Numerical methods for nonsmooth mechanical systems

Vincent Acary
INRIA Rhône–Alpes, Grenoble.

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Objectives of the lecture

- Principles and Design of Event–tracking (Event–Driven) schemes. Pros and cons.
- Principles and Design of Event–capturing (Time–stepping) schemes. Pros and cons.
- Comparison between Event–tracking and Event–capturing schemes
- Newmark-type schemes for flexible multibody systems and FEM applications.
- Toward higher order schemes and adaptive time–step strategies

Event-tracking schemes

Nonsmooth Lagrangian Dynamics

Definition (Nonsmooth Lagrangian Dynamics)

$$\begin{cases} M(q)dv + F(t, q, v^+)dt = di \\ \\ v^+ = \dot{q}^+ \end{cases} \tag{1}$$

where di is the reaction measure and dt is the Lebesgue measure.

Decomposition of measure

$$\begin{cases} dv = & \gamma\, dt + & (v^+ - v^-)\, d\nu + & dv_s \\ di = & f\, dt + & p\, d\nu + & di_s \end{cases} \tag{2}$$

Impact equations and Smooth Lagrangian dynamics

Substituting the decomposition of measures into the nonsmooth Lagrangian Dynamics, one obtains

Definition (Impact equations)

$$M(q)(v^+ - v^-)d\nu = pd\nu, \tag{3}$$

or

$$M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \tag{4}$$

Definition (Smooth Dynamics between impacts)

$$M(q)\gamma dt + F(t,q,v)dt = fdt \tag{5}$$

or

$$M(q)\gamma^+ + F(t,q,v^+) \quad = \quad f^+ \quad [dt - a.e.] \tag{6}$$

The smooth dynamics and the impact equations

The impact equations

The impact equations can be written at the time, t_i of discontinuities:

$$M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \tag{7}$$

This equation will be solved at the time of impact together with an impact law. That is for an Newton impact law

$$\begin{cases} M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \\ \\ U_{\mathrm{N}}^+(t_i) = H^T(q(t_i))v^+(t_i) \\ U_{\mathrm{N}}^-(t_i) = H^T(q(t_i))v^-(t_i) \\ p_i = H(q(t_i))P_{\mathrm{N},i} \\ \\ 0 \leqslant U_{\mathrm{N}}^+(t_i) + eU_{\mathrm{N}}^-(t_i) \perp P_{\mathrm{N},i} \geqslant 0 \end{cases} \tag{8}$$

The smooth dynamics and the impact equations

The impact equations reduced on the local unknowns

One obtains the following LCP at time t_i of discontinuities of v:

$$\begin{cases} U_{\mathrm{N}}^{+}(t_i) = H(q(t_i))(M(q(t_i)))^{-1}H(q(t_i))P_{\mathrm{N},i} + U_{\mathrm{N}}^{-}(t_i) \\ 0 \leqslant U_{\mathrm{N}}^{+}(t_i) + eU_{\mathrm{N}}^{-}(t_i) \perp P_{\mathrm{N},i} \geqslant 0 \end{cases} \quad (9)$$

if the matrix $M(q(t_i))$ is assumed to be invertible.

The smooth dynamics and the impact equations

The smooth dynamics

The following smooth system are then to be solved $(dt - a.e.)$:

$$\begin{cases} M(q(t))\gamma^+(t) + F(t, q, v^+) = f^+(t) \\ g = g(q(t)) \\ f^+ = H(q)F^+(t) \\ 0 \leqslant g \perp F^+(t) \geqslant 0 \end{cases} \tag{10}$$

Reformulations of the unilateral constraints on Different kinematics levels

Differentiation of the constraints w.r.t time

The constraints $g = g(q(t))$ can de differentiate with respect to time as follows in the Lagrangian setting:

$$\begin{cases} \dot{g}(q(t^+)) = U_{\mathrm{N}}^+(t) = \nabla g^T(q(t))v^+(t) \\ \\ \ddot{g}(q(t^+)) = \dot{U}_{\mathrm{N}}^+(t) = \Gamma_{\mathrm{N}}(t^+) = \nabla g^T(q(t))\gamma^+(t) + \frac{d}{dt}(\nabla g^T(q(t)))v^+(t) \end{cases} \tag{11}$$

Comments. Index reduction techniques.

Solving the smooth dynamics requires that the complementarity condition $0 \leqslant g \perp F^+(t) \geqslant 0$ must be written now at different kinematic level, i.e. in terms of right velocity U_{N}^+ and in terms of accelerations Γ_{N}^+.

Reformulations of the unilateral constraints on Different kinematics levels

At the velocity level

Assuming that U_{N}^{+} is right-continuous by definition of the right limit of a B.V. function, the complementarity condition implies, in terms of velocity, the following relation,

$$-F^{+} \in \begin{cases} 0 & \text{if } g > 0 \\ 0 & \text{if } g = 0, U_{\mathrm{N}}^{+} > 0 \\ ]-\infty, 0] & \text{if } g = 0, U_{\mathrm{N}}^{+} = 0 \end{cases} \quad . \tag{12}$$

A rigorous proof of this assertion can be found in (Glocker, 2001).

Reformulations of the unilateral constraints on Different kinematics levels

Equivalent formulations

- Inclusion into $N_{\mathbb{R}^+}(U_{\mathrm{N}}^+)$

$$-F^+ \in \begin{cases} 0 & \text{if } g > 0 \\ N_{\mathbb{R}^+}(U_{\mathrm{N}}^+) & \text{if } g = 0 \end{cases} \tag{12}$$

- Inclusion into $N_{T_{\mathbb{R}^+(g)}}(U_{\mathrm{N}}^+)$

$$-F^+ \in N_{T_{\mathbb{R}^+(g)}}(U_{\mathrm{N}}^+) \tag{13}$$

- In a complementarity formalism

$$\begin{array}{ll} \text{if } g = 0 & 0 \leqslant U_{\mathrm{N}}^+ \perp F^+ \geqslant 0 \\ \text{if } g > 0 & F^+ = 0 \end{array} \tag{14}$$

Reformulations of the unilateral constraints on Different kinematics levels

At the acceleration level

In the same way, the complementarity condition can be written at the acceleration level as follows.

$$-F^{+} \in \begin{cases} 0 & \text{if } g > 0 \\ 0 & \text{if } g = 0, U_{\mathrm{N}}^{+} > 0 \\ 0 & \text{if } g = 0, U_{\mathrm{N}}^{+} = 0, \Gamma_{\mathrm{N}}^{+} > 0 \\ ]-\infty, 0] & \text{if } g = 0, U_{\mathrm{N}}^{+} = 0, \Gamma_{\mathrm{N}}^{+} = 0 \end{cases} \tag{15}$$

A rigorous proof of this assertion can be found in (Glocker, 2001).

