The nonsmooth dynamics framework for the analysis and simulation of multi-body systems.

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Objectives of this lecture

Workshop/school ... a lecture between a course and a research talk.

- Formulation of nonsmooth dynamical systems
- Basics on Mathematical properties
- Formulation of unilateral contact, Coulomb's friction and impacts.
- Flavor of enhanced nonsmooth laws
- Principles and Design of Event–capturing (Time–stepping) schemes.
- Newmark-type schemes for flexible multibody systems and FEM applications.

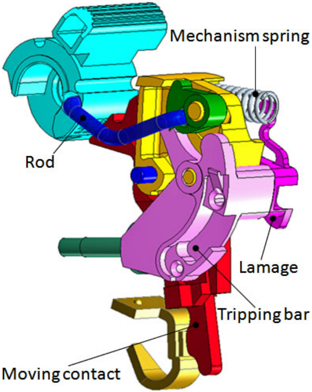
Strengths and advantages of the nonsmooth approach

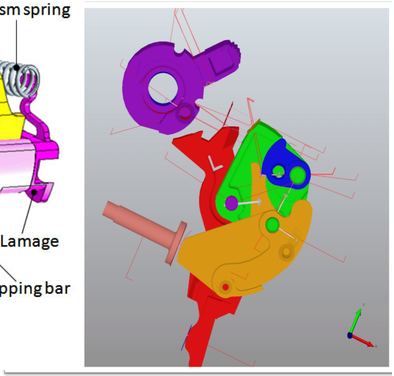
Nonsmooth approach is crucial for an efficient and robust simulation

- Compliant contacts imply stiff problems. Nonsmooth approach removes numerical stiffness.
  ➜ Remove (not needed) artificial stiffness and damping at contact.
  ➜ Time integrators can use large time-steps and are robust
- Small number of parameters at contacts
  ➜ Facilitate the parameter identification
- Deal with large number of nonsmooth events and contacts (accumulation of impacts, clearances or granular material)
  ➜ Time integrators do not stop on events
  ➜ Deal with large systems using optimization techniques
- Quality of solutions:
  - Threshold effect and inequality are strictly modeled.
    ➜ Users can have digital diagnostics on discrete status of variables (contact/no contact, sliding/sticking, on/off).
  - Possibility to play with the trade-off accuracy/performance for high-level and optimization design.

Multi-body systems

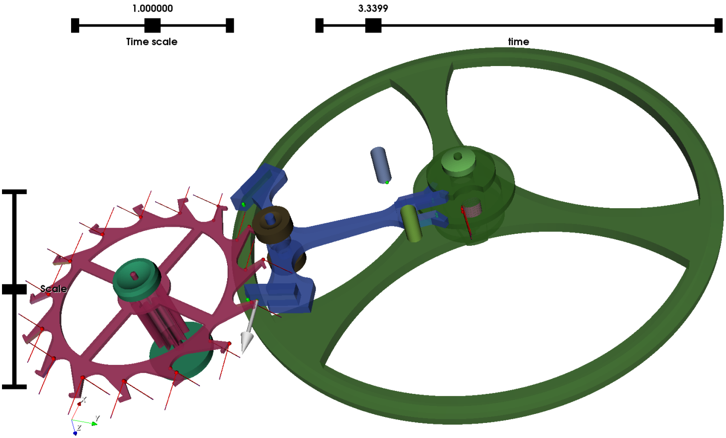
Siconos simulation of circuits breakers with clearances in joints





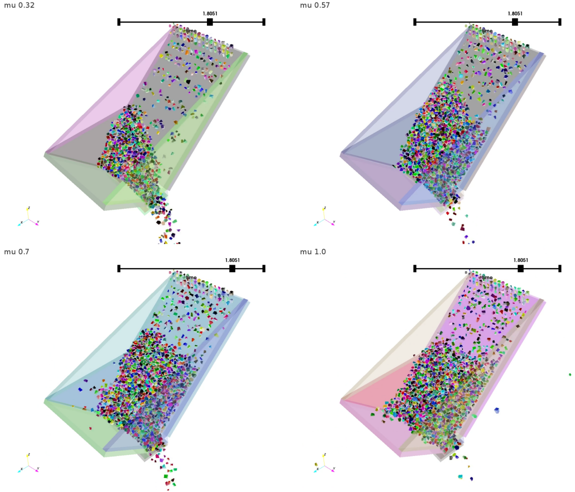
Multi-body systems

Siconos simulation of watch escapement mechanism.



Multi-body systems

Siconos simulation of granular flows for mines applications.



Constrained Smooth Lagrangian Dynamics

Smooth multibody dynamics

Definition (Equations of motion)

$$\begin{cases} M(q(t))\dfrac{dv(t)}{dt} + F(t,q,v) = 0, \\ v(t) = \dot{q}(t) \end{cases} \tag{1}$$

where

- $F(t,q,v) = N(q,v) + F_{int}(t,q,v) - F_{ext}(t)$

Definition (Boundary conditions)

- Initial Value Problem (IVP):

$$t_0 \in \mathbb{R}, \quad q(t_0) = q_0 \in \mathbb{R}^n, \quad v(t_0) = v_0 \in \mathbb{R}^n, \tag{2}$$

- Boundary Value Problem (BVP):

$$(t_0, T) \in \mathbb{R} \times \mathbb{R}, \quad \Gamma(q(t_0), v(t_0), q(T), v(T)) = 0 \tag{3}$$

Perfect bilateral constraints, joints, liaisons and spatial boundary conditions

Bilateral constraints

- Finite set of m bilateral constraints on the generalized coordinates :

$$h(q,t) = \left[h_j(q,t) = 0, \quad j \in \{1 \dots m\}\right]^T. \tag{4}$$

where h_j are sufficiently smooth with regular gradients, $\nabla_q(h_j)$.

- Configuration manifold, $\mathcal{M}(t)$

$$\mathcal{M}(t) = \{q(t) \in \mathbb{R}^n, h(q,t) = 0\} \subset \mathbb{R}^n, \tag{5}$$

Tangent and normal space

- Tangent space to the manifold $\mathcal{M}$ at q

$$T_{\mathcal{M}}(q) = \{\xi \mid \nabla h(q)^T \xi = 0\} \tag{6}$$

- Normal space as the orthogonal to the tangent space

$$N_{\mathcal{M}}(q) = \{\eta \mid \eta^T \xi = 0, \forall \xi \in T_{\mathcal{M}}\} \tag{7}$$

Bilateral constraints as inclusion

Definition (Perfect bilateral holonomic constraints on the smooth dynamics)

$$\begin{cases} \dot{q} = v \\ M(q)\dfrac{dv}{dt} + F(q,v) = r \\ -r \in N_{\mathcal{M}}(q) \end{cases} \tag{8}$$

where r is the generalized force or generalized reaction due to the constraints.

Remark

- The formulation as an inclusion is very useful in practice
- The constraints are said to be perfect due to the normality condition.

Bilateral constraints as inclusion

Lagrange multipliers

When the manifold is defined by smooth constraints

$$\mathcal{M} = \{q(t) \in \mathbb{R}^n, h(q,t) = 0\}$$

with some constraints qualification, the multipliers $\mu \in \mathbb{R}^m$ can be introduced and we get

$$r = \nabla_q h(q,t)\,\mu.$$

The equations of motion are

$$\begin{cases} \dot{q} = v \\ M(q)\dfrac{dv}{dt} + F(q,v) = \nabla_q h(q,t)\,\mu \\ h(q,t) = 0, \quad \mu \end{cases} \tag{8}$$

Perfect unilateral constraints

Unilateral constraints

- Finite set of ν unilateral constraints on the generalized coordinates :

$$g(q,t) = [g_\alpha(q,t) \geqslant 0, \quad \alpha \in \{1\ldots\nu\}]^T . \tag{9}$$

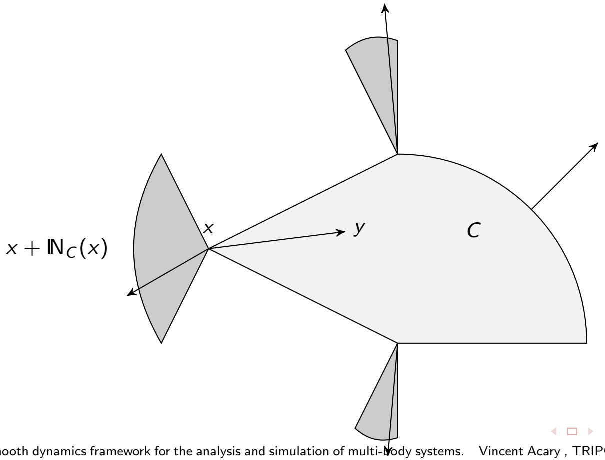
- Admissible set $\mathcal{C}(t)$

$$\mathcal{C}(t) = \{q \in \mathbb{R}^n, g_\alpha(q,t) \geqslant 0, \alpha \in \{1\ldots\nu\}\} . \tag{10}$$

Unilateral constraints as an inclusion

$\mathcal{C}(t)$ a closed convex set

$$N_{\mathcal{C}(t)}(q(t)) = \left\{ s \in \mathbb{R}^n \mid s^\top (y - q(t)) \leqslant 0 \text{ for all } y \in \mathcal{C}(t) \right\} \tag{11}$$



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Definition (Perfect unilateral constraints on the smooth dynamics)

$$\begin{cases} \dot{q} = v \\ \\ M(q)\dfrac{dv}{dt} + F(t,q,v) = r \\ \\ -r \in N_{\mathcal{C}(t)}(q(t)) \end{cases} \tag{12}$$

where r it the generalized force or generalized reaction due to the constraints.

Remark

- The unilateral constraints are said to be perfect due to the normality condition.
- Notion of normal cones can be extended to more general sets. see (Clarke, 1975, 1983 ; Mordukhovich, 1994)

Unilateral constraints as an inclusion

Normal cone to $\mathcal{C}(t)$ finitely represented

Under qualification conditions, we have

$$N_{\mathcal{C}(t)}(q(t)) = \left\{ y \in \mathbb{R}^n, y = -\sum_{\alpha} \lambda_\alpha \nabla g_\alpha(q,t),\ \lambda_\alpha \geqslant 0,\ \lambda_\alpha g_\alpha(q,t) = 0 \right\} \tag{11}$$

Equations of motion

$$\begin{cases} \dot{q} = v \\ M(q)\dfrac{dv}{dt} + F(t,q,v) = \nabla g(q,t)\lambda \\ 0 \leqslant g(q,t) \perp \lambda \geqslant 0 \end{cases} \tag{12}$$

Notation (Complementarity)

$$0 \leqslant x \perp y \geqslant 0 \Longleftrightarrow x \geqslant 0, y \geqslant 0, x^\top y = 0 \tag{13}$$

Smooth dynamics as a DI

Differential Inclusion

$$-\left[M(q)\frac{dv}{dt} + F(t,q,v)\right] \in N_{\mathcal{C}(t)}(q(t)), \tag{14}$$

with

$$\dot{q} = v.$$

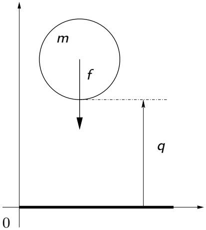
Remark

- The right hand side is neither bounded (and then nor compact).
- The inclusion and the constraints concern the second order time derivative of q.

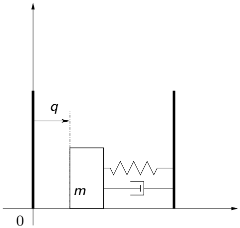
➜ Standard Analysis of DI does no longer apply.

