A monolithic computational method for elasto-dynamics with plasticity and contact based on variational approach.

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Motivations and Objectives

Elasto-dynamics with plasticity and unilateral contact

1. A single differential variational inequality
   - pioneering work of J.J. Moreau (1974), Halphen & Nguyen (1975)
   - multi-criteria elasto-plastic flow rules with contact constraints.
2. A nonsmooth dynamical framework for finite-dimensional systems
   - dealing with jumps in velocities and impulsive forces (FEM discretized)
3. A Moreau-Jean type time–stepping method,
   - enabling the consistent integration of the nonsmooth dynamics.
4. A discrete energy balance
   - a practically stable scheme with positive dissipation.
5. Variational approach:
   - formulation of a saddle point problem (min-max problem) and a convex quadratic program
   - well-posedness results (existence and uniqueness)
   - numerical optimization methods as an alternative to return-mapping algorithm

Plasticity in the generalized standard materials framework

Simplest framework: Small perturbation, associative plasticity and linear hardening

- Small perturbations hypothesis with additive decomposition of the strain

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p. \tag{1}$$

- Linear elasticity and hardening laws

$$\boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\varepsilon}^e \quad \text{and } \boldsymbol{a} = -\mathbf{D} \cdot \boldsymbol{\alpha}. \tag{2}$$

- Generalized standard material (GSM) (associative plasticity)

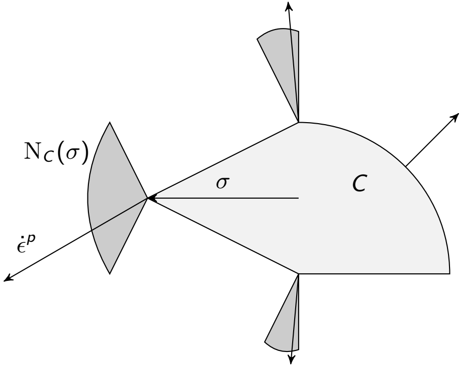
$$\begin{pmatrix} \dot{\boldsymbol{\varepsilon}}^p \\ \dot{\boldsymbol{\alpha}} \end{pmatrix} \in \mathrm{N}_{\boldsymbol{C}} \begin{pmatrix} \boldsymbol{\sigma} \\ \boldsymbol{a} \end{pmatrix}. \tag{3}$$

$\boldsymbol{C}(\boldsymbol{\sigma}, \boldsymbol{a})$ a convex set of admissible stresses $\boldsymbol{\sigma}$ and hardening forces $\boldsymbol{a}$

- Clausius-Duhem dissipation inequality is automatically satisfied

$$d = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p + \boldsymbol{a} \cdot \dot{\boldsymbol{\alpha}} \geqslant 0 \text{ if } 0 \in C. \tag{4}$$

Plasticity in the generalized standard materials framework



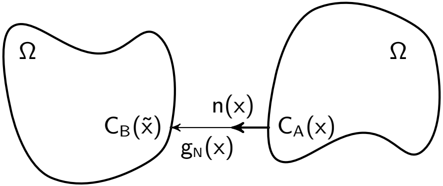
$$C = \{(\sigma, a), f(\sigma, a) \geqslant 0\}$$

$$\begin{pmatrix} \dot{\varepsilon}^p \\ \dot{\alpha} \end{pmatrix} \in \mathrm{N}_{\boldsymbol{C}} \begin{pmatrix} \sigma \\ a \end{pmatrix} \iff \begin{cases} \dot{\varepsilon}^p = -\nabla_\sigma^\top f(\sigma, a)\lambda \\ \dot{\alpha} = -\nabla_a^\top f(\sigma, a)\lambda \\ 0 \leqslant \lambda \perp f(\sigma, a) \geqslant 0, \end{cases}$$

$$\Updownarrow$$

$$\begin{pmatrix} \sigma \\ a \end{pmatrix} = \mathrm{proj}_C \left(\begin{pmatrix} \sigma \\ a \end{pmatrix} - \rho \begin{pmatrix} \dot{\varepsilon}^p \\ \dot{\alpha} \end{pmatrix} \right), \rho > 0 \tag{5}$$