Reformulations of the unilateral constraints on Different kinematics levels

Equivalent formulations

- Inclusion into a cone $N_{\mathbb{R}^+}(\Gamma_{\mathrm{N}}^+)$

$$-F^+ \in \begin{cases} 0 & \text{if } g > 0 \\ 0 & \text{if } g = 0, U_{\mathrm{N}}^+ > 0 \\ N_{\mathbb{R}^+}(\Gamma_{\mathrm{N}}^+) \end{cases} \tag{15}$$

- Inclusion into $N_{T_{T_{\mathbb{R}^+}(g)}(U_{\mathrm{N}}^+)}(\Gamma_n^+)$

$$-F^+ \in N_{T_{T_{\mathbb{R}^+}(g)}(U_{\mathrm{N}}^+)}(\Gamma_n^+) \tag{16}$$

- In the complementarity formalism,

$$\begin{array}{ll} \text{if } g = 0, U_{\mathrm{N}}^+ = 0 & 0 \leqslant \Gamma_{\mathrm{N}}^+ \perp F^+ \geqslant 0 \\ \text{otherwise} & F^+ = 0 \end{array} \tag{17}$$

Reformulations of the unilateral constraints on Different kinematics levels

Trivial inclusions

$$N_K(g(q)) \supset N_{T_{\mathbb{R}^+}(g(q))}(U_{\mathrm{N}}^+) \supset N_{T_{T_{\mathbb{R}^+}(g(q))}(U_{\mathrm{N}}^+)}(\Gamma_n^+) \qquad (18)$$

Reformulations of the smooth dynamics at acceleration level.

The smooth dynamics as an inclusion

$$\begin{cases} M(q(t))\gamma^+(t) + F(t, q, v^+) = f^+(t) \\ \Gamma_{\mathrm{N}} = \nabla_q^T g(q)\gamma^+ + \frac{d}{dt}(\nabla_q^T g(q))v^+ \\ f^+ = \nabla_q g(q(t))F^+ \\ -F^+ \in N_{T_{T_{\mathbb{R}^+}(g)}(U_{\mathrm{N}}^+)}(\Gamma_n) \end{cases} \tag{19}$$

Reformulations of the smooth dynamics at acceleration level.

The smooth dynamics as a LCP

When the condition, $g = 0, U_{\mathrm{N}}^{+} = 0$ is satisfied, we obtain the following LCP

$$\begin{cases} M(q(t))\gamma^{+}(t) + F(t,q,v^{+}) = \nabla_q g(q(t))F^{+}(t) \\ \\ \Gamma_{\mathrm{N}}^{+} = \nabla_q g^T(q)\gamma^{+} + \frac{d}{dt}(\nabla_q g^T(q))v^{+} \\ \\ 0 \leqslant \Gamma_{\mathrm{N}}^{+} \perp F^{+} \geqslant 0 \end{cases} \quad (20)$$

which can be reduced on variable Γ_{N}^{+} and F^{+}, if $M(q(t))$ is invertible,

$$\begin{cases} \Gamma_{\mathrm{N}}^{+} = \nabla_q g^T(q)M^{-1}(q(t))(-F(t,q,v^{+})) + \frac{d}{dt}(\nabla_q g^T(q))v^{+} \\ \qquad\qquad + \nabla_q g(q)M^{-1}\nabla_q g(q(t))F^{+}(t) \\ \\ 0 \leqslant \Gamma_{\mathrm{N}}^{+} \perp F^{+} \geqslant 0 \end{cases} \quad (21)$$

The case of a single contact.

Two modes for the nonsmooth dynamics

1. *The constraint is not active.* $F^+ = 0$

$$M(q)\gamma^+ + F(\cdot, q, v) = 0 \tag{22}$$

In this case, we associate to this step an integer, $status_k = 0$.

2. *The constraint is active.* Bilateral constraint $\Gamma_{\mathrm{N}}^+ = 0$,

$$\begin{bmatrix} M(q) & -\nabla_q g(q) \\ \nabla_q g^T(q) & 0 \end{bmatrix} \begin{bmatrix} \gamma^+ \\ F^+ \end{bmatrix} = \begin{bmatrix} -F(\cdot, q, v) \\ \nabla_q \dot{g}^T(q) v^+ \end{bmatrix} \tag{23}$$

In this case, we associate to this step an integer, $status_k = 1$.

The case of a single contact.

[Case 1] $status_k = 0$.

Integrate the system (22) on the time interval $[t_k, t_{k+1}]$

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Case 1.1 $g_{k+1} > 0$. — The constraint is still not active

$status_{k+1} \leftarrow 0$

Case 1.2 $g_{k+1} = 0, U_{\mathrm{N},k+1} < 0$ — An impact occurs

Solve the impact equation (9) with $U^- \leftarrow U_{\mathrm{N},k+1} < 0$

$U_{\mathrm{N},k+1} \leftarrow U^+$.

Two cases are then possible:

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Case 1.2.1 $U_+ > 0$. — The constraint ceases to be active

$status_{k+1} \leftarrow 0$.

Case 1.2.2 $U_+ = 0$. — The relative post-impact velocity vanishes

Solve the LCP (20) to obtain the new status.

Three cases are then possible:

Case 1.2.2.1 $\Gamma_{\mathrm{N},k+1} > 0, F_{k+1} = 0$ — The constraint is still not active

$status_{k+1} \leftarrow 0$.

Case 1.2.2.2 $\Gamma_{\mathrm{N},k+1} = 0, F_{k+1} > 0$ — The constraint has to be activated $status_{k+1} \leftarrow 1$.