Academic examples

The bouncing Ball and the linear impacting oscillator



(a) Bouncing ball example



(b) Linear Oscillator example

Figure: Academic test examples with analytical solutions

NonSmooth Multibody Systems (NSMBS)

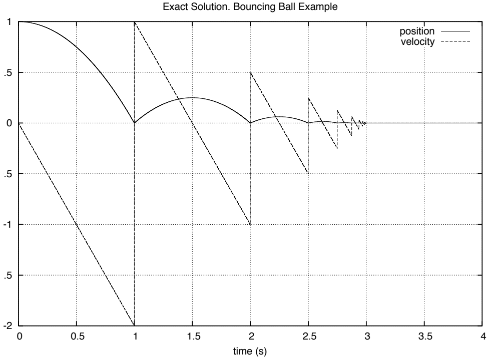


Figure: Analytical solution. Bouncing ball example

NonSmooth Multibody Systems (NSMBS)

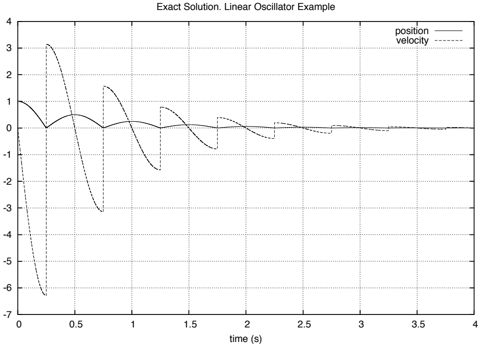


Figure: Analytical solution. Linear Oscillator

Nonsmooth Lagrangian Dynamics

Nonsmooth Lagrangian Dynamics

Fundamental assumptions.

- The velocity $v = \dot{q}$ is of Bounded Variations (B.V)
  ➜ The equation are written in terms of a right continuous B.V. (R.C.B.V.) function, v^+ such that
$$v^+ = \dot{q}^+ \tag{15}$$
- q is related to this velocity by
$$q(t) = q(t_0) + \int_{t_0}^{t} v^+(t)\,dt \tag{16}$$
- The acceleration, ($\ddot{q}$ in the usual sense) is hence a differential measure dv associated with v such that
$$dv(]a,b]) = \int_{]a,b]} dv = v^+(b) - v^+(a) \tag{17}$$

Nonsmooth Lagrangian Dynamics

Definition (Nonsmooth Lagrangian Dynamics)

$$\begin{cases} M(q)dv + F(t,q,v^+)dt = di \\ \\ v^+ = \dot{q}^+ \end{cases} \tag{18}$$

where di is the reaction measure and dt is the Lebesgue measure.

Remarks

- The nonsmooth Dynamics contains the impact equations and the smooth evolution in a single equation.
- The formulation allows one to take into account very complex behaviors, especially, finite accumulation (Zeno-state).
- This formulation is sound from a mathematical Analysis point of view.

References

(Schatzman, 1973, 1978 ; Moreau, 1983, 1988)

Nonsmooth Lagrangian Dynamics

Measures Decomposition (for dummies)

$$\begin{cases} dv = & \gamma\, dt + & (v^+ - v^-)\, d\nu + & dv_s \\ di = & f\, dt + & p\, d\nu + & di_s \end{cases} \tag{19}$$

where

- $\gamma = \ddot{q}$ is the acceleration defined in the usual sense.
- f is the Lebesgue measurable force,
- $v^+ - v^-$ is the difference between the right continuous and the left continuous functions associated with the B.V. function $v = \dot{q}$,
- $d\nu$ is a purely atomic measure concentrated at the time t_i of discontinuities of v, i.e. where $(v^+ - v^-) \neq 0$,i.e. $d\nu = \sum_i \delta_{t_i}$
- p is the purely atomic impact percussions such that $pd\nu = \sum_i p_i \delta_{t_i}$
- dv_S and di_S are singular measures with the respect to $dt + d\eta$.

Impact equations and Smooth Lagrangian dynamics

Substituting the decomposition of measures into the nonsmooth Lagrangian Dynamics, one obtains

Definition (Impact equations)

$$M(q)(v^+ - v^-)d\nu = pd\nu, \tag{20}$$

or

$$M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \tag{21}$$

Definition (Smooth Dynamics between impacts)

$$M(q)\gamma dt + F(t,q,v)dt = fdt \tag{22}$$

or

$$M(q)\gamma^+ + F(t,q,v^+) \quad = \quad f^+ \quad [dt - a.e.] \tag{23}$$

The Moreau's sweeping process of second order

Definition (Moreau (1983, 1988))

A key stone of this formulation is the inclusion in terms of velocity. Indeed, the inclusion (**??**) is "replaced" by the following inclusion

$$\begin{cases} M(q)dv + F(t,q,v^+)dt = di \\ \\ v^+ = \dot{q}^+ \\ \\ -di \in N_{T_C(q)}(v^+) \end{cases} \tag{24}$$

Comments

This formulation provides a common framework for the nonsmooth dynamics containing inelastic impacts without decomposition.

➜ Foundation for the time–stepping approaches.

The Moreau's sweeping process of second order

Comments

- *The inclusion concerns measures.* Therefore, it is necessary to define what is the inclusion of a measure into a cone.
- *The inclusion in terms of velocity* v^+ rather than of the coordinates q.

Interpretation

- Inclusion of measure, $-di \in K$
  - Case $di = r'dt = fdt$.
$$-f \in K \tag{25}$$
  - Case $di = p_i\delta_i$.
$$-p_i \in K \tag{26}$$
- Inclusion in terms of the velocity. Viability Lemma
  If $q(t_0) \in C(t_0)$, then

$$v^+ \in T_C(q), t \geqslant t_0 \Rightarrow q(t) \in C(t), t \geqslant t_0$$

➜ The unilateral constraints on q are satisfied. The equivalence needs at least an impact inelastic rule.

The Moreau's sweeping process of second order

The Newton-Moreau impact rule

$$- di \in N_{T_C(q(t))}(v^+(t) + ev^-(t)) \qquad (27)$$

where e is a coefficient of restitution.

Velocity level formulation. Index reduction

$$\begin{array}{c} 0 \leqslant y \perp \lambda \geqslant 0 \\ \Updownarrow \\ -\lambda \in N_{\mathbb{R}+}(y) \\ \Uparrow \\ -\lambda \in N_{T_{\mathbb{R}+}(y)}(\dot{y}) \\ \Updownarrow \\ \text{if } y \leqslant 0 \text{ then } 0 \leqslant \dot{y} \perp \lambda \geqslant 0 \end{array} \qquad (28)$$

The Moreau's sweeping process of second order

Summary for perfect schleronomic constraints

$$\begin{cases} M(q)dv + F(t,q,v^+)dt = di \\ v^+ = \dot{q}^+ \\ \\ di = H(q)d\lambda \\ U^+ = H(q)^T v^+ \\ \\ \text{ if } g_\alpha(q) \leqslant 0, \text{ then } 0 \leqslant U^+_\alpha \perp d\lambda_\alpha \geqslant 0 \end{cases} \tag{29}$$

where $H(q)$ is the transpose of the Jacobian matrix of the constraints,

$$H(q) = \nabla_q g(q)$$

.

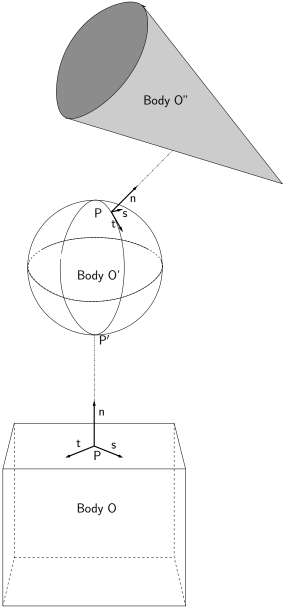
Local coordinates system at contact

Lagrangian approach of constraints is not sufficient.

The elegant Lagrangian approach of unilateral constraints and their associated multipliers is not sufficient for describing more complex behavior of the contact :

- The Lagrange multipliers have no physical dimensions
- The constraints can be multiplied by a positive constant.

For a mechanical description of the behaviour of the contact interface, a (set-valued) force laws needs to be introduced together with a coordinate systems at contact.



Definition of a contact frame

Assume that we have defined

- P and P' proximal points between O and O'
- $\mathbf{n}$ an outward unit normal vector along $P'P$
- $\mathbf{t}$ and $\mathbf{s}$ two unit tangent vectors
- $g(q)$ a gap function, i.e., the signed distance $\overline{P'P}$

Remark

This definition is not trivial for a nonsmooth or nonconvex surfaces.

Local coordinates system at contact

Relative local velocity

The relative local velocity U is defined by

$$U = V_P - V_{P'} \tag{30}$$

and is decomposed in the frame $(P', \mathbf{n}, \mathbf{t}, \mathbf{s})$ as

$$U = U_{\mathrm{N}}\mathbf{n} + U_{\mathrm{T}}, \quad U_{\mathrm{N}} \in \mathbb{R}, U_{\mathrm{T}} \in \mathbb{R}^2 \tag{31}$$

Link with the gap function

The derivative with respect to time of the gap function $t \to g(q(t))$ is the normal relative velocity U_{N}

$$\dot{g}(\cdot) = U_{\mathrm{N}}(\cdot) = \nabla g^T(q)v(\cdot) \tag{32}$$

Local reaction force at contact

The relative local velocity R acts from O' to O and is also decomposed as

$$R = R_{\mathrm{N}}\mathbf{n} + R_{\mathrm{T}}, \quad R_{\mathrm{N}} \in \mathbb{R}, R_{\mathrm{T}} \in \mathbb{R}^2 \tag{33}$$

Local coordinates system at contact

Relations with global/generalized coordinates

Is is assumed that there exists a relation between the local relative velocity U and the velocity of bodies v such that

$$U = H^T(q)v \tag{34}$$

By duality (expressed in terms of power) we get

$$r = H(q)R \tag{35}$$

Unilateral contact in terms of local variables

$$\text{if } g(q) \leqslant 0, \text{ then } 0 \leqslant U_{\mathrm{N}} \perp R_{\mathrm{N}} \geqslant 0 \tag{36}$$

Coulomb's friction

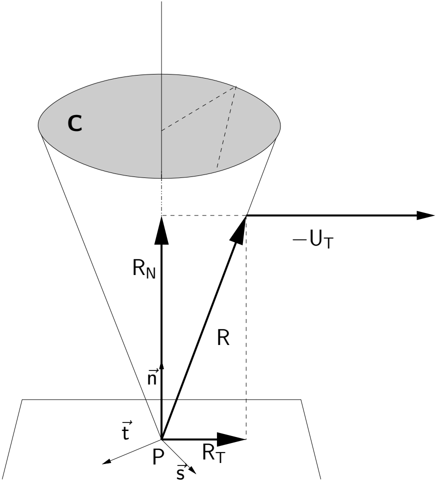


Figure: Coulomb's friction. The sliding case.