Unilateral contact



$g_N(x)$: gap function
$v_N(x) = \dfrac{d}{dt} g_N(x)$: relative velocity

- Signorini contact law:

$$0 \leqslant g_N \perp r_N \geqslant 0 \Longleftrightarrow -r_N \in \mathrm{N}_{\mathbb{R}_+}(g_N). \tag{6}$$

- Signorini contact law at the velocity level:

$$\begin{array}{c} 0 \leqslant r_N \perp v_N \geqslant 0 \text{ if } g_N = 0, \text{ else } r_N = 0 \\ \Updownarrow \\ -r_N \in N_{T_{\mathbb{R}_+}(g_N)}(v_N), \end{array} \tag{7}$$

where $T_{\mathbb{R}_+}(g_N)$ is the tangent cone of $\mathbb{R}_+$ at g_N

Space discretization

Standard FEM discretization to make is as simple as possible.

Finite dimensional smooth linear dynamics:

$$M\dot{v}(t) + B^T\sigma(t) = f_{\rm ext}(t) + r(t). \tag{8}$$

- u, v nodal displacement and velocity vector,
- σ stress at Gauss points,
- M constant mass matrix,
- B^T discrete divergence operator, B is the discrete gradient.

Elasto-plastic relation at Gauss points and contact kinematics

$$\begin{cases} \sigma = E\varepsilon^e = E(\varepsilon - \varepsilon^p) \\ a = -D\alpha \\ \begin{pmatrix} \dot{\varepsilon}^p \\ \dot{\alpha} \end{pmatrix} \in N_C \begin{pmatrix} \sigma \\ a \end{pmatrix}, \end{cases} \tag{9}$$

$$\begin{aligned} v_{\rm N} &= H^\top(u)v \\ r &= H(u)r_{\rm N}. \end{aligned} \tag{10}$$

Finite dimensional systems and nonsmooth dynamics

Measure differential equation

Ability to deal with velocity jumps and impulsive forces in discrete systems

$$M\mathrm{d}v + B^\top \sigma(t)\mathrm{d}t = f_{\mathrm{ext}}(t)\mathrm{d}t + H(u(t))\mathrm{d}i_{\mathsf{N}}. \tag{11}$$

- $\mathrm{d}v$ differential measure ("acceleration as measure")
- $\mathrm{d}i_{\mathsf{N}}$ contact reaction measure

Unilateral contact and Newton impact law

$$-\mathrm{d}i_{\mathsf{N}} \in \mathrm{N}_{T_{\mathbb{R}^m_+}(g_{\mathsf{N}}(t))}(v_{\mathsf{N}}(t) + e v_{\mathsf{N}}^-(t)). \tag{12}$$

where e is the coefficient of restitution

Standard FEM discretization

After the differentiation of the constitutive law with $S = E^{-1}$(similar to incremental formulation) and the introduction of slack variable

$$y = -\dot{\alpha} \text{ and } z = -\dot{\varepsilon}^{p}$$

we get a differential measure variational inequality

$$\begin{cases} M\mathrm{d}v + B^{\top}\sigma(t)\mathrm{d}t = f_{\mathrm{ext}}(t)\mathrm{d}t + H(u(t))\mathrm{d}i_{\mathrm{N}} \\ \dot{u}(t) = v(t) \\ S\dot{\sigma}(t) = Bv(t) + z(t) \\ D^{-1}\dot{a}(t) = y(t) \\ v_{\mathrm{N}}(t) = H^{\top}(u(t))v(t) \\ \\ -\begin{pmatrix} z(t) \\ y(t) \\ \mathrm{d}i_{\mathrm{N}} \end{pmatrix} \in \mathrm{N}_{C\times T_{\mathbb{R}^m_+}(g_{\mathrm{N}}(t))}\begin{pmatrix} \sigma(t) \\ a(t) \\ (v_{\mathrm{N}}(t) + ev_{\mathrm{N}}^{-}(t)) \end{pmatrix}. \end{cases} \tag{13}$$

A single variational inequality.