Case 1.2.2.3 $\Gamma_{\mathrm{N},k+1} = 0, F_{k+1} = 0$ — This case is undetermined.

We need to know the value of $\dot{\Gamma}^+_{\mathrm{N}}$.

The case of a single contact.

[Case 1] $status_k = 0$.

Integrate the system (22) on the time interval $[t_k, t_{k+1}]$

-25pt-

Case 1.3 $g_{k+1} = 0, U_{\mathrm{N},k+1} = 0$ — we have grazing constraint

Solve the LCP (20) to obtain the new status assuming that $U^+ = U^- = U_{\mathrm{N},k+1}$.

Three cases are then possible:

- **Case 1.3.1** $\Gamma_{\mathrm{N},k+1} > 0, F_{k+1} = 0$ — The constraint is still not active
  $status_{k+1} \leftarrow 0$.
- **Case 1.3.2** $\Gamma_{\mathrm{N},k+1} = 0, F_{k+1} > 0$ — The constraint has to be activated $status_{k+1} \leftarrow 1$.
- **Case 1.3.3** $\Gamma_{\mathrm{N},k+1} = 0, F_{k+1} = 0$ — This case is undetermined.
  We need to know the value of $\dot{\Gamma}^+_{\mathrm{N}}$.

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Case 1.4 $g_{k+1} = 0, U_{\mathrm{N},k+1} > 0$ — Activation of constraints not detected.

Seek for the first time t_* such that $g(q(t_*)) = 0$.

$t_{k+1} \leftarrow t_*$.

Perform all of this procedure keeping with $status_k \leftarrow 0$.

Case 1.5 $g_{k+1} < 0$ — Activation of constraints not detected.

Seek for the first time t_* such that $g(q(t_*)) = 0$.

$t_{k+1} \leftarrow t_*$.

Perform all of this procedure keeping with $status_k \leftarrow 0$.

The case of a single contact.

[Case 2] $status_k = 1$

Integrate the system (23) on the time interval $[t_k, t_{k+1}]$

-25pt-

Case 2.1 $g_{k+1} \neq 0$ or $U_{\mathrm{N},k+1} = 0$
Something is wrong in the time integration or the drift from the constraints is too huge.

Case 2.2 $g_{k+1} = 0, U_{\mathrm{N},k+1} = 0$
In this case, we assume that $U^+ = U^- = U_{\mathrm{N},k+1}$ and we compute $\Gamma_{\mathrm{N},k+1}, F_{k+1}$ thanks to the LCP (20) assuming that $U^+ = U^- = U_{\mathrm{N},k+1}$. Three cases are then possible

25pt

- Case 2.2.1 $\Gamma_{\mathrm{N},k+1} = 0, F_{k+1} > 0$
  The constraint is still active. We set $status_{k+1} = 1$.
- Case 2.2.2 $\Gamma_{\mathrm{N},k+1} > 0, F_{k+1} = 0$
  The bilateral constraint is no longer valid. We seek for the time t_* such that $F^+ = 0$. We set $t_{k+1} = t_*$ and we perform the integration up to this instant. We perform all of these procedure at this new time t_{k+1}
- Case 2.2.3 $\Gamma_{\mathrm{N},k+1} = 0, F_{k+1} = 0$
  This case is undetermined. We need to know the value of $\dot{\Gamma}^+_{\mathrm{N}}$.

The case of a single contact.

Comments

- The Delassus example.
  In the one-contact case, a naive approach consists in to suppressing the constraint if $F_{k+1} < 0$ after a integration with a bilateral constraints.
  ➜ Work only for the one contact case.
- The role of the "ε"
  In practical situation, all of the test are made up to an accuracy threshold. All statements of the type $g = 0$ are replaced by $|g| < \varepsilon$. The role of these epsilons can be very important and they are quite difficult to size.

The case of a single contact.

Comments

- If the ODE solvers is able to perform the root finding of the function $g = 0$ for $status_k = 0$ and $F^+ = 0$ for $status_k = 1$
  ➜ the case 1.4, 1.5 and the case 2.2.2 can be suppressed.
- If the drift from the constraints is also controlled into the ODE solver by a error computation,
  ➜ the case 2.1 can also be suppressed
- Most of the case can be resumed into the following step
  - Continue with the same status
  - Compute $U_{N,k+1}, P_{k+1}$ thanks to the LCP (9)(impact equations).
  - Compute $\Gamma_{N,k+1}, F_{k+1}$ thanks to the LCP (20) (Smooth dynamics)

➜ Rearranging the cases, we obtain the following algorithm.

The case of a single contact. An algorithm

Require: $(g_k, U_{\mathrm{N},k}, status_k)$
Ensure: $(g_{k+1}, U_{\mathrm{N},k+1}, status_{k+1})$
  Time-integration of the system on $[t_k, t_{k+1}]$(22) if $status_k = 0$ or of the system (23) if $status_k = 1$ up to an event.
  if $g_{k+1} > 0$ **then**
    $status_{k+1} = 0$ *//The constraint is still not active. (case 1.1)*
  end if
  if $g_{k+1} = 0, U_{\mathrm{N},k+1} < 0$ **then**
    //The constraint is active $g_{k+1} = 0$ *and an impact occur* $U_{\mathrm{N},k+1} < 0$ *(case 1.2)*
    Solve the LCP (9) for $U_{\mathrm{N}}^- = U_{\mathrm{N},k+1}$; $U_{\mathrm{N},k+1} = U_{\mathrm{N}}^+$
    if $U_{\mathrm{N},k+1} > 0$ **then** $status_{k+1} = 0$
  end if
  if $g_{k+1} = 0, U_{\mathrm{N},k+1} = 0$ **then**
    //The constraint is active $g_{k+1} = 0$ *without impact (case 1.2.2, case 1.3, case 2.2)*
    solve the LCP (21)
    if $\Gamma_{\mathrm{N},k+1} = 0, F_{k+1} > 0$ **then**
      $status_{k+1} = 1$
    else if $\Gamma_{\mathrm{N},k+1} > 0, F_{k+1} = 0$ **then**
      $status_{k+1} = 0$
    else if $\Gamma_{\mathrm{N},k+1} = 0, F_{k+1} = 0$ **then**
      //Undetermined case.
    end if
  end if