Coulomb's friction

Definition (Coulomb's friction)

Coulomb's friction says the following. If $g(q) = 0$ then:

$$\begin{cases} \text{If } U_{\mathrm{T}} = 0 \quad \text{then} \quad R \in \mathbf{C} & \text{(sticking)} \\ \text{If } U_{\mathrm{T}} \neq 0 \quad \text{then} \quad \|U_{\mathrm{T}}\| R_{\mathrm{T}} = -\|R_t\| U_{\mathrm{T}} & \text{(sliding)} \end{cases} \tag{37}$$

where $C = \{R, \|R_{\mathrm{T}}|\| \leqslant \mu |R_{\mathrm{N}}| \ \}$ is the Coulomb friction cone

Coulomb's friction

Definition (Coulomb's friction as an inclusion into a disk)

Let us introduce the following inclusion (Moreau, 1988) using the indicator function $\psi_{\mathbf{D}}(\cdot)$:

$$-U_{\mathrm{T}} \in \mathbb{N}_{\mathbf{D}(\mu R_{\mathrm{N}})}(R_{\mathrm{T}}) \tag{38}$$

where $\mathbf{D}(\mu R_{\mathrm{N}}) = \{R_{\mathrm{T}}, \|R_{\mathrm{T}}(t)\| \leqslant \mu |R_{\mathrm{N}}| \ \}$ is the Coulomb friction disk

Definition (Coulomb's friction as a variational inequality (VI))

Then (38) appears to be equivalent to

$$\begin{cases} R_{\mathrm{T}} \in \mathbf{D}(\mu R_{\mathrm{N}}) \\ \\ \langle U_{\mathrm{T}}, z - R_{\mathrm{T}} \rangle \geqslant 0 \text{ for all } z \in \mathbf{D}(\mu R_{\mathrm{N}}) \end{cases} \tag{39}$$

and to

$$R_{\mathrm{T}} = \mathrm{proj}_{\mathbf{D}(\mu R_{\mathrm{N}})}[R_{\mathrm{T}} - \rho U_{\mathrm{T}}], \text{ for all } \rho > 0 \tag{40}$$

Definition (Coulomb's Friction as a Second–Order Cone Complementarity Problem)

Let us introduce the modified velocity $\widehat{U}$ defined by

$$\widehat{U} = \begin{bmatrix} U_{\mathrm{N}} + \mu \| U_{\mathrm{T}} \| \\ U_{\mathrm{T}}. \end{bmatrix} \tag{41}$$

This notation provides us with a synthetic form of the Coulomb friction as

$$-\widehat{U} \in \mathbb{N}_{\mathbf{C}}(R), \tag{42}$$

or

$$\mathbf{C}^* \ni \widehat{U} \perp R \in \mathbf{C}. \tag{43}$$

where $\mathbf{C}^* = \{ v \in \mathbb{R}^n \mid r^T v \geqslant 0, \forall r \in \mathbf{C} \}$ is the dual cone.

Coulomb's friction

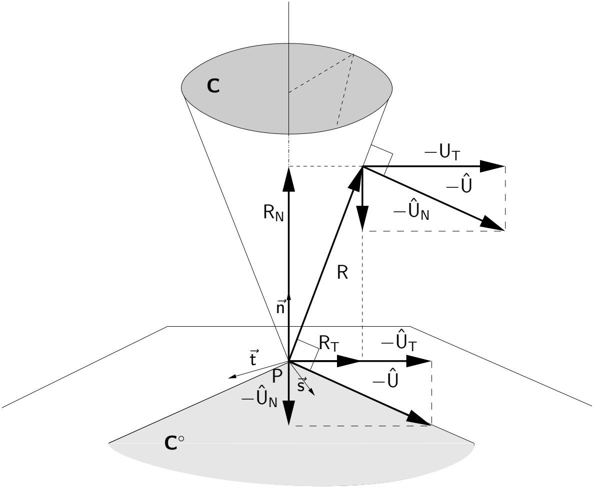


Figure: Coulomb's friction and the modified velocity $\widehat{U}$. The sliding case.

Coulomb's friction with impacts

It is for instance proposed in (Moreau, 1988) to extend (38) (**??**) to densities, i.e. to impulses with a tangential restitution

$$\left\{\begin{array}{l} -P_{\mathsf{N}} \in \partial\psi^*_{\mathbb{R}^-}\left(\dfrac{1}{1+\rho}U_{\mathsf{N}}^+(t) + \dfrac{\rho}{1+\rho}U_{\mathsf{N}}^-(t)\right) \\ \\ -P_{\mathsf{T}} \in \partial\psi^*_{\mathbf{D}}\left(\dfrac{1}{1+\tau}U_{\mathsf{T}}^+(t) + \dfrac{\tau}{1+\tau}U_{\mathsf{T}}^-(t)\right). \end{array}\right. \tag{44}$$

with ρ and τ are constants with values in the interval $[0,1]$ or

$$\left\{\begin{array}{l} -P_{\mathsf{N}} \in \partial\psi^*_{\mathbb{R}^-}(U_{\mathsf{N}}^+(t) + e_{\mathsf{N}}U_{\mathsf{N}}^-(t)) \\ \\ -P_{\mathsf{T}} \in \partial\psi^*_{\mathbf{D}}(U_{\mathsf{T}}^+(t) + e_{\mathsf{T}}U_{\mathsf{T}}^-(t)) \end{array}\right. \tag{45}$$

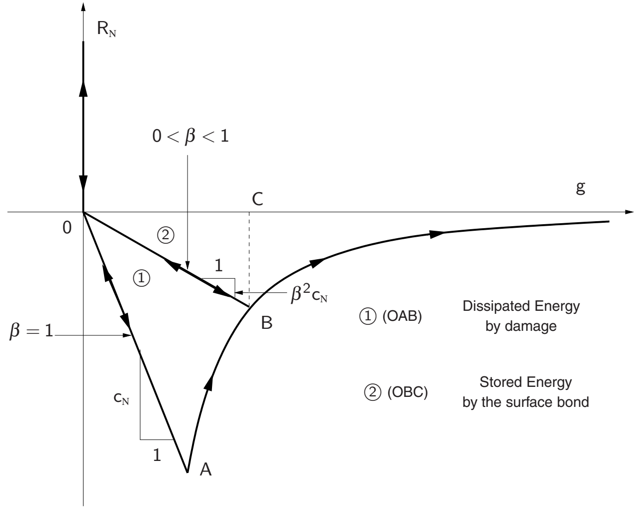
where $e_{\mathsf{N}} \in [0,1)$ and $e_{\mathsf{T}} \in (-1,1)$.

Other contact models

Many other contact models can be constructed starting from the unilateral and Coulomb's friction laws:

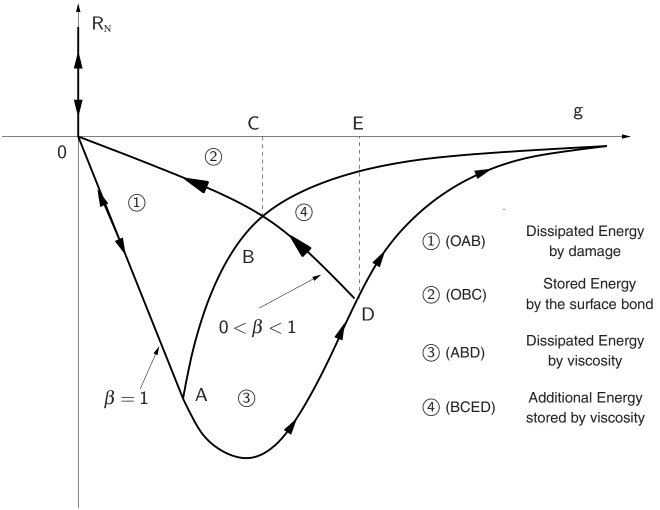
- Rolling friction and spinning friction
- Rate and state laws (varying coefficient of friction)
- Cohesive Zone Model (damage and plasticity at interface)

Cohesive zone model with damage, contact and friction.



(a) Rate independent law

Cohesive zone model with damage, contact and friction.



(b) Rate dependent law (viscosity)

Event–Capturing (Time-stepping) schemes

Time Discretization of the nonsmooth dynamics

For sake of simplicity, the linear time invariant case is only considered.

$$\begin{cases} Mdv + (Kq + Cv^+)\, dt = F_{ext}\, dt + di. \\ v^+ = \dot{q}^+ \end{cases} \tag{46}$$

Integrating both sides of this equation over a time step $]t_k, t_{k+1}]$ of length h,

$$\begin{cases} \displaystyle\int_{]t_k,t_{k+1}]} Mdv + \int_{t_k}^{t_{k+1}} Cv^+ + Kq\, dt = \int_{t_k}^{t_{k+1}} F_{ext}\, dt + \int_{]t_k,t_{k+1}]} di\ , \\[2ex] \displaystyle q(t_{k+1}) = q(t_k) + \int_{t_k}^{t_{k+1}} v^+\, dt\ . \end{cases} \tag{47}$$

By definition of the differential measure dv,

$$\int_{]t_k,t_{k+1}]} M\, dv = M \int_{]t_k,t_{k+1}]} dv = M\,(v^+(t_{k+1}) - v^+(t_k))\ . \tag{48}$$

Note that the right velocities are involved in this formulation.

Time Discretization of the nonsmooth dynamics

The equation of the nonsmooth motion can be written under an integral form as:

$$\begin{cases} M\,(v^+(t_{k+1}) - v^+(t_k)) = \displaystyle\int_{t_k}^{t_{k+1}} -Cv^+ - Kq + F_{ext}\,dt + \int_{]t_k,t_{k+1}]} di\,, \\[2ex] q(t_{k+1}) = q(t_k) + \displaystyle\int_{t_k}^{t_{k+1}} v^+\,dt\,. \end{cases} \tag{49}$$

The following notations will be used:

- $q_k \approx q(t_k)$ and $q_{k+1} \approx q(t_{k+1})$,
- $v_k \approx v^+(t_k)$ and $v_{k+1} \approx v^+(t_{k+1})$,

Impulse as primary unknown

The impulse $\displaystyle\int_{]t_k,t_{k+1}]} di$ of the reaction on the time interval $]t_k, t_{k+1}]$ emerges as a natural unknown. we denote

$$p_{k+1} \approx \int_{]t_k,t_{k+1}]} di$$

Time Discretization of the nonsmooth dynamics

Interpretation

The measure di may be decomposed as follows :

$$di = f\,dt + p d\nu$$

where

- $f\,dt$ is the abs. continuous part of the measure di, and
- $pd\nu$ the atomic part.

Two particular cases:

- Impact at $t_* \in]t_k, t_{k+1}]$: If $f = 0$ and $pd\nu = p\delta_{t_{k+1}}$ then

$$p_{k+1} = p$$

- Continuous force over $]t_k, t_{k+1}]$: If $di = fdt$ and $p = 0$ then

$$p_{k+1} = \int_{t_k}^{t_{k+1}} f(t)\,dt$$

Time Discretization of the nonsmooth dynamics

Remark

- A pointwise evaluation of a (Dirac) measure is a non sense. It practice using the value

$$f_{k+1} \approx f(t_{k+1})$$

yield severe numerical inconsistencies, since

$$\lim_{h \to 0} f_{k+1} = +\infty$$

- Since discontinuities of the derivative v are to be expected if some shocks are occurring, i.e. di has some Dirac atoms within the interval $]t_k, t_{k+1}]$, it is not relevant to use high order approximations integration schemes for di. It may be shown on some examples that, on the contrary, such high order schemes may generate artefact numerical oscillations.