Time stepping scheme

Extension of the Moreau–Jean scheme with $\theta \in (0,1]$

$$
\begin{cases}
M(v_{k+1} - v_k) + hB^\top \sigma_{k+\theta} = hf_{\mathrm{ext},k+\theta} + Hp_{\mathrm{N},k+1} \\
S(\sigma_{k+1} - \sigma_k) - hBv_{k+\theta} = hz_{k+\theta}, \\
D^{-1}(a_{k+1} - a_k) = hy_{k+\theta} \\
v_{\mathrm{N},k+1} = H^\top v_{k+1} \\
\\
-\begin{pmatrix} z_{k+\theta} \\ y_{k+\theta} \\ p_{\mathrm{N},k+1} \end{pmatrix} \in \mathrm{N}_{C \times \mathbb{R}^m_+} \begin{pmatrix} \sigma_{k+\theta} \\ a_{k+\theta} \\ v_{\mathrm{N},k+1} + ev_{\mathrm{N},k} \end{pmatrix}
\end{cases}
\tag{14}
$$

where

- notation: h time step, $x_{k+\theta} = \theta x_{k+1} + (1-\theta)x_k$
- the Signorini condition only on the active contact ($g_{\mathrm{N},k} \leqslant 0$)
- displacements are updated afterwards $u_{k+1} = u_k + hv_{k+\theta}$
- impulses $p_{\mathrm{N},k+1}$ as primary variable
- elasto-plastic law with hardening is solved at $k+\theta$.

Variational approach: saddle point problem

Proposition (Saddle point problem)

The solutions of $(v, \dot{\varepsilon}, \sigma, a, v_N)$ *of the first order optimality conditions of*

$$
\begin{array}{ll}
\min_{v,\dot{\varepsilon}} \max_{\sigma,a} & \frac{1}{2}(v - v_k)^\top M(v - v_k) \\
& \quad -\frac{1}{2}(\sigma - \sigma_k)^\top S(\sigma - \sigma_k) - \frac{1}{2}(a - a_k)^\top D^{-1}(a - a_k) \\
& \qquad + h\theta\sigma^\top \dot{\varepsilon} - h\theta f_{\mathrm{ext},k+1}^\top v \\
\\
\text{s.t.} & Bv = \dot{\varepsilon} \\
& \theta v_N = H^\top v - (1-\theta) v_{N,k} \\
& \begin{pmatrix} \sigma \\ a \\ v_N + e v_{N,k} \end{pmatrix} \in C \times \mathbb{R}^m_+.
\end{array}
\tag{15}
$$

are solutions of (14) *for* $(v_{k+\theta}, z_{k+\theta}, \sigma_{k+\theta}, a_{k+\theta}, v_{N,k+1})$.

- A kind of discrete ∂'Alembert principle for elastoplasticity with contact.

Variational approach: saddle point problem

Assumption (1)

The matrices M, S and D are symmetric definite positive matrices.

Assumption (2)

It exists v^0, σ^0, a^0 such that

$$\left\{\begin{array}{lr} \begin{pmatrix}\sigma^0 \\ a^0\end{pmatrix} \in C & 0 \textit{ is at least in } C \\ \\ H^\top v^0 + (\theta(1+e) - 1) v_{N,k} \geqslant 0 & \textit{standard feasibility condition} \end{array}\right. \tag{16}$$

Proposition

Under Assumptions 1 and 2, the saddle-point problem (15) *has a unique solution* $(v, \dot{\varepsilon}, \sigma, a, v_N)$.

Variational approach: convex quadratic problem

Substitution of v_{k+1} in the linear equations in (14)

Reduction to local variables.

$$-\left(Q\begin{pmatrix}\sigma_{k+\theta}\\ a_{k+\theta}\\ p_{\mathrm{N},k+1}\end{pmatrix}+p\right)\in N_{C\times\mathbb{R}^m_+}\begin{pmatrix}\sigma_{k+\theta}\\ a_{k+\theta}\\ p_{\mathrm{N},k+1}\end{pmatrix}, \tag{17}$$

with

$$Q=\begin{pmatrix}U & 0 & -V\\ 0 & D^{-1} & 0\\ -V^\top & 0 & W\end{pmatrix} \text{ and } p=\begin{pmatrix}s\\ D^{-1}a_k\\ r\end{pmatrix}. \tag{18}$$

and

$$\begin{aligned}
W &= \theta^2 H^\top M^{-1} H && \text{Delassus matrix}\\
U &= S + h^2\theta^2 B M^{-1} B^\top\\
V^\top &= h\theta^2 H^\top M^{-1} B^\top\\
s &= -S\sigma_k - h\theta B\left(v_k+\theta h M^{-1} f_{\mathrm{ext},k+\theta}\right)\\
r &= \theta^2\left(e v_{\mathrm{N},k} + H\left(v_k+\theta M^{-1}\left(h f_{\mathrm{ext},k+\theta}\right)\right)\right)
\end{aligned} \tag{19}$$

