\ \Updownarrow \\ -\mathbf{r}_{\mathrm{N}} \in N_{\mathbb{R}^+}(g_{\mathrm{N}}) \\ \Uparrow \\ -\mathbf{r}_{\mathrm{N}} \in N_{T_{\mathbb{R}^+}(g_{\mathrm{N}})}(u_{\mathrm{N}}) \\ \Updownarrow \\ \text{if } g_{\mathrm{N}} \leq 0 \text{ then } 0 \leq u_{\mathrm{N}} \perp r_{\mathrm{N}} \geq 0 \end{array} \tag{12}$$

Principles of event–capturing schemes

1. A unified formulation

$$\begin{cases} -mdv + fdt = \iota \\ \dot{q} = v^+ \\ 0 \leq \iota \perp v^+ \geq 0 \text{ si } q \leq 0 \end{cases} \tag{13}$$

2. A consistent integration

$$\int_{]t_k,t_{k+1}]} mdv = \int_{]t_k,t_{k+1}]} m\,dv = m(v^+(t_{k+1}) - v^+(t_k)) \approx m(v_{k+1} - v_k) \tag{14}$$

3. An consistent approximation with the measure differential inclusion-measure

$$0 \leq \iota \perp v^+ \geq 0 \text{ si } q \leq 0 \quad \rightarrow \quad \begin{cases} p_{k+1} \approx \int_{]t_k,t_{k+1}]} \iota \\ \\ 0 \leq p_{k+1} \perp v_{k+1} \geq 0 \quad \text{if } \tilde{q}_k \leq 0 \end{cases} \tag{15}$$

Moreau-Jean's scheme

[Jean and Moreau, 1987; Moreau, 1988; Jean, 1999]

$$\begin{cases} M(q_{k+\theta})(v_{k+1} - v_k) - hF_{k+\theta} = p_{k+1} = G(q_{k+\theta})P_{k+1}, \\ q_{k+1} = q_k + hv_{k+\theta}, \\ u_{k+1} = G^T(q_{k+\theta})\, v_{k+1} \\ \begin{array}{ll} 0 \leq u^\alpha_{k+1} + eU^\alpha_k \perp P^\alpha_{k+1} \geq 0 & \text{if} \quad \bar{g}^\alpha_{k,\gamma} \leq 0 \\ P^\alpha_{k+1} = 0 & \text{otherwise} \end{array} \end{cases} \tag{16}$$

with

- $G(q) = \nabla_q g(q)$
- $\theta \in [0, 1]$
- $x_{k+\theta} = (1 - \theta)x_{k+1} + \theta x_k$
- $F_{k+\theta} = F(t_{k+\theta}, q_{k+\theta}, v_{k+\theta})$
- $\bar{g}_{k,\gamma} = g_k + \gamma h U_k, , \gamma \geq 0$

An optimization problem is solved at each time–step.

Moreau-Jean's scheme

Advantages

a consistent and stable scheme that is robust
that satisfies some invariants in discrete time:
equilibrium, energy, dissipation, ...

Recent improvements

- Nonsmooth generalized–α schemes [Chen et al., 2013; Brüls et al., 2014]
- Time discontinuous Galerkin methods [Schindler and Acary, 2013; Schindler et al., 2015]
- Stabilized index-2 formulation [Acary, 2014, 2013]
- Stabilized index-1 formulation [Brüls et al., 2018]
- Discrete variational integrators, geometric and symplectic properties [Capobianco and R. Eugster, 2016; Capobianco and Eugster, 2018]

Introduction & illustrations

One–sided and nonsmooth mechanics

Nonsmooth Dynamics

Principles of event–capturing time-stepping schemes

Natural gravity-driven risks

- Louis Guillet PhD: MPM, plasticity and contact for debris flows
- Mattéo Oziol PhD: Smooth and nonsmoth DEM, validation of $\mu(I)$ model on obstacles
- Florian Vincent PhD: Thermodynamical admissible neural networks for granular materials
- Chloé Gergely PhD: Instability phenomena linked to warming in ice–filled permafrost rock

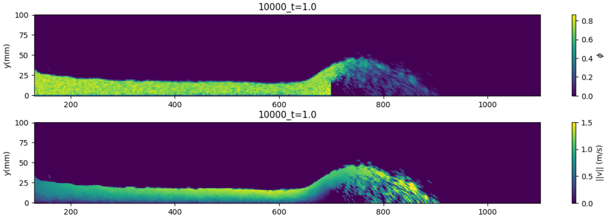
Conclusions

Louis Guillet PhD:
MPM, plasticity and contact for debris flows

Objectives

- Simulation of landslides and debris flows (elasto-plastic fluids + rocks + debris)
- Non-associative plasticity (Drucker-Prager, Mohr Coulomb) with controlled dilatency
- Contact, impact and Coulomb's friction (non-associated)
- Transition from instability to flows
- Monolithic solver based on (non-monotone) variational inequalities and complementarity problems: semi-smooth Newton methods, interior point methods, first-order accelerated methods
- Existence, convergence, energy consistency
- Software code

Mattéo Oziol PhD:
Smooth and nonsmoth DEM, validation of $\mu(I)$ model on obstacles



Objectives

- Modeling of heavy, wet and dense avalanches by DEM
- Impact on obstacles (protective structures) as an continuum media by FEM
- Comparison of smooth DEM (Yade) and nonsmooth DEM (Siconos): computation of stresses, strains, velocity profiles, porosity, forces on obstacles.
- Validation of $\mu(I)$, $\phi(I)$ model in the flow, near the obstacle (dead zone) and in the gaseous jets.
- 3D Simulation on real case obstacles.

Florian Vincent PhD:
Thermodynamical admissible neural networks for granular materials

Objectives

- Application of SciML to constitutive modeling of materials.
- Granular materials: non-associated plasticity, localisation of strains, transition solids/fluids/gas
- Constitutive modeling is difficult ans still open
- Use of data (experimental (X-Ray) and synthetic) trough machine learning.
- NN that respect thermodynamics principles to learn models, rather than computing solutions
- Upscaling of reliable granular models to the scale of mountains by FEM or MPM

Chloé Gergely PhD
Instability phenomena linked to warming in ice–filled permafrost rock

Rockfall at Mel de la Niva. Evolène, Switzerland, October 18, 2015.

Objectives

- Run-out of the granular flows with fragmentation.
- Particle breakage modeling cohesive zone model(CZM) and with unilateral contact and friction
- Characterize the shape and the volume of the "big blocks" after fragmentation
- understanding and quantifying the effect of temperature on the stability of permafrost rock mass
- Extrinsic CZM models taking into account the effect of heat (and temperature) on the mechanical properties of interface
- Coupling with heat equation in the rock mass
- Understanding whether other phenomena need to be added (freeze/thaw cycle, water flow and porous media, ...)

Conclusions

Nonsmooth dynamics a framework :

- to model one-sided and threshold effects,
- to give a rigorous mathematical setting prone to results, and
- to enable the design of powerful numerical tools,

with relevant application to gravity flows.

Thank you for your attention

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