Time Discretization of the nonsmooth dynamics

Discretization of smooth terms, for instance θ-method

θ-method is used for the term supposed to be sufficiently smooth,

$$\begin{aligned}\int_{t_k}^{t_{k+1}} Cv + Kq\,dt &\approx h\left[\theta(Cv_{k+1} + Kq_{k+1}) + (1-\theta)(Cv_k + Kq_k)\right]\\ \int_{t_k}^{t_{k+1}} F_{ext}(t)\,dt &\approx h\left[\theta(F_{ext})_{k+1} + (1-\theta)(F_{ext})_k\right]\end{aligned}$$

The displacement, assumed to be absolutely continuous is approximated by:

$$q_{k+1} = q_k + h\,\left[\theta v_{k+1} + (1-\theta)v_k\right]\ .$$

Time Discretization of the nonsmooth dynamics

Finally, introducing the expression of q_{k+1} in the first equation of (48), one obtains:

$$\begin{aligned}\left[M + h\theta C + h^2\theta^2 K\right](v_{k+1} - v_k) &= -hCv_k - hKq_k - h^2\theta K v_k \\ &+h\left[\theta(F_{ext})_{k+1}) + (1-\theta)(F_{ext})_k\right] + p_{k+1}\,,\end{aligned} \tag{50}$$

which can be written :

$$v_{k+1} = v_{free} + \widehat{M}^{-1} p_{k+1} \tag{51}$$

where,

- the matrix $\widehat{M} = \left[M + h\theta C + h^2\theta^2 K\right]$ is usually called the iteration matrix and,
- The vector

$$\begin{aligned} v_{free} \quad = \quad v_k + \widehat{M}^{-1}\Big[& -hCv_k - hKq_k - h^2\theta K v_k \\ & +h\left[\theta(F_{ext})_{k+1}) + (1-\theta)(F_{ext})_k\right]\Big]\end{aligned}$$

  is the so-called "free" velocity, i.e. the velocity of the system when reaction forces are null.

Time Discretization of the kinematics relations

According to the implicit mind, the discretization of kinematic laws is proposed as follows.
For a constraint α,

$$U^{\alpha}_{k+1} = H^{\alpha\,T}(q_k)\, v_{k+1}\,,$$

$$p^{\alpha}_{k+1} = H^{\alpha}(q_k)\, P^{\alpha}_{k+1}\,, \quad p_{k+1} = \sum_{\alpha} p^{\alpha}_{k+1}\,,$$

where

$$P^{\alpha}_{k+1} \approx \int_{]t_k,t_{k+1}]} d\lambda^{\alpha}.$$

For the unilateral constraints, it is proposed

$$g^{\alpha}_{k+1} = g^{\alpha}_{k} + h\left[\theta U^{\alpha}_{k+1} + (1-\theta) U^{\alpha}_{k}\right]\,.$$

Discretization of the unilateral constraints

Recall that the unilateral constraint is expressed in terms of velocity as

$$-di \in N_{T_C(q)}(v^+) \tag{52}$$

or in local coordinates as

$$-d\lambda^\alpha \in N_{T_{\mathbb{R}_+}(g(q))}(U^{\alpha,+}) \tag{53}$$

The time discretization is performed by

$$-P^\alpha_{k+1} \in N_{T_{\mathbb{R}+}(g^\alpha(\tilde{q}_{k+1}))}(U^\alpha_{k+1}) \tag{54}$$

where $\tilde{q}_{k+1}$ is a forecast of the position for the activation of the constraints, for instance,

$$\tilde{q}_{k+1} = q_k + \frac{h}{2} v_k$$

In the complementarity formalism, we obtain

$$\text{if } g^\alpha(\tilde{q}_{k+1}) \leqslant 0, \text{ then } \quad 0 \leqslant U^\alpha_{k+1} \perp P^\alpha_{k+1} \geqslant 0$$

Summary of the time discretized equations

One step linear problem
$$\begin{cases} v_{k+1} = v_{free} + \widehat{M}^{-1} p_{k+1} \\ q_{k+1} = q_k + h\,[\theta v_{k+1} + (1-\theta) v_k] \end{cases}$$

Relations
$$\begin{cases} U^{\alpha}_{k+1} = H^{\alpha\,T}(q_k)\, v_{k+1} \\ p^{\alpha}_{k+1} = H^{\alpha}(q_k)\, P^{\alpha}_{k+1} \end{cases}$$

Nonsmooth Law
$$\begin{cases} \text{ if } g^{\alpha}(\tilde{q}_{k+1}) \leqslant 0, \text{ then} \\ 0 \leqslant U^{\alpha}_{k+1} \perp P^{\alpha}_{k+1} \geqslant 0 \end{cases}$$

One step LCP

$$U_{k+1} = H^T(q_k) v_{free} + H^T(q_k) \widehat{M}^{-1} H(q_k)\, P_{k+1}$$

$$\text{if } g^{\alpha}_p \leqslant 0, \text{ then } 0 \leqslant U^{\alpha}_{k+1} \perp P^{\alpha}_{k+1} \geqslant 0$$

Moreau–Jean's Time stepping scheme (Jean and Moreau, 1987 ; Moreau, 1988 ; Jean, 1999)

$$\begin{cases} M(q_{k+\theta})(v_{k+1}-v_k) - h\tilde{F}_{k+\theta} = H(q_{k+\theta})P_{k+1}, & (55a) \\ q_{k+1} = q_k + hv_{k+\theta}, & (55b) \\ U_{k+1} = H^T(q_{k+\theta})\, v_{k+1} & (55c) \\ -P_{k+1} \in \mathbb{N}_{T_{\mathbb{R}^m_+}(\tilde{y}_{k+\gamma})}(U_{k+1}+eU_k), & (55d) \\ \tilde{y}_{k+\gamma} = y_k + h\gamma U_k, \quad \gamma \in [0,1]. & (55e) \end{cases}$$

with $\theta \in [0,1], \gamma \geqslant 0$ and $x_{k+\alpha} = (1-\alpha)x_{k+1} + \alpha x_k$ and $\tilde{y}_{k+\gamma}$ is a prediction of the constraints.

Properties

- Convergence results for one constraints
- Convergence results for multiple constraints problems with acute kinetic angles
- No theoretical proof of order

Schatzman–Paoli's Time stepping scheme

$$\begin{cases} M(q_k+1)(q_{k+1}-2q_k+q_{k-1}) - h^2 F(t_{k+\theta}, q_{k+\theta}, v_{k+\theta}) = p_{k+1}, & (56a)\\[2ex] v_{k+1} = \dfrac{q_{k+1}-q_{k-1}}{2h}, & (56b)\\[2ex] -p_{k+1} \in N_K\left(\dfrac{q_{k+1}+eq_{k-1}}{1+e}\right), & (56c)\end{cases}$$

where N_K defined the normal cone to K.
For $K = \{q \in \mathbb{R}^n, y = g(q) \geqslant 0\}$

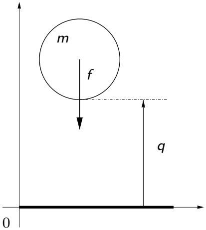
$$0 \leqslant g\left(\frac{q_{k+1}+eq_{k-1}}{1+e}\right) \perp \nabla g\left(\frac{q_{k+1}+eq_{k-1}}{1+e}\right) P_{k+1} \geqslant 0 \qquad (57)$$

Properties

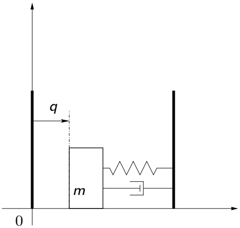
- Convergence results for one constraints
- Convergence results for multiple constraints problems with acute kinetic angles
- No theoretical proof of order

Academic examples

The bouncing Ball and the linear impacting oscillator



(a) Bouncing ball example



(b) Linear Oscillator example

Figure: Academic test examples with analytical solutions

Academic examples

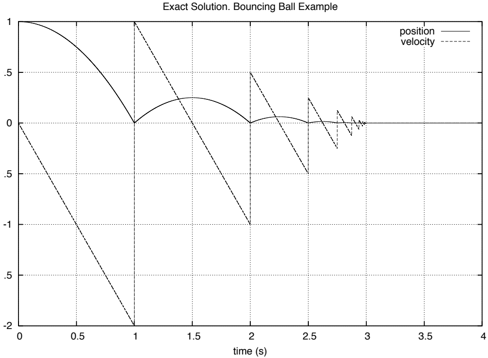


Figure: Analytical solution. Bouncing ball example

Academic examples

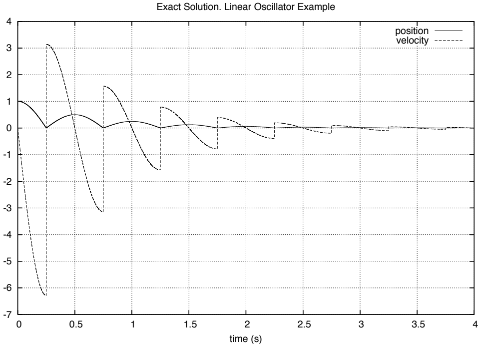


Figure: Analytical solution. Linear Oscillator

Measuring error and convergence

Convergence in the sense of filled-in graph (Moreau (1978))

$$gr^\star(f) = \{(t,x) \in [0,T] \times \mathbb{R}^n, 0 \leqslant t \leqslant T \text{ and } x \in [f(t^-), f(t^+)]\}. \quad (58)$$

Such graphs are closed bounded subsets of $[0,T] \times \mathbb{R}^n$, hence, we can use the Hausdorff distance between two such sets with a suitable metric:

$$d((t,x),(s,y)) = \max\{|t-s|, \|x-y\|\}. \quad (59)$$

Defining the excess of separation between two graphs by

$$e(gr^\star(f), gr^\star(g)) = \sup_{(t,x)\in gr^\star(f)} \inf_{(s,y)\in gr^\star(g)} d((t,x),(s,y)), \quad (60)$$

the Hausdorff distance between two filled-in graphs $h^\star$ is defined by

$$h^\star(gr^\star(f), gr^\star(g)) = \max\{e(gr^\star(f), gr^\star(g)), e(gr^\star(g), gr^\star(f))\}. \quad (61)$$

Measuring error and convergence

An equivalent grid-function norm to the function norm in $\mathcal{L}_1$

$$\|e\|_1 = h \sum_{i=0}^{N} |f_i - f(t_i)| \tag{62}$$

In the same way, the $p-norm$ can be defined by

$$\|e\|_p = \left(h \sum_{i=0}^{N} |f_i - f(t_i)|^p \right)^{1/p} \tag{63}$$

The computation of this two last norm is easier to implement for piecewise continuous analytical function than the Hausdorff distance.

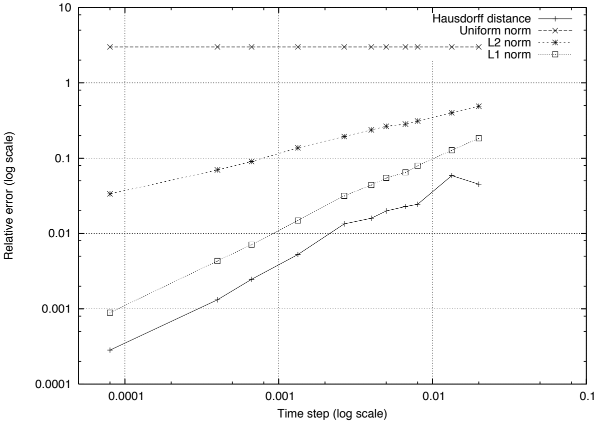
Global order of convergence.

Definition

A one-step time–integration scheme is of order q for a given norm $\|\cdot\|$ if there exists a constant C such that

$$\|e\| = Ch^q + \mathcal{O}(h^{q+1}) \tag{64}$$

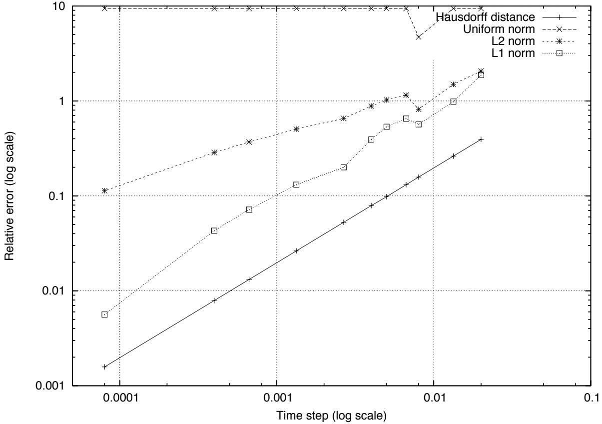
Empirical order of convergence. Moreau–Jean's time–stepping scheme



(a) The bouncing ball example

Figure: Empirical order of convergence of the Moreau–Jean's time-stepping scheme.