The multi-contact case and the index-sets

Index sets

The index set I is the set of all unilateral constraints in the system

$$I = \{1 \dots \nu\} \subset \mathbb{N} \tag{24}$$

The index-set I_c is the set of all active constraints of the system,

$$I_c = \{\alpha \in I, g^{\alpha} = 0\} \subset I \tag{25}$$

and the index-set I_s is the set of all active constraints of the system with a relative velocity equal to zero,

$$I_s = \{\alpha \in I_c, U_{\mathrm{N}}^{\alpha} = 0\} \subset I_c \tag{26}$$

The multi-contact case and the index-sets

Impact equations

$$\begin{cases} M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \\ \\ U_{\mathrm{N}}^+(t_i) = \nabla_q g^T(q(t_i))v^+(t_i) \\ U_{\mathrm{N}}^-(t_i) = \nabla_q g^T(q(t_i))v^-(t_i) \\ p_i = \nabla_q g(q(t_i))P_{\mathrm{N},i} \\ \\ P_{\mathrm{N},i}^{\alpha} = 0; U_{\mathrm{N}}^{\alpha,+}(t_i) = U_{\mathrm{N}}^{\alpha,-}(t_i), \quad \forall \alpha \in I \setminus I_c \\ \\ 0 \leqslant U_{\mathrm{N}}^{+,\alpha}(t_i) + eU_{\mathrm{N}}^{-,\alpha}(t_i) \perp P_{\mathrm{N},i}^{\alpha} \geqslant 0, \quad \forall \alpha \in I_c \end{cases} \tag{27}$$

Using the fact that $P_{\mathrm{N},i}^{\alpha} = 0$ for $\alpha \in I \setminus I_c$, this problem can be reduced on the local unknowns $U_{\mathrm{N}}^+(t_i), P_{\mathrm{N},i} \quad \forall \alpha \in I_c$.

The multi-contact case and the index-sets

Modes for the smooth Dynamics

- The smooth unilateral dynamics as a LCP

$$\begin{cases} M(q)\gamma^+ + F_{int}(\cdot,q,v) = F_{ext} + \nabla_q g(q) F^+ \\ \Gamma_{\rm N}^+ = \nabla_q g^T(q)\gamma^+ + \frac{d}{dt}(\nabla_q g^T(q))v^+ \\ F^{+,\alpha} = 0, \quad \forall \alpha \in I \setminus I_s \\ 0 \leqslant \Gamma_{\rm N}^{+,\alpha} \perp F^{+,\alpha} \geqslant 0 \quad \forall \alpha \in I_s \end{cases} \tag{28}$$

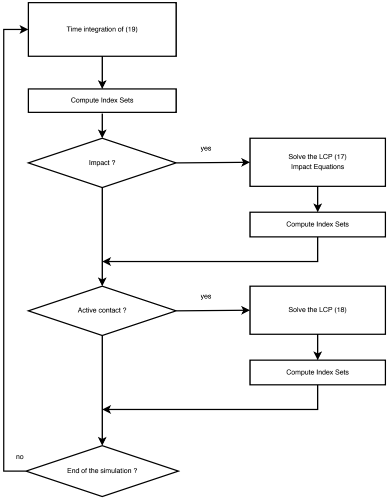
- The smooth bilateral dynamics

$$\begin{cases} M(q)\gamma^+ + F_{int}(\cdot,q,v) = F_{ext} + \nabla_q g(q) F^+ \\ \Gamma_{\rm N}^+ = \nabla_q g^T(q)\gamma^+ + \frac{d}{dt}(\nabla_q g^T(q))v^+ \\ F^{+,\alpha} = 0, \quad \forall \alpha \in I \setminus I_s \\ \Gamma_{\rm N}^{+,\alpha} = 0, \quad \forall \alpha \in I_s \end{cases} \tag{29}$$

The multi-contact case and the index-sets. an algorithm

Require: $(g_k, U_{\mathrm{N},k}, I_{c,k}, I_{s,k})$,
Ensure: $(g_{k+1}, U_{\mathrm{N},k+1}, I_{c,k+1}, I_{s,k+1})$
  Time-integration on $[t_k, t_{k+1}]$ of the system (29) according to $I_{c,k}$ and $I_{s,k}$ up to an event.
  Compute the temporary index-sets $I_{c,k+1}$ and $I_{s,k+1}$.
  if $I_{c,k+1} \setminus I_{s,k+1} \neq \emptyset$ **then**
    //Impacts occur.
    Solve the LCP (27).
    Update the index-set $I_{c,k+1}$ and temporary $I_{s,k+1}$
    Check that $I_{c,k+1} \setminus I_{s,k+1} = \emptyset$
  end if
  if $I_{s,k+1} \neq \emptyset$ **then**
    Solve the LCP (28)
    for $\alpha \in I_{s,k+1}$ **do**
      if $\Gamma_{\mathrm{N},\alpha,k+1} > 0, F_{\alpha,k+1} = 0$ **then**
        remove α from $I_{s,k+1}$ and $I_{c,k+1}$
      else if $\Gamma_{\mathrm{N},\alpha,k+1} = 0, F_{\alpha,k+1} = 0$ **then**
        //Undetermined case.
      end if
    end for
  end if
  // Go to the next time step

The multi-contact case and the index-sets



Comments and extensions

Extensions to Coulomb's friction

The set I_r is the set of sticking or rolling contact:

$$I_r = \{\alpha \in I_s, U_{\mathrm{N}}^{\alpha} = 0, \|U_{\mathrm{T}}\| = 0\} \subset I_s, \tag{30}$$

is the set of sticking or rolling contact, and

$$I_t = \{\alpha \in I_s, U_{\mathrm{N}}^{\alpha} = 0, \|U_{\mathrm{T}}\| > 0\} \subset I_s, \tag{31}$$

is the set of slipping or sliding contact.

Remarks

In the 3D case, checking the events and the transition sticking/sliding and sliding/sticking is not a easy task.

Comments

Advantages and Weaknesses and the Event Driven schemes

- Advantages :
  - Low cost implementation of time integration solvers (re-use of existing ODE solvers).
  - Higher-order accuracy on free motion.
  - Pseudo-localization of the time of events with finite time-step.
- Weaknesses
  - Numerous events in short time.
  - Accumulation of impacts.
  - No convergence proof
  - Robustness with the respect to thresholds "ε". Tuning codes is difficult.