Variational approach: convex quadratic problem

Equivalent convex minimization problem:

Equivalent convex minimization problem:

$$
\begin{aligned}
\min_{\sigma, a, p_{\mathrm{N}}} \quad & \frac{1}{2}\begin{pmatrix}\sigma\\ a\\ p_{\mathrm{N}}\end{pmatrix}^{\top} Q \begin{pmatrix}\sigma\\ a\\ p_{\mathrm{N}}\end{pmatrix} + p^{\top}\begin{pmatrix}\sigma\\ a\\ p_{\mathrm{N}}\end{pmatrix} \\
\text{s.t.} \quad & \begin{pmatrix}\sigma\\ a\\ p_{\mathrm{N}}\end{pmatrix} \in C \times \mathbb{R}^m_+.
\end{aligned}
\tag{20}
$$

Assumption (3)

The matrix H has full rank.

Proposition

Under Assumptions 1 and 3 and for a sufficiently small time step, the problem (20) *has a unique solution* (σ, a, p_N) *if the set* $0 \in C$.

Convex Quadratic problem

C finitely represented

$$C = \{(\sigma, a) \mid f(\sigma, a) \geqslant 0\}, \tag{21}$$

f is a smooth vector-valued function with non-vanishing gradients

$$\begin{aligned} \min_{\sigma, a, p_{\mathrm{N}}} \quad & \frac{1}{2} \begin{pmatrix} \sigma \\ p_{\mathrm{N}} \\ a \end{pmatrix}^{\top} Q \begin{pmatrix} \sigma \\ p_{\mathrm{N}} \\ a \end{pmatrix} + p^{\top} \begin{pmatrix} \sigma \\ p_{\mathrm{N}} \\ a \end{pmatrix} \\ \text{s.t.} \quad & f(\sigma, a) \geqslant 0 \\ & p_{\mathrm{N}} \geqslant 0 \end{aligned} \tag{22}$$

Numerical methods from optimization

- Direct pivoting techniques (Active set, Lemke, PATH)
  Direct solution if C is polyhedral
- First order (Nesterov) accelerated techniques (ADMM, APGD, ...)
- Second order methods:
  1. interior point methods
  2. semi-smooth Newton method

$$0 = \begin{pmatrix} \sigma \\ a \\ p_{\mathrm{N}} \end{pmatrix} - \mathrm{proj}_{C \times \mathbb{R}^m_+} \left[\begin{pmatrix} \sigma \\ a \\ p_{\mathrm{N}} \end{pmatrix} - \rho \left(Q \begin{pmatrix} \sigma \\ a \\ p_{\mathrm{N}} \end{pmatrix} + p \right) \right], \quad \rho > 0$$

Comments on semi-smooth Newton methods

Similarity with return-mapping algorithm [Hager, Wolhmuth (2009), Christensen(2009)]
but with more flexibility:

- optimal choice of ρ with self-adaptation (proximal technique)
- rescaling technique are easier
- no explicit need of a consistent tangent operator

Discrete energy balance

- Energy balance for discrete systems:

$$\mathrm{dE}(t) = \mathrm{d}(\mathrm{T}(t) + \Psi(t)) = -\mathrm{D}(t)\mathrm{d}t + \mathrm{P}_{\mathrm{ext}}(t)\mathrm{d}t + \mathrm{dP}_{\mathrm{impact}}, \tag{23}$$

with

$$\mathrm{T}(t) = \frac{1}{2}v^\top(t)Mv(t), \quad \text{and } \Psi(t) = \frac{1}{2}\varepsilon^{e\top}(t)E\varepsilon^{e}(t) + \frac{1}{2}\alpha^\top(t)D\alpha(t) \tag{24}$$