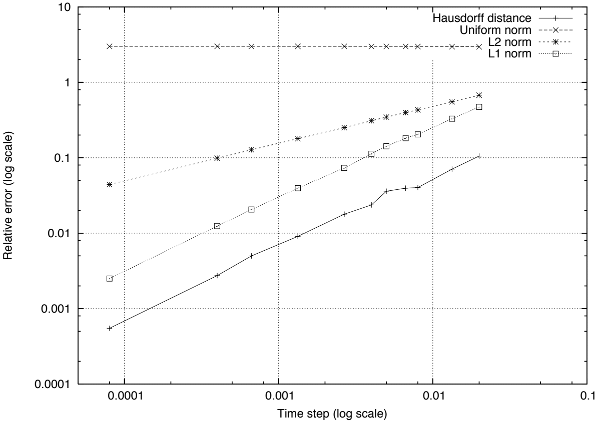
Empirical order of convergence. Moreau–Jean's time–stepping scheme



(a) The linear oscillator example

Figure: Empirical order of convergence of the Moreau–Jean's time-stepping scheme.

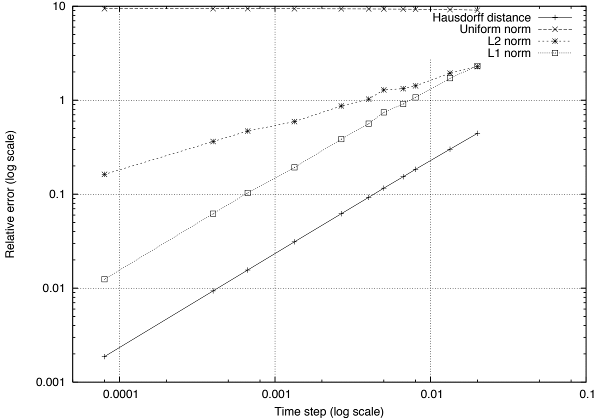
Empirical order of convergence. Schatzman–Paoli's time–stepping scheme



(a) The bouncing ball example

Figure: Empirical order of convergence of the Schatzman-Paoli's time-stepping scheme.

Empirical order of convergence. Schatzman–Paoli's time–stepping scheme



(a) The linear oscillator example

Figure: Empirical order of convergence of the Schatzman-Paoli's time-stepping scheme.

Newmark-type schemes for flexible multibody systems and FEM applications.

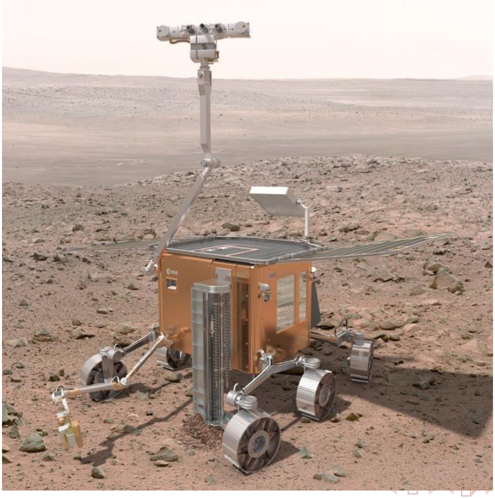
Joint work with

- O. Brüls, Q.Z. Chen and G. Virlez (Université de Liège, Belgium)
- A. Cardona (Cimec, Santa Fe, Argentina.)

Mechanical systems with contact, impact and friction

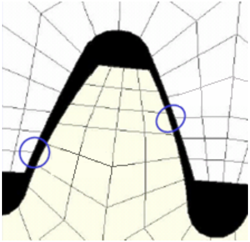
Simulation of flexible multibody systems.

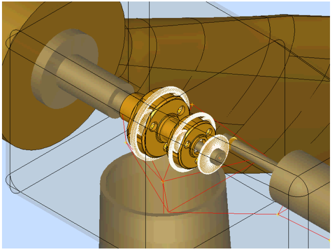
Simulation of the ExoMars Rover (INRIA/Trasys Space/ESA)

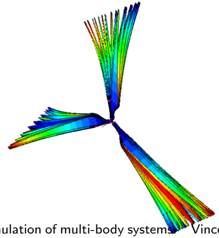


Mechanical systems with contact, impact and friction

- Simulation of flexible multibody systems.
- Simulation of wind turbines (DYNAWIND project)
- Joint work with O. Brüls, Q.Z. Chen and G. Virlez (Université de Liège)

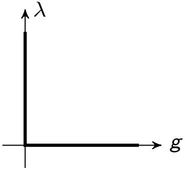






NonSmooth Multibody Systems

Scleronomous holonomic perfect unilateral constraints and joints



- Nonsmooth equations of motion

$$
\begin{aligned}
\dot{\mathbf{q}}^{+} &= \mathbf{v}^{+} && \text{(65a)}\\
\mathbf{M}(\mathbf{q})\,\mathrm{d}\mathbf{v} - \mathbf{g}_{\mathbf{q}}^{T}\,\mathrm{d}\mathbf{i} &= \mathbf{f}(\mathbf{q},\mathbf{v},t)\,\mathrm{d}t && \text{(65b)}\\
\mathbf{g}^{\overline{\mathcal{U}}}(\mathbf{q}) &= \mathbf{0} && \text{(65c)}\\
\mathbf{0} \leqslant \mathbf{g}^{\mathcal{U}}(\mathbf{q}) &\perp \mathrm{d}\mathbf{i}^{\mathcal{U}} \geqslant \mathbf{0} && \text{(65d)}
\end{aligned}
$$

  where
  - $\mathbf{g}_{\mathbf{q}} = \nabla g(q)$.
  - $\mathcal{U}$ index set of indices of the unilateral constraints,
  - $\overline{\mathcal{U}}$ the set of bilateral constraints,
  - $\mathcal{C} = \mathcal{U} \cup \overline{\mathcal{U}}$
- Newton Impact law $\mathbf{g}_{\mathbf{q}}^{\mathcal{U}}\mathbf{v}^{+}(t) = -e\,\mathbf{g}_{\mathbf{q}}^{\mathcal{U}}\mathbf{v}^{-}(t)$
  e is the coefficient of restitution.

The Moreau's sweeping process of second order

Definition (Moreau (1983, 1988))

A key stone of this formulation is the inclusion in terms of velocity.

$$
\begin{aligned}
\dot{\mathbf{q}} &= \mathbf{v} && (66a)\\
\mathbf{M}(\mathbf{q})\,\mathrm{d}\mathbf{v} - \mathbf{g}_{\mathbf{q}}^{T}\,\mathrm{d}\mathbf{i} &= \mathbf{f}(\mathbf{q},\mathbf{v},t)\,\mathrm{d}t && (66b)\\
\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}\,\mathbf{v} &= \mathbf{0} && (66c)\\
\text{if } g^{j}(\mathbf{q}) \leqslant 0 \text{ then } 0 \leqslant g_{\mathbf{q}}^{j}\,\mathbf{v} + e\,g_{\mathbf{q}}^{j}\,\mathbf{v}^{-} &\perp \mathrm{d}i^{j} \geqslant 0, \quad \forall j \in \mathcal{U} && (66d)
\end{aligned}
$$

Comments

This formulation provides a common framework for the nonsmooth dynamics containing inelastic impacts without decomposition.

➜ Foundation for the Moreau–Jean time–stepping approach.

Moreau–Jean time stepping scheme (Jean and Moreau, 1987 ; Moreau, 1988 ; Jean, 1999)

Principle

$$P_{n+1} \approx di((t_n, t_{n+1}]) = \int_{(t_n,t_{n+1}]} \mathrm{d}i \tag{67}$$

$$\mathbf{q}_{n+1} = \mathbf{q}_n + h\mathbf{v}_{n+\theta}, \tag{68a}$$

$$M(\mathbf{q}_{n+\theta})(\mathbf{v}_{n+1} - \mathbf{v}_n) - hf_{n+\theta} = g_q(\mathbf{q}_{n+\theta})P_{n+1}, \tag{68b}$$

$$\text{if} \quad \bar{g}_n^j \leqslant 0, 0 \leqslant g^j_{\mathbf{q},n+1}\,\mathbf{v}_{n+1} + e\, g^j_{\mathbf{q},n}\,\mathbf{v}_n \perp P^j_{n+1} \geqslant 0 \tag{68c}$$

(68d)

with

- $\theta \in [0,1]$
- $x_{n+\theta} = (1-\theta)x_{n+1} + \theta x_n$
- $f_{n+\theta} = f(t_{n+\theta}, \mathbf{q}_{n+\theta}, \mathbf{v}_{n+\theta})$
- $\bar{g}_n$ is a prediction of the constraints, e.g. $\bar{g}_n = g_n + h/2 g^j_{\mathbf{q},n}\,\mathbf{v}_n$

Objectives & Motivations

Limitations of the Moreau–Jean scheme

- Moreau–Jean time–stepping : strong numerical damping for $\theta \gg 1/2$.
  ➜ Improve numerical damping with a controlled damping of high frequencies.
- Constraint treated at the velocity level : penetration at the position level.
  ➜ solve the constraints at position level.
- Rough activation of constraints at the velocity level

Means

- Splitting between impulsive and non impulsive terms and use of α–scheme. (Chen et al., 2013)
- Gear–Gupta–Leimkuhler (GGL) enforcement of the unilateral constraint at the position level (Acary, 2013).
- Nonsmooth Newton method viewed as an active set method.

A first naive approach

Direct Application of the HHT scheme to Linear Time "Invariant" Dynamics with contact

$$\begin{cases} M\dot{v}(t) + Kq(t) + Cv(t) = f(t) + r(t), \text{a.e} \\ \dot{q}(t) = v(t) \\ r(t) = G(q)\,\lambda(t) \\ g(t) = g(q(t)), \quad \dot{g}(t) = G^T(q(t))v(t), \\ 0 \leqslant g(t) \perp \lambda(t) \geqslant 0, \end{cases} \tag{69}$$

results in

$$\begin{cases} M\ddot{q}_{k+1} = -(Kq_{k+1} + Cv_{k+1}) + F_{k+1} + r_{k+1} \\ r_{k+1} = G_{k+1}\lambda_{k+1} \end{cases} \tag{70}$$

$$\begin{cases} Ma_{k+1} = M\ddot{q}_{k+1+\alpha} \\ v_{k+1} = v_k + ha_{k+\gamma} \\ q_{k+1} = q_k + hv_k + \dfrac{h^2}{2}a_{k+2\beta} \\ 0 \leqslant g_{k+1} \perp \lambda_{k+1} \geqslant 0, \end{cases} \tag{71}$$

A first naive approach

Direct Application of the HHT scheme to Linear Time "Invariant" Dynamics with contact

The scheme is not consistent for mainly two reasons:

- If an impact occur between rigid bodies, or if a restitution law is needed which is mandatory between semidiscrete structure, the impact law is not taken into account by the discrete constraint at position level
- Even if the constraint is discretized at the velocity level, i.e.