Event–Capturing (Time-stepping) schemes

Time Discretization of the nonsmooth dynamics

For sake of simplicity, the linear time invariant case is only considered.

$$\begin{cases} Mdv + (Kq + Cv^+)\,dt = F_{ext}\,dt + di. \\ v^+ = \dot{q}^+ \end{cases} \tag{32}$$

Integrating both sides of this equation over a time step $]t_k, t_{k+1}]$ of length h,

$$\begin{cases} \displaystyle\int_{]t_k,t_{k+1}]} Mdv + \int_{t_k}^{t_{k+1}} Cv^+ + Kq\,dt = \int_{t_k}^{t_{k+1}} F_{ext}\,dt + \int_{]t_k,t_{k+1}]} di\;, \\[2ex] \displaystyle q(t_{k+1}) = q(t_k) + \int_{t_k}^{t_{k+1}} v^+\,dt\;. \end{cases} \tag{33}$$

By definition of the differential measure dv,

$$\int_{]t_k,t_{k+1}]} M\,dv = M\int_{]t_k,t_{k+1}]} dv = M\left(v^+(t_{k+1}) - v^+(t_k)\right). \tag{34}$$

Note that the right velocities are involved in this formulation.

Time Discretization of the nonsmooth dynamics

The equation of the nonsmooth motion can be written under an integral form as:

$$\begin{cases} M\,(v(t_{k+1}) - v(t_k)) = \displaystyle\int_{t_k}^{t_{k+1}} -Cv^+ - Kq + F_{ext}\,dt + \int_{]t_k,t_{k+1}]} di\;, \\ q(t_{k+1}) = q(t_k) + \displaystyle\int_{t_k}^{t_{k+1}} v^+\,dt\;. \end{cases} \tag{35}$$

The following notations will be used:

- $q_k \approx q(t_k)$ and $q_{k+1} \approx q(t_{k+1})$,
- $v_k \approx v^+(t_k)$ and $v_{k+1} \approx v^+(t_{k+1})$,

Impulse as primary unknown

The impulse $\displaystyle\int_{]t_k,t_{k+1}]} di$ of the reaction on the time interval $]t_k, t_{k+1}]$ emerges as a natural unknown. we denote

$$p_{k+1} \approx \int_{]t_k,t_{k+1}]} di$$

Time Discretization of the nonsmooth dynamics

Interpretation

The measure di may be decomposed as follows :

$$di = f\,dt + pd\nu$$

where

- $f\,dt$ is the abs. continuous part of the measure di, and
- $pd\nu$ the atomic part.

Two particular cases:

- Impact at $t_* \in]t_k, t_{k+1}]$: If $f = 0$ and $pd\nu = p\delta_{t_{k+1}}$ then

$$p_{k+1} = p$$

- Continuous force over $]t_k, t_{k+1}]$: If $di = fdt$ and $p = 0$ then

$$p_{k+1} = \int_{t_k}^{t_{k+1}} f(t)\,dt$$

Time Discretization of the nonsmooth dynamics

Remark

- A pointwise evaluation of a (Dirac) measure is a non sense. It practice using the value

$$f_{k+1} \approx f(t_{k+1})$$

yield severe numerical inconsistencies, since

$$\lim_{h \to 0} f_{k+1} = +\infty$$

- Since discontinuities of the derivative v are to be expected if some shocks are occurring, i.e. di has some Dirac atoms within the interval $]t_k, t_{k+1}]$, it is not relevant to use high order approximations integration schemes for di. It may be shown on some examples that, on the contrary, such high order schemes may generate artefact numerical oscillations.

Time Discretization of the nonsmooth dynamics

Discretization of smooth terms

θ-method is used for the term supposed to be sufficiently smooth,

$$\int_{t_k}^{t_{k+1}} Cv + Kq \, dt \approx h \left[\theta(Cv_{k+1} + Kq_{k+1}) + (1-\theta)(Cv_k + Kq_k)\right]$$

$$\int_{t_k}^{t_{k+1}} F_{ext}(t) \, dt \approx h \left[\theta(F_{ext})_{k+1} + (1-\theta)(F_{ext})_k\right]$$

The displacement, assumed to be absolutely continuous is approximated by:

$$q_{k+1} = q_k + h \left[\theta v_{k+1} + (1-\theta) v_k\right] .$$

Time Discretization of the nonsmooth dynamics

Finally, introducing the expression of q_{k+1} in the first equation of (34), one obtains:

$$\left[M + h\theta C + h^2\theta^2 K\right](v_{k+1} - v_k) = -hCv_k - hKq_k - h^2\theta K v_k \\ +h\left[\theta(F_{ext})_{k+1}) + (1-\theta)(F_{ext})_k\right] + p_{k+1}\ , \quad (36)$$

which can be written :

$$v_{k+1} = v_{free} + \widehat{M}^{-1} p_{k+1} \quad (37)$$

where,

- the matrix $\widehat{M} = \left[M + h\theta C + h^2\theta^2 K\right]$ is usually called the iteration matrix and,
- The vector

$$\begin{aligned} v_{free} \quad = \quad & v_k + \widehat{M}^{-1}\Big[- hCv_k - hKq_k - h^2\theta K v_k \\ & +h\left[\theta(F_{ext})_{k+1}) + (1-\theta)(F_{ext})_k\right]\Big] \end{aligned}$$

  is the so-called "free" velocity, i.e. the velocity of the system when reaction forces are null.