- The dissipation and the power of external forces are

$$\mathrm{D}(t) = \sigma^\top(t)\dot{\varepsilon}^p(t) + a^\top(t)\dot{\alpha}(t) \text{ and } \mathrm{P}_{\mathrm{ext}}(t) = f_{\mathrm{ext}}^\top(t)v(t). \tag{25}$$

- The power of the reaction impulse is given by

$$\mathrm{dP}_{\mathrm{impact}} = \frac{1}{2}(v_{\mathrm{N}}^+ + v_{\mathrm{N}}^-)\mathrm{d}i_{\mathrm{N}}. \tag{26}$$

Discrete energy balance

Approximation of works by the θ-method as

$$\mathrm{W}_{\mathrm{ext}}{}_{k}^{k+1} := h v_{k+\theta}^{\top} f_{\mathrm{ext},k+\theta} \approx \int_{t_k}^{t_{k+1}} \mathrm{P}_{\mathrm{ext}}(t)\mathrm{d}t, \tag{27}$$

and

$$\mathrm{W}_{p}{}_{k}^{k+1} := h\sigma_{k+\theta}^{\top}\dot{\varepsilon}_{k+\theta}^{p} - h a_{k+\theta}^{\top} y_{k+\theta} \approx \int_{t_k}^{t_{k+1}} \mathrm{D}(t)\mathrm{d}t, \tag{28}$$

Approximation of the work dissipated by the percussion

$$\mathrm{W}_{c}{}_{k}^{k+1} := v_{\mathrm{N},k+\theta}^{\top} p_{\mathrm{N},k+1} = (1-\theta(1+e))v_{\mathrm{N},k}^{\top} p_{\mathrm{N},k+1}, \tag{29}$$

the increment of total energy is then given by

$$\begin{aligned}\Delta\mathrm{E}_{k}^{k+1} - \mathrm{W}_{\mathrm{ext}}{}_{k}^{k+1} + \mathrm{W}_{p}{}_{k}^{k+1} - \mathrm{W}_{c}{}_{k}^{k+1} = \\ \left(\tfrac{1}{2}-\theta\right)\left(\|v_{k+1}-v_k\|_M^2 + \|\varepsilon_{k+1}^{e}-\varepsilon_{k}^{e}\|_E^2 + \|\alpha_{k+1}-\alpha_k\|_D^2\right).\end{aligned} \tag{30}$$

Discrete energy balance

Proposition

Under Assumption 1, energy dissipation of the scheme is as follows

1. *When $\theta = \frac{1}{2}$, the time–stepping scheme satisfies the approximation of the discrete energy balance :*

$$\Delta \mathsf{E}_k^{k+1} - \mathsf{W}_{\mathrm{ext}}{}_k^{k+1} = -\mathsf{W}_p{}_k^{k+1} + \mathsf{W}_c{}_k^{k+1}. \tag{31}$$

2. *The dissipated work due to plasticity is always positive*
3. *When $\theta \leqslant \frac{1}{1+e}$, the dissipated work due to impact is also positive.*
4. *When $\frac{1}{2} \leqslant \theta \leqslant \frac{1}{1+e} \leqslant 1$, we have the following dissipation inequality*

$$\Delta \mathsf{E}_k^{k+1} - \mathsf{W}_{\mathrm{ext}}{}_k^{k+1} \leqslant 0. \tag{32}$$

Conclusions & perspectives

Conclusions

A monolithic solver for elastodynamics with contact, impact and plasticity:

- A practically stable scheme with a discrete energy balance
- A variational formulation (optimization problem) of plasticity with contact
- A gateway to a host of multiple optimisation algorithms
- Useful also for quasi-static application, even with perfect plasticity

Vincent Acary, Franck Bourrier, Benoit Viano.
Variational approach for nonsmooth elasto-plastic dynamics with contact and impacts. 2023.
⟨hal-03978387v1⟩ to appear in Computer Methods in Applied Mechanics and Engineering

Perspectives

- Non associated plasticity and Coulomb friction
  - Implicit standard materials (De Saxcé) ➜ ~~quasi~~-variational inequality
- Finite strain plasticity and mortar method ([Seitz, Popp, Wall (2015)])
- Material point method (PhD Louis Guillet)
  - Application to gravity-driven flows of geomaterials in mountains (mud and debris flows).

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