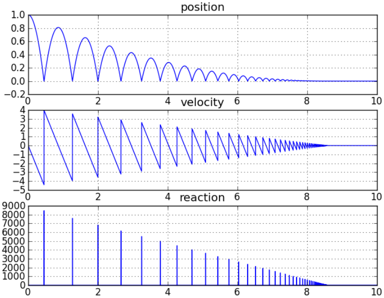
$$\text{if } \bar{g}_{k+1}, \text{ then } 0 \leqslant \dot{g}_{k+1} + e g_k \perp \lambda_{k+1} \geqslant 0 \tag{72}$$

  the scheme is consistent only for $\gamma = 1$ and $\alpha = 0$ (first order approximation.)

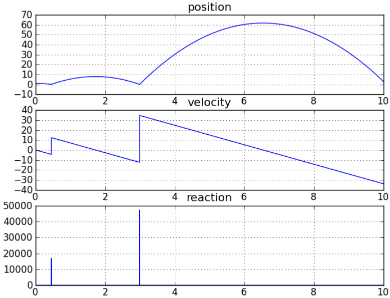
A first naive approach

Velocity based constraints with standard Newmark scheme ($\alpha = 0.0$)

Bouncing ball example. $m = 1$, $g = 9.81$, $x_0 = 1.0$ $v_0 = 0.0$, $e = 0.9$



$h = 0.001$, $\gamma = 1.0$, $\beta = \gamma/2$

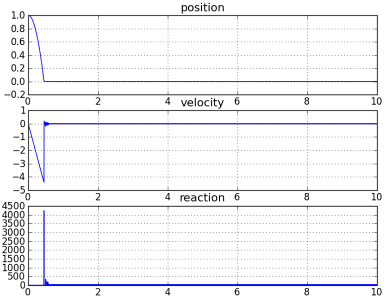


$h = 0.001$, $\gamma = 1/2$, $\beta = \gamma/2$

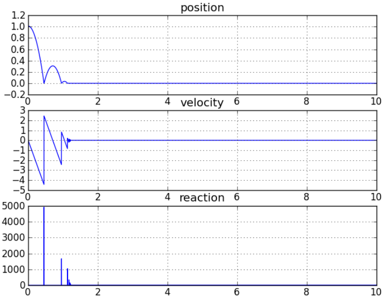
A first naive approach

Position based constraints with standard Newmark scheme ($\alpha = 0.0$)

Bouncing ball example. $m = 1$, $g = 9.81$, $v_0 = 0.0$, $e = 0.9$, $h = 0.001$, $\gamma = 1.0$, $\beta = \gamma/2$



$x_0 = 1.0$



$x_0 = 1.01$

The nonsmooth generalized α scheme

Splitting the dynamics between smooth and nonsmooth part

$$\mathrm{d}\mathbf{w} = \mathrm{d}\mathbf{v} - \dot{\tilde{\mathbf{v}}}\,\mathrm{d}t \tag{73}$$

Index sets of constraints

$\mathcal{U}$ index set of indices of the unilateral constraints,
$\overline{\mathcal{U}}$ the set of bilateral constraints,

Smooth (non-impulsive) part

Solutions of the following DAE

$$\begin{aligned}
\dot{\tilde{\mathbf{q}}} &= \tilde{\mathbf{v}} && \text{(74a)}\\
\mathbf{M}(\mathbf{q})\,\dot{\tilde{\mathbf{v}}} - \mathbf{g}_{\mathbf{q}}^{T}(\mathbf{q})\,\tilde{\boldsymbol{\lambda}} &= \mathbf{f}(\mathbf{q},\mathbf{v},t) && \text{(74b)}\\
\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}(\mathbf{q})\,\tilde{\mathbf{v}} &= \mathbf{0} && \text{(74c)}\\
\tilde{\boldsymbol{\lambda}}^{\mathcal{U}} &= \mathbf{0} && \text{(74d)}
\end{aligned}$$

with the initial value $\tilde{\mathbf{v}}(t_n) = \mathbf{v}(t_n)$, $\tilde{\mathbf{q}}(t_n) = \mathbf{q}(t_n)$.

The nonsmooth generalized α scheme

Splitting the dynamics between smooth and nonsmooth part

$$\begin{aligned}
\dot{\mathbf{q}} &= \mathbf{v} && \text{(75a)}\\
\mathrm{d}\mathbf{v} &= \mathrm{d}\mathbf{w} + \dot{\tilde{\mathbf{v}}}\,\mathrm{d}t && \text{(75b)}\\
\mathbf{M}(\mathbf{q})\,\dot{\tilde{\mathbf{v}}} - \mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}},T}\,\tilde{\boldsymbol{\lambda}}^{\overline{\mathcal{U}}} &= \mathbf{f}(\mathbf{q},\mathbf{v},t) && \text{(75c)}\\
\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}\,\tilde{\mathbf{v}} &= \mathbf{0} && \text{(75d)}\\
\tilde{\boldsymbol{\lambda}}^{\mathcal{U}} &= \mathbf{0} && \text{(75e)}\\
\mathbf{M}(\mathbf{q})\,\mathrm{d}\mathbf{w} - \mathbf{g}_{\mathbf{q}}^{T}\,(\mathrm{d}\mathbf{i} - \tilde{\boldsymbol{\lambda}}\,\mathrm{d}t) &= \mathbf{0} && \text{(75f)}\\
\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}\mathbf{v} &= \mathbf{0} && \text{(75g)}\\
\text{if } g^j(\mathbf{q}) \leqslant 0 \text{ then } 0 \leqslant g_{\mathbf{q}}^j\,\mathbf{v} + e\,g_{\mathbf{q}}^j\,\mathbf{v}^- &\perp \mathrm{d}i^j \geqslant 0, \quad \forall j \in \mathcal{U} && \text{(75h)}
\end{aligned}$$

The nonsmooth generalized α scheme

GGL approach to stabilize the constraints at the position level

The equations of motion become

$$\cancel{\dot{\mathbf{q}} = \mathbf{v}} \rightarrow$$

$$\begin{aligned}
\mathbf{M}(\mathbf{q})\,\dot{\mathbf{q}} - \mathbf{g}_{\mathbf{q}}^{T}\,\boldsymbol{\mu} &= \mathbf{M}(\mathbf{q})\,\mathbf{v} && (76a)\\
\mathbf{g}^{\overline{\mathcal{U}}}(\mathbf{q}) &= \mathbf{0} && (76b)\\
\mathbf{0} \leqslant \mathbf{g}^{\mathcal{U}}(\mathbf{q}) &\perp \boldsymbol{\mu}^{\mathcal{U}} \geqslant \mathbf{0} && (76c)\\
\mathrm{d}\mathbf{v} &= \mathrm{d}\mathbf{w} + \dot{\tilde{\mathbf{v}}}\,\mathrm{d}t && (76d)\\
\mathbf{M}(\mathbf{q})\,\dot{\tilde{\mathbf{v}}} - \mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}},T}\,\tilde{\boldsymbol{\lambda}}^{\overline{\mathcal{U}}} &= \mathbf{f}(\mathbf{q},\mathbf{v},t) && (76e)\\
\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}\,\tilde{\mathbf{v}} &= \mathbf{0} && (76f)\\
\tilde{\boldsymbol{\lambda}}^{\mathcal{U}} &= \mathbf{0} && (76g)\\
\mathbf{M}(\mathbf{q})\,\mathrm{d}\mathbf{w} - \mathbf{g}_{\mathbf{q}}^{T}\,(\mathrm{d}\mathbf{i} - \tilde{\boldsymbol{\lambda}}\,\mathrm{d}t) &= \mathbf{0} && (76h)\\
\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}\mathbf{v} &= \mathbf{0} && (76i)\\
\text{if } g^{j}(\mathbf{q}) \leqslant 0 \text{ then } 0 \leqslant g_{\mathbf{q}}^{j}\,\mathbf{v} + e\,g_{\mathbf{q}}^{j}\,\mathbf{v}^{-} &\perp \mathrm{d}i^{j} \geqslant 0, \quad \forall j \in \mathcal{U} && (76j)
\end{aligned}$$

The nonsmooth generalized α scheme

Velocity jumps and position correction

The multipliers $\mathbf{\Lambda}(t_n; t)$ and $\boldsymbol{\nu}(t_n; t)$ are defined as

$$\mathbf{\Lambda}(t_n; t) = \int_{(t_n,t]} (\mathrm{d}\mathbf{i} - \tilde{\boldsymbol{\lambda}}(\tau)\,\mathrm{d}\tau) \tag{77a}$$

$$\boldsymbol{\nu}(t_n; t) = \int_{t_n}^{t} (\boldsymbol{\mu}(\tau) + \mathbf{\Lambda}(t_n; \tau))\,\mathrm{d}\tau \tag{77b}$$

with $\mathbf{\Lambda}(t_n; t_n) = \boldsymbol{\nu}(t_n; t_n) = \mathbf{0}$.
The velocity jump and position correction variables

$$\mathbf{W}(t_n; t) = \int_{(t_n,t]} \mathrm{d}\mathbf{w} = \mathbf{v}(t) - \tilde{\mathbf{v}}(t) \tag{78a}$$

$$\mathbf{U}(t_n; t) = \int_{t_n}^{t} (\dot{\mathbf{q}} - \tilde{\mathbf{v}})\mathrm{d}t = \mathbf{q}(t) - \tilde{\mathbf{q}}(t) \tag{78b}$$

➜ Low-order approximation of impulsive terms.
➜ Higher–order approximation of non impulsive terms.

The nonsmooth generalized α scheme

$$\mathbf{M}(\mathbf{q}_{n+1})\mathbf{U}_{n+1} - \mathbf{g}^{T}_{\mathbf{q},n+1}\,\boldsymbol{\nu}_{n+1} = \mathbf{0} \tag{79a}$$

$$\mathbf{g}^{\overline{\mathcal{U}}}(\mathbf{q}_{n+1}) = \mathbf{0} \tag{79b}$$

$$\mathbf{0} \leqslant \mathbf{g}^{\mathcal{U}}(\mathbf{q}_{n+1}) \perp \boldsymbol{\nu}^{\mathcal{U}}_{n+1} \geqslant \mathbf{0} \tag{79c}$$

$$\mathbf{W}_{n+1} - \mathbf{v}_{n+1} - \tilde{\mathbf{v}}_{n+1} = \mathbf{0} \tag{79d}$$

$$\mathbf{M}(\mathbf{q}_{n+1})\dot{\tilde{\mathbf{v}}}_{n+1} - \mathbf{f}(\mathbf{q}_{n+1}, \mathbf{v}_{n+1}, t_{n+1}) - \mathbf{g}^{\overline{\mathcal{U}},T}_{\mathbf{q},n+1}\,\tilde{\boldsymbol{\lambda}}^{\overline{\mathcal{U}}}_{n+1} = \mathbf{0} \tag{79e}$$

$$\mathbf{g}^{\overline{\mathcal{U}}}_{\mathbf{q},n+1}\,\tilde{\mathbf{v}}_{n+1} = \mathbf{0} \tag{79f}$$

$$\mathbf{M}(\mathbf{q}_{n+1})\mathbf{W}_{n+1} - \mathbf{g}^{T}_{\mathbf{q},n+1}\boldsymbol{\Lambda}_{n+1} = \mathbf{0} \tag{79g}$$

$$\mathbf{g}^{\overline{\mathcal{U}}}_{\mathbf{q},n+1}\mathbf{v}_{n+1} = \mathbf{0} \tag{79h}$$

$$\text{if } g^{j}(\mathbf{q}^{*}_{n+1}) \leqslant 0 \text{ then } 0 \leqslant g^{j}_{\mathbf{q},n+1}\,\mathbf{v}_{n+1} + e\,g^{j}_{\mathbf{q},n}\,\mathbf{v}_{n} \perp \Lambda^{j}_{n+1} \geqslant 0, \forall j \in \mathcal{U}$$