Time Discretization of the kinematics relations

According to the implicit mind, the discretization of kinematic laws is proposed as follows.
For a constraint α,

$$U^{\alpha}_{k+1} = H^{\alpha\,T}(q_k)\; v_{k+1}\;,$$

$$p^{\alpha}_{k+1} = H^{\alpha}(q_k)\; P^{\alpha}_{k+1}\;, \quad p_{k+1} = \sum_{\alpha} p^{\alpha}_{k+1}\;,$$

where

$$P^{\alpha}_{k+1} \approx \int_{]t_k, t_{k+1}]} d\lambda^{\alpha}.$$

For the unilateral constraints, it is proposed

$$g^{\alpha}_{k+1} = g^{\alpha}_{k} + h\left[\theta U^{\alpha}_{k+1} + (1-\theta) U^{\alpha}_{k}\right]\;.$$

Discretization of the unilateral constraints

Recall that the unilateral constraint is expressed in terms of velocity as

$$-di \in N_{T_C(q)}(v^+) \tag{38}$$

or in local coordinates as

$$-d\lambda^\alpha \in N_{T_{\mathbb{R}_+}(g(q))}(U^{\alpha,+}) \tag{39}$$

The time discretization is performed by

$$-P^\alpha_{k+1} \in N_{T_{\mathbb{R}^+}(g^\alpha(\tilde{q}_{k+1}))}(U^\alpha_{k+1}) \tag{40}$$

where $\tilde{q}_{k+1}$ is a forecast of the position for the activation of the constraints, for instance,

$$\tilde{q}_{k+1} = q_k + \frac{h}{2} v_k$$

In the complementarity formalism, we obtain

$$\text{if } g^\alpha(\tilde{q}_{k+1}) \leqslant 0, \text{ then } \quad 0 \leqslant U^\alpha_{k+1} \perp P^\alpha_{k+1} \geqslant 0$$

Summary of the time discretized equations

One step linear problem
$$\begin{cases} v_{k+1} = v_{free} + \widehat{M}^{-1} p_{k+1} \\ q_{k+1} = q_k + h\,[\theta v_{k+1} + (1-\theta) v_k] \end{cases}$$

Relations
$$\begin{cases} U^{\alpha}_{k+1} = H^{\alpha\,T}(q_k)\; v_{k+1} \\ p^{\alpha}_{k+1} = H^{\alpha}(q_k)\; P^{\alpha}_{k+1} \end{cases}$$

Nonsmooth Law
$$\begin{cases} \text{ if } g^{\alpha}(\tilde{q}_{k+1}) \leqslant 0, \text{ then} \\ 0 \leqslant U^{\alpha}_{k+1} \perp P^{\alpha}_{k+1} \geqslant 0 \end{cases}$$

One step LCP

$$U_{k+1} = H^T(q_k) v_{free} + H^T(q_k) \widehat{M}^{-1} H(q_k)\; P_{k+1}$$

$$\text{if } g^{\alpha}_p \leqslant 0, \text{ then } 0 \leqslant U^{\alpha}_{k+1} \perp P^{\alpha}_{k+1} \geqslant 0$$

Moreau's Time stepping scheme

$$\begin{cases} M(q_{k+\theta})(v_{k+1} - v_k) - h\tilde{F}_{k+\theta} = H(q_{k+\theta})P_{k+1}, & (41a) \\ q_{k+1} = q_k + hv_{k+\theta}, & (41b) \\ U_{k+1} = H^T(q_{k+\theta})\, v_{k+1} & (41c) \\ -P_{k+1} \in \partial\psi_{T_{\mathbb{R}^m_+}(\tilde{y}_{k+\gamma})}(U_{k+1} + eU_k), & (41d) \\ \tilde{y}_{k+\gamma} = y_k + h\gamma U_k, \quad \gamma \in [0,1]. & (41e) \end{cases}$$

with $\theta \in [0,1], \gamma \geqslant 0$ and $x_{k+\alpha} = (1-\alpha)x_{k+1} + \alpha x_k$ and $\tilde{y}_{k+\gamma}$ is a prediction of the constraints.

Properties

- Convergence results for one constraints
- Convergence results for multiple constraints problems with acute kinetic angles
- No theoretical proof of order

Schatzman–Paoli's Time stepping scheme

$$\begin{cases} M(q_k+1)(q_{k+1}-2q_k+q_{k-1}) - h^2 F(t_{k+\theta}, q_{k+\theta}, v_{k+\theta}) = p_{k+1}, & (42a) \\ v_{k+1} = \dfrac{q_{k+1}-q_{k-1}}{2h}, & (42b) \\ -p_{k+1} \in N_K\left(\dfrac{q_{k+1}+eq_{k-1}}{1+e}\right), & (42c) \end{cases}$$

where N_K defined the normal cone to K.
For $K = \{q \in \mathbb{R}^n, y = g(q) \geqslant 0\}$

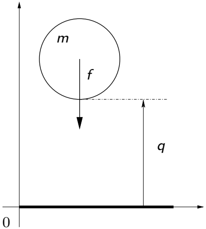
$$0 \leqslant g\left(\frac{q_{k+1}+eq_{k-1}}{1+e}\right) \perp \nabla g\left(\frac{q_{k+1}+eq_{k-1}}{1+e}\right) P_{k+1} \geqslant 0 \tag{43}$$

Properties

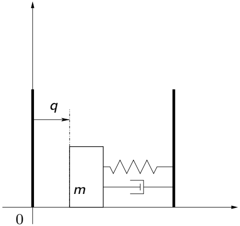
- Convergence results for one constraints
- Convergence results for multiple constraints problems with acute kinetic angles
- No theoretical proof of order

Academic examples

The bouncing Ball and the linear impacting oscillator



(a) Bouncing ball example



(b) Linear Oscillator example

Figure: Academic test examples with analytical solutions

Academic examples

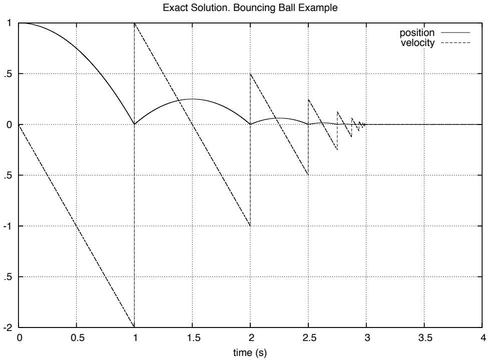


Figure: Analytical solution. Bouncing ball example

Academic examples

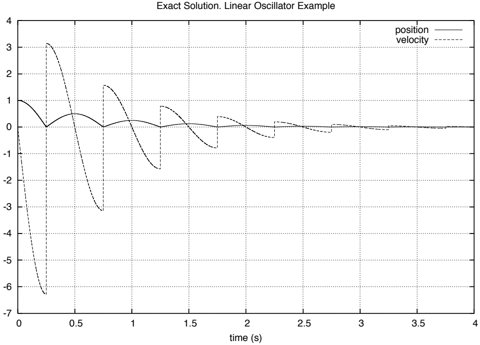


Figure: Analytical solution. Linear Oscillator

Measuring error and convergence

Convergence in the sense of filled-in graph (Moreau (1978))

$$gr^{\star}(f) = \{(t,x) \in [0,T] \times \mathbb{R}^n, 0 \leqslant t \leqslant T \text{ and } x \in [f(t^-), f(t^+)]\}. \tag{44}$$