The nonsmooth generalized α scheme

Nonsmooth generalized α-scheme

$$\begin{aligned}
\tilde{\mathbf{q}}_{n+1} &= \mathbf{q}_n + h\mathbf{v}_n + h^2(0.5-\beta)\mathbf{a}_n + h^2\beta\mathbf{a}_{n+1} && (80a)\\
\mathbf{q}_{n+1} &= \tilde{\mathbf{q}}_{n+1} + \mathbf{U}_{n+1} && (80b)\\
\tilde{\mathbf{v}}_{n+1} &= \mathbf{v}_n + h(1-\gamma)\mathbf{a}_n + h\gamma\mathbf{a}_{n+1} && (80c)\\
\mathbf{v}_{n+1} &= \tilde{\mathbf{v}}_{n+1} + \mathbf{W}_{n+1} && (80d)\\
(1-\alpha_m)\mathbf{a}_{n+1} + \alpha_m\mathbf{a}_n &= (1-\alpha_f)\dot{\tilde{\mathbf{v}}}_{n+1} + \alpha_f\dot{\tilde{\mathbf{v}}}_n && (80e)
\end{aligned}$$

Special cases

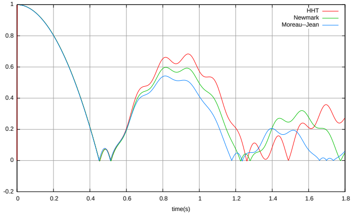
- $\alpha_m = \alpha_f = 0$ ➜ Nonsmooth Newmark
- $\alpha_m = 0, \alpha_f \in [0, 1/3]$ ➜ Nonsmooth Hilber-Hughes–Taylor (HHT)

Spectral radius at infinity $\rho_\infty \in [0,1]$

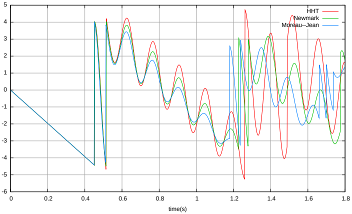
$$\alpha_m = \frac{2\rho_\infty - 1}{\rho_\infty + 1}, \quad \alpha_f = \frac{\rho_\infty}{\rho_\infty + 1}, \quad \beta = \frac{1}{4}(\gamma + \frac{1}{2})^2. \tag{81}$$

Numerical Illustrations

Two ball oscillator with impact.

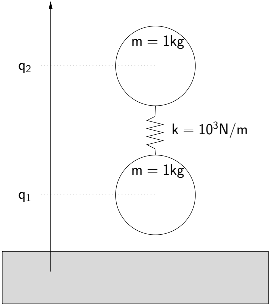


Position of the first ball



Velocity of the first ball

Time–step : $h = 2e-3$.
Moreau ($\theta = 1.0$).
Newmark ($\gamma = 1.0, \beta = 0.5, \alpha_m = \alpha_f = 0$).
HHT ($\gamma = 1.0, \beta = 0.5, \alpha_f = 0.1, \alpha_m = 0$)



Numerical Illustrations

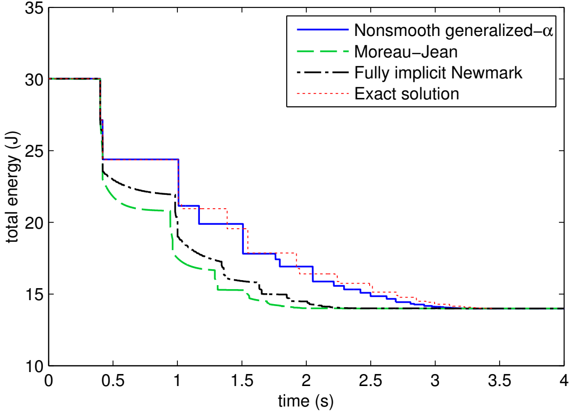
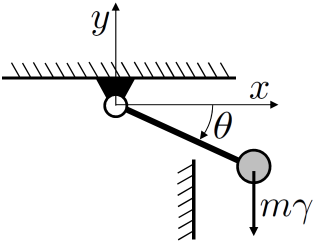


Figure 7. Numerical results for the total energy of the bouncing oscillator.

Numerical Illustrations

Bouncing Pendulum

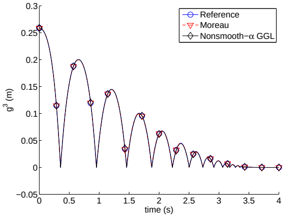
$$\mathbf{q} = [x, y, \theta]^T$$
$$g_1(\mathbf{q}) = x - l\cos\theta = 0$$
$$g_2(\mathbf{q}) = y - l\sin\theta = 0$$
$$g_3(\mathbf{q}) = x - \sqrt{2}/2 \geqslant 0$$

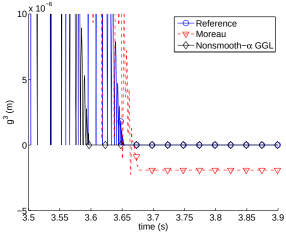


Time–step : $h = 2e-3$.
Moreau ($\theta = 1/1.8$).
α-schemes ($\rho_\infty = 0.8$)

$e = 0.8$

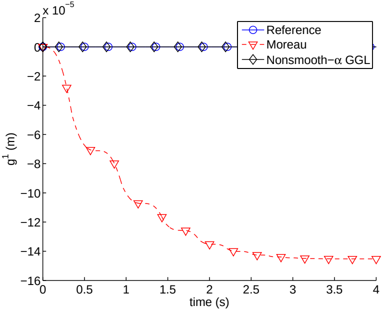
Unilateral constraint

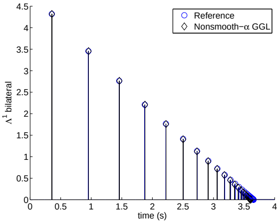


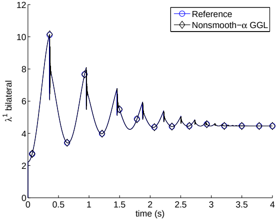


Numerical Illustrations

Bouncing Pendulum

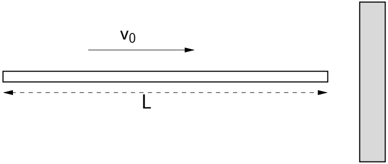




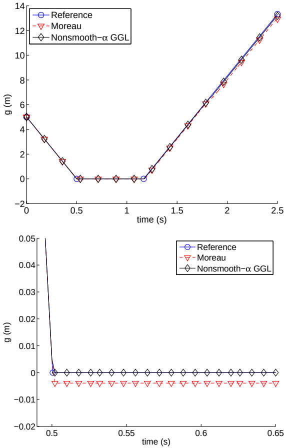


Numerical Illustrations

Impacting elastic bar



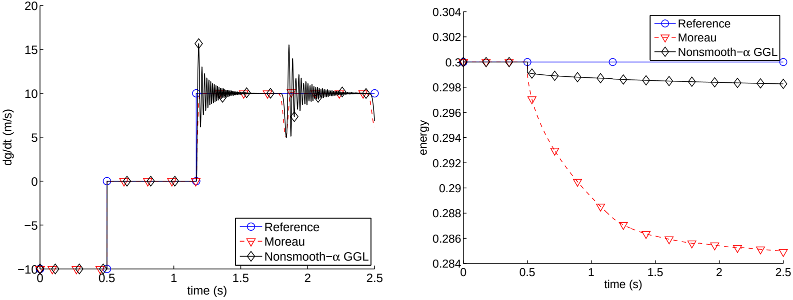
Unilateral constraint



$g_3(\mathbf{q}) = x_1 \geqslant 0$
$e = 0.0$
200 finite elements
Time–step : $h = 2e - 3$.
Moreau ($\theta = 1/1.8$).
α-schemes ($\rho_\infty = 0.8$)

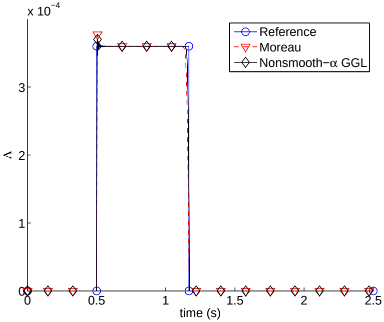
Numerical Illustrations

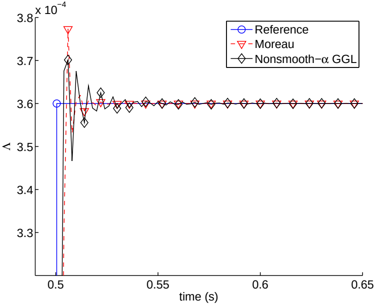
Impacting elastic bar



Numerical Illustrations

Impacting elastic bar





Energy analysis

Time–continuous energy balance equations

Let us start with the “LTI” Dynamics

$$\begin{cases} M\,dv + (Kq + Cv)\;dt = F\;\;dt + \;\;di \\ dq = v^{\pm}\,dt \end{cases} \tag{82}$$

we get for the Energy Balance

$$d(v^\top Mv) + (v^+ + v^-)(Kq + Cv)\;dt \quad = \quad (v^+ + v^-)F\;\;dt + (v^+ + v^-)\;di \tag{83}$$

that is

$$2d\mathcal{E} := d(v^\top Mv) + 2q^\top Kdq \quad = \quad 2v^\top F\;\;dt - 2v^\top Cv\;\;dt + (v^+ + v^-)^\top\;\;di \tag{84}$$

with

$$\mathcal{E} := \frac{1}{2}v^\top Mv + \frac{1}{2}q^\top Kq. \tag{85}$$

Energy analysis

Time–continuous energy balance equations

If we split the differential measure in $di = \lambda \;\; dt + \sum_i p_i \delta_{t_i}$, we get

$$2d\mathcal{E} = \;\; = \;\; 2v^\top (F + \lambda) \;\; dt - 2v^\top C v \;\; dt + (v^+ + v^-)^\top p_i \delta_{t_i} \tag{86}$$

By integration over a time interval $[t_0, t_0]$ such that $t_i \in [t_0, t_1]$, we obtain an energy balance equation as

$$\begin{aligned} \Delta\mathcal{E} \;\; &:= \;\; \mathcal{E}(t_1) - \mathcal{E}(t_0) \\ &= \underbrace{\int_{t_0}^{t_1} v^\top F \;\; dt}_{W^{\text{ext}}} - \underbrace{\int_{t_0}^{t_1} v^\top C v \;\; dt}_{W^{\text{damping}}} + \underbrace{\int_{t_0}^{t_1} v^\top \lambda \;\; dt}_{W^{\text{con}}} + \underbrace{\frac{1}{2}\sum_i (v^+(t_i) + v^-(t_i))^\top p_i}_{W^{\text{impact}}} \end{aligned} \tag{87}$$

Energy analysis

Work performed by the reaction impulse di

- The term
$$W^{\text{con}} = \int_{t_0}^{t_1} v^\top \lambda \; dt \tag{88}$$
is the work done by the contact forces within the time–step. If we consider perfect unilateral constraints, we have $W^{\text{con}} = 0$.
- The term
$$W^{\text{impact}} = \frac{1}{2}\sum_i (v^+(t_i) + v^-(t_i))^\top p_i \tag{89}$$
represents the work done by the contact impulse p_i at the time of impact t_i. Since $p_i = G(t_i)P_i$ and if we consider the Newton impact law, we have
$$\begin{aligned} W^{\text{impact}} &= \frac{1}{2}\sum_i (v^+(t_i) + v^-(t_i))^\top G(t_i)P_i \\ &= \frac{1}{2}\sum_i (U^+(t_i) + U^-(t_i))^\top P_i \\ &= \frac{1}{2}\sum_i ((1-e)U^-(t_i))^\top P_i \leqslant 0 \text{ for } 0 \leqslant e \leqslant 1 \end{aligned} \tag{90}$$
with the local relative velocity defines as $U(t) = G^\top(t)v(t)$