Such graphs are closed bounded subsets of $[0,T] \times \mathbb{R}^n$, hence, we can use the Hausdorff distance between two such sets with a suitable metric:

$$d((t,x),(s,y)) = \max\{|t-s|, \|x-y\|\}. \tag{45}$$

Defining the excess of separation between two graphs by

$$e(gr^{\star}(f), gr^{\star}(g)) = \sup_{(t,x)\in gr^{\star}(f)} \inf_{(s,y)\in gr^{\star}(g)} d((t,x),(s,y)), \tag{46}$$

the Hausdorff distance between two filled-in graphs $h^{\star}$ is defined by

$$h^{\star}(gr^{\star}(f), gr^{\star}(g)) = \max\{e(gr^{\star}(f), gr^{\star}(g)), e(gr^{\star}(g), gr^{\star}(f))\}. \tag{47}$$

Measuring error and convergence

An equivalent grid-function norm to the function norm in $\mathcal{L}_1$

$$\|e\|_1 = h \sum_{i=0}^{N} |f_i - f(t_i)| \tag{48}$$

In the same way, the $p-norm$ can be defined by

$$\|e\|_p = \left(h \sum_{i=0}^{N} |f_i - f(t_i)|^p \right)^{1/p} \tag{49}$$

The computation of this two last norm is easier to implement for piecewise continuous analytical function than the Hausdorff distance.

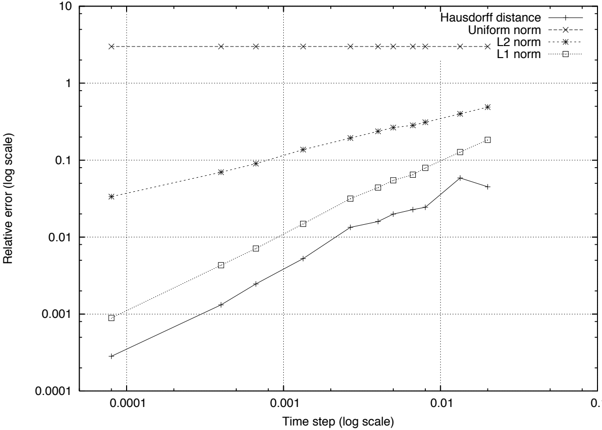
Global order of convergence.

Definition

A one-step time–integration scheme is of order q for a given norm $\|\cdot\|$ if there exists a constant C such that

$$\|e\| = Ch^q + \mathcal{O}(h^{q+1}) \tag{50}$$

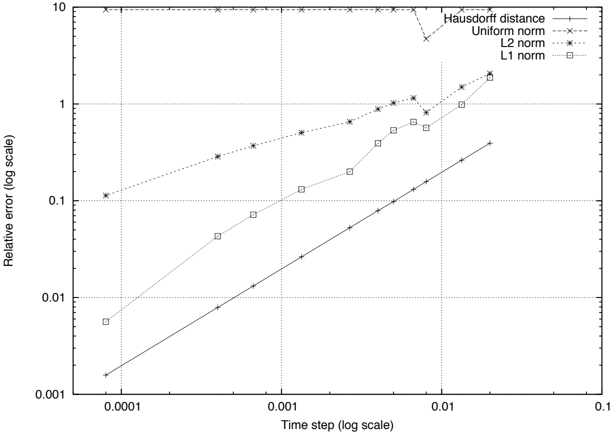
Empirical order of convergence. Moreau's time–stepping scheme



(a) The bouncing ball example

Figure: Empirical order of convergence of the Moreau's time-stepping scheme.

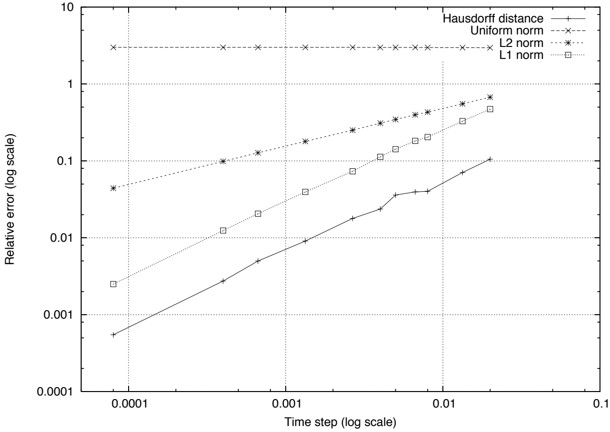
Empirical order of convergence. Moreau's time–stepping scheme



(a) The linear oscillator example

Figure: Empirical order of convergence of the Moreau's time-stepping scheme.

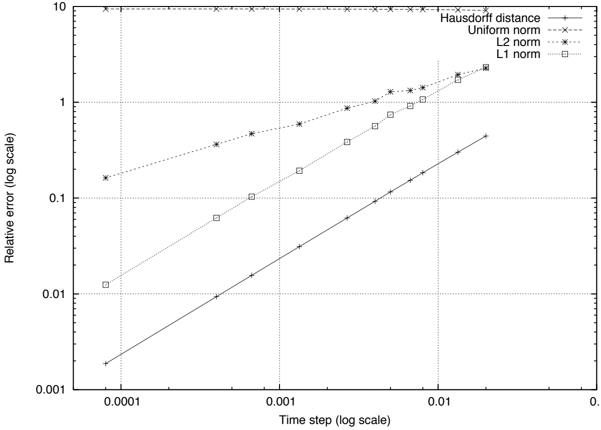
Empirical order of convergence. Schatzman–Paoli's time–stepping scheme



(a) The bouncing ball example

Figure: Empirical order of convergence of the Schatzman-Paoli's time-stepping scheme.