Energy analysis for Moreau–Jean scheme

Let us define the discrete approximation of the work done by the external forces within the step by

$$W^{\text{ext}}_{k+1} = h v^{\top}_{k+\theta} F_{k+\theta} \approx \int_{t_k}^{t_{k+1}} F v \; dt, \tag{91}$$

and the discrete approximation of the work done by the damping term by

$$W^{\text{damping}}_{k+1} = -h v^{\top}_{k+\theta} C v_{k+\theta} \approx -\int_{t_k}^{t_{k+1}} v^T C v \; dt. \tag{92}$$

Lemma

The variation of the total mechanical energy over a time–step $[t_k, t_{k+1}]$ performed by the Moreau–Jean scheme (4) is

$$\Delta\mathcal{E} - W^{\text{ext}}_{k+1} - W^{\text{damping}}_{k+1} \quad = \quad (\frac{1}{2} - \theta) \left[\|v_{k+1} - v_k\|^2_M + \|(q_{k+1} - q_k)\|^2_K \right] + U^{\top}_{k+\theta} P_{k+1} \tag{93}$$

Energy analysis for Moreau–Jean scheme

Proposition

The Moreau–Jean scheme dissipates energy in the sense that

$$\mathcal{E}(t_{k+1}) - \mathcal{E}(t_k) \leqslant W^{\text{ext}}_{k+1} + W^{\text{damping}}_{k+1}, \tag{94}$$

if

$$\frac{1}{2} \leqslant \theta \leqslant \frac{1}{1+e} \leqslant 1. \tag{95}$$

where $e = \max e^{\alpha}, \alpha \in \mathcal{I}$. *In particular, for* $e = 0$, *we get* $\frac{1}{2} \leqslant \theta \leqslant 1$ *and or* $e = 1$, *we get* $\theta = \frac{1}{2}$.

Energy analysis for Moreau–Jean scheme

Variant of the Moreau scheme that always dissipates energy

Let us consider the variant of the Moreau scheme

$$\begin{cases} M(v_{k+1}-v_k) + hKq_{k+\theta} - hF_{k+\theta} = p_{k+1} = GP_{k+1}, & (96a)\\ q_{k+1} = q_k + hv_{k+1/2}, & (96b)\\ U_{k+1} = G^\top v_{k+1} & (96c)\\ \left.\begin{array}{l}\text{if} \quad \bar{g}^\alpha_{k+1} \leqslant 0 \text{ then } 0 \leqslant U^\alpha_{k+1} + eU^\alpha_k \perp P^\alpha_{k+1} \geqslant 0,\\ \text{otherwise } P^\alpha_{k+1} = 0.\end{array}\right. , \alpha \in \mathcal{I} & (96d)\end{cases}$$

Energy analysis for Moreau–Jean scheme

Lemma

The variation of the total mechanical energy performed by the scheme (96) over a time–step is

$$\Delta\mathcal{E} - \bar{W}^{\text{ext}}_{k+1} - W^{\text{damping}}_{k+1} \quad = \quad (\frac{1}{2} - \theta)\|(q_{k+1} - q_k)\|^2_K + U^\top_{k+1/2} P_{k+1} \tag{97}$$

If $\theta \geqslant 1/2$, then the scheme (96) dissipates energy in the sense that

$$\mathcal{E}(t_{k+1}) - \mathcal{E}(t_k) \leqslant \bar{W}^{\text{ext}}_{k+1} + W^{\text{damping}}_{k+1}. \tag{98}$$

Energy analysis for nonsmooth Newmark scheme

Let us define the discrete approximation of the work done by the external forces within the step by

$$W^{\text{ext}}_{k+1} = (q_{k+1} - q_k)^\top F_{k+\gamma} \approx \int_{t_k}^{t_{k+1}} Fv \; dt \tag{99}$$

and the discrete approximation of the work done by the damping term by

$$W^{\text{damping}}_{k+1} = -(q_{k+1} - q_k)^\top C v_{k+\gamma} \approx -\int_{t_k}^{t_{k+1}} v^T C v \; dt. \tag{100}$$

Lemma

The variation of energy over a time–step performed by the scheme is

$$\begin{aligned}\Delta\mathcal{E} - W^{\text{ext}}_{k+1} \quad - \quad & W^{\text{damping}}_{k+1} = (\frac{1}{2} - \gamma)\|(q_{k+1} - q_k)\|^2_K + P^\top_{k+1} U_{k+1/2} \\ & + \frac{h}{2}(2\beta - \gamma)\left[(q_{k+1} - q_k)^\top K (v_{k+1} - v_k) + \|(v_{k+1} - v_k)\|^2_C\right] \\ & - \frac{h}{2}(2\beta - \gamma)\left[(v_{k+1} - v_k)^\top (F_{k+1} - F_k) - (a_{k+1} - a_k)^\top G P_{k+1}\right].\end{aligned} \tag{101}$$

Energy analysis for Newmark's scheme

Define an discrete "algorithmic energy" (discrete storage function) of the form

$$\mathcal{K}(q, v, a) = \mathcal{E}(q, v) + \frac{h^2}{4}(2\beta - \gamma)a^\top Ma. \tag{102}$$

Proposition

The variation of the "algorithmic" energy $\Delta\mathcal{K}$ over a time–step performed by the nonsmooth Newmark scheme is

$$\begin{aligned}\Delta\mathcal{K} - W^{\text{ext}}_{k+1} - W^{\text{damping}}_{k+1} \quad = \quad & (\frac{1}{2} - \gamma)\left[\|q_{k+1} - q_k\|^2_K + \frac{h}{2}(2\beta - \gamma)\|(a_{k+1} - a_k)\|^2_M\right] \\ & + U^\top_{k+1/2} P_{k+1}.\end{aligned} \tag{103}$$

Moreover, the nonsmooth Newmark scheme dissipates the "algorithmic" energy $\mathcal{K}$ in the following sense

$$\Delta\mathcal{K} - W^{\text{ext}}_{k+1} - W^{\text{damping}}_{k+1} \leqslant 0, \tag{104}$$

for

$$2\beta \geqslant \gamma \geqslant \frac{1}{2}. \tag{105}$$

Energy analysis for HHT scheme

Augmented dynamics

Let us introduce the modified dynamics

$$Ma(t) + Cv(t) + Kq(t) = F(t) + \frac{\alpha}{\nu}[Kw(t) + Cx(t) - y(t)] \tag{106}$$

and the following auxiliary dynamics that filter the previous one

$$\begin{array}{rcl} \nu h\dot{w}(t) + w(t) & = & \nu h\dot{q}(t) \\ \nu h\dot{x}(t) + x(t) & = & \nu h\dot{v}(t) \\ \nu h\dot{y}(t) + y(t) & = & \nu h\dot{F}(t) \end{array} \tag{107}$$

Energy analysis for HHT scheme

Discretized Augmented dynamics

The equation (107) are discretized as follows

$$\begin{aligned}
\nu(w_{k+1} - w_k) + \frac{1}{2}(w_{k+1} + w_k) &= \nu(q_{k+1} - q_k) \\
\nu(x_{k+1} - x_k) + \frac{1}{2}(x_{k+1} + x_k) &= \nu(v_{k+1} - v_k) \\
\nu(y_{k+1} - y_k) + \frac{1}{2}(y_{k+1} + y_k) &= \nu(F_{k+1} - F_k)
\end{aligned} \tag{108}$$

or rearranging the terms

$$\begin{aligned}
(\frac{1}{2} + \nu)w_{k+1} + (\frac{1}{2} - \nu)w_k &= \nu(q_{k+1} - q_k) \\
(\frac{1}{2} + \nu)x_{k+1} + (\frac{1}{2} - \nu)x_k &= \nu(v_{k+1} - v_k) \\
(\frac{1}{2} + \nu)y_{k+1} + (\frac{1}{2} - \nu)y_k &= \nu(F_{k+1} - F_k)
\end{aligned} \tag{109}$$

With the special choice $\nu = \frac{1}{2}$, we obtain the HHT scheme collocation that is

$$Ma_{k+1} + (1-\alpha)[Kq_{k+1} + Cv_{k+1}] + \alpha[Kq_k + Cv_k] = (1-\alpha)F_{k+1} + \alpha F_k \tag{110}$$

Energy analysis for nonsmooth HHT scheme

Define an discrete "algorithmic energy" (discrete storage function) of the form

$$\mathcal{H}(q,v,a,w) = \mathcal{E}(q,v) + \frac{h^2}{4}(2\beta-\gamma)a^\top Ma + 2\alpha(1-\gamma)w^\top Kw. \tag{111}$$

Let us define the approximation of works as follows:

$$W^{\text{ext}}_{k+1} = (q_{k+1}-q_k)^\top\left[(1-\alpha)F_{k,\gamma}+\alpha F_{k-1,\gamma}\right] \approx \int_{t_k}^{t_{k+1}} Fv\ dt \tag{112}$$

and

$$W^{\text{damping}}_{k+1} = -(q_{k+1}-q_k)^\top C\left[(1-\alpha)v_{k,\gamma}+\alpha v_{k-1,\gamma}\right] \approx -\int_{t_k}^{t_{k+1}} v^T Cv\ dt. \tag{113}$$

Energy analysis for nonsmooth HHT scheme

Proposition

The variation of the "algorithmic" energy $\Delta\mathcal{H}$ over a time–step performed by the nonsmooth HHT scheme is

$$\begin{array}{rcl}\Delta\mathcal{H} - W^{\text{ext}}_{k+1} - W^{\text{damping}}_{k+1} & = & U^\top_{k+1/2}P_{k+1} - \frac{1}{2}h^2(\gamma-\frac{1}{2})(2\beta-\gamma)\|(a_{k+1}-a_k)\|^2_M \\ & & -(\gamma-\frac{1}{2}-\alpha)\|q_{k+1}-q_k\|^2_K - 2\alpha(1-\gamma)\|z_{k+1}-z_k\|^2_K.\end{array} \tag{114}$$

Moreover, the nonsmooth HHT scheme dissipates the "algorithmic" energy $\mathcal{H}$ in the following sense

$$\Delta\mathcal{H} - W^{\text{ext}}_{k+1} - W^{\text{damping}}_{k+1} \leqslant 0, \tag{115}$$

if

$$2\beta \geqslant \gamma \geqslant \frac{1}{2} \quad \textit{and} \quad 0 \leqslant \alpha \leqslant \gamma - \frac{1}{2} \leqslant \frac{1}{2}. \tag{116}$$

Energy analysis for nonsmooth schemes

Conclusions

- For the Moreau–Jean, a simple variant allows us to obtain a scheme which always dissipates energy.
- For the Newmark and the HHT scheme with retrieve the dissipation properties as the smooth case. The term associated with impact is added is the balance.
- For the generalized–α, similar analysis can be performed but some issues in the interpretation of results. New variant of the generalized–α scheme has been proposed
- Open Problem: We get dissipation inequality for discrete with quadratic storage function and plausible supply rate. The rest step is to conclude to the stability of the scheme with this argument. At least, we can bound discrete variable and conclude to the convergence of the scheme.

Conclusions

- Last developments: Nonsmooth generalized with stabilization of position, velocity and acceleration constraints (Bru, 2018)
- Most of the time integrator schemes and discrete solvers can be found in

Siconos. an open-source software for modeling, simulation and control of nonsmooth dynamical systems
http://github.com/siconos/siconos
Thank you for your attention.

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