Empirical order of convergence. Schatzman–Paoli's time–stepping scheme



(a) The linear oscillator example

Figure: Empirical order of convergence of the Schatzman-Paoli's time-stepping scheme.

Comparison

State–of–the–art

Numerical time–integration methods for Nonsmooth Multibody systems (NSMBS):

Nonsmooth event capturing methods (Time–stepping methods)

- ⊕ robust, stable and proof of convergence
- ⊕ low kinematic level for the constraints
- ⊕ able to deal with finite accumulation
- ⊖ very low order of accuracy even in free flight motions

Nonsmooth event tracking methods (Event–driven methods)

- ⊕ high level integration of free flight motions
- ⊖ no proof of convergence
- ⊖ sensibility to numerical thresholds
- ⊖ reformulation of constraints at higher kinematic levels.
- ⊖ unable to deal with finite accumulation

Newmark-type schemes for flexible multibody systems and FEM applications.

Joint work with O. Brüls, Q.Z. Chen and G. Virlez (Université de Liège)

The Newmark scheme

Linear Time "Invariant" Dynamics without contact

$$\begin{cases} M\dot{v}(t) + Kq(t) + Cv(t) = f(t) \\ \dot{q}(t) = v(t) \end{cases} \tag{51}$$

The Newmark scheme (Newmark, 1959)

Principle

Given two parameters γ and β

$$\begin{cases} Ma_{k+1} = f_{k+1} - Kq_{k+1} - Cv_{k+1} \\ v_{k+1} = v_k + ha_{k+\gamma} \\ q_{k+1} = q_k + hv_k + \dfrac{h^2}{2} a_{k+2\beta} \end{cases} \tag{52}$$

Notations

$$f(t_{k+1}) = f_{k+1}, \quad x_{k+1} \approx x(t_{k+1}),$$
$$x_{k+\gamma} = (1-\gamma)x_k + \gamma x_{k+1} \tag{53}$$

The Newmark scheme

Implementation

Let us consider the following explicit prediction

$$\begin{cases} v_k^* = v_k + h(1-\gamma)a_k \\ q_k^* = q_k + hv_k + \frac{1}{2}(1-2\beta)h^2 a_k \end{cases} \quad (54)$$

The Newmark scheme may be written as

$$\begin{cases} a_{k+1} = \hat{M}^{-1}(-Kq_k^* - Cv_k^* + f_{k+1}) \\ v_{k+1} = v_k^* + h\gamma a_{k+1} \\ q_{k+1} = q_k^* + h^2\beta a_{k+1} \end{cases} \quad (55)$$

with the iteration matrix

$$\hat{M} = M + h^2\beta K + \gamma hC \quad (56)$$

The Newmark scheme

Properties

- One–step method in state. (Two steps in position)
- Second order accuracy if and only if $\gamma = \frac{1}{2}$
- Unconditional stability for $2\beta \geqslant \gamma \geqslant \frac{1}{2}$

Average acceleration (Trapezoidal rule)	implicit	$\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$
central difference	explicit	$\gamma = \frac{1}{2}$ and $\beta = 0$
linear acceleration	implicit	$\gamma = \frac{1}{2}$ and $\beta = \frac{1}{6}$
Fox–Goodwin (Royal Road)	implicit	$\gamma = \frac{1}{2}$ and $\beta = \frac{1}{12}$

Table: Standard value for Newmark scheme ((Hughes, 1987, p 493)Géradin and Rixen (1993))

The Newmark scheme

High frequencies dissipation

- In flexible multibody Dynamics or in standard structural dynamics discretized by FEM, high frequency oscillations are artifacts of the semi-discrete structures.
- In Newmark's scheme, maximum high frequency damping is obtained with

$$\gamma \gg \frac{1}{2}, \quad \beta = \frac{1}{4}(\gamma + \frac{1}{2})^2 \tag{57}$$

example for $\gamma = 0.9$, $\beta = 0.49$

The Newmark scheme

From (Hughes, 1987) :

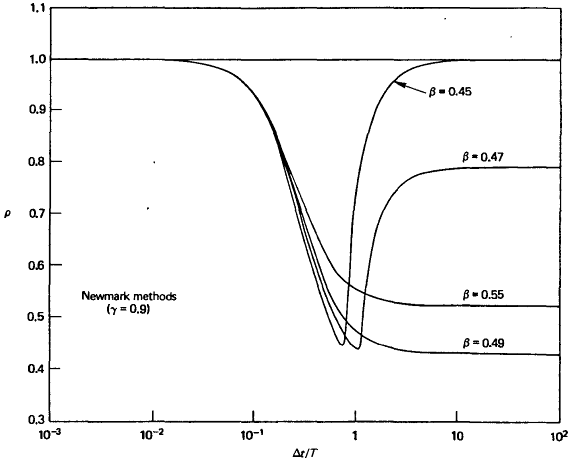


Figure 9.1.3 Spectral radii for Newmark methods for varying β [9].

The Hilber–Hughes–Taylor scheme. Hilber et al. (1977)

Objectives

- to introduce numerical damping without dropping the order to one.

Principle

Given three parameters γ, β and α and the notation

$$M\ddot{q}_{k+1} = -(Kq_{k+1} + Cv_{k+1}) + F_{k+1} \tag{58}$$

$$\begin{cases} Ma_{k+1} = M\ddot{q}_{k+1+\alpha} = -(Kq_{k+1+\alpha} + Cv_{k+1+\alpha}) + F_{k+1+\alpha} \\ v_{k+1} = v_k + ha_{k+\gamma} \\ q_{k+1} = q_k + hv_k + \dfrac{h^2}{2} a_{k+2\beta} \end{cases} \tag{59}$$

Standard parameters (Hughes, 1987, p532) are

$$\alpha \in [-1/3, 0], \gamma = (1 - 2\alpha/2) \text{ and } \beta = (1-\alpha)^2/4 \tag{60}$$

Warning

The notation are abusive. a_{k+1} is not the approximation of the acceleration at t_{k+1}

The HHT scheme

Properties

- Two–step method in state. (Three–steps method in position)
- Unconditional stability and second order accuracy with the previous rule. (60)
- For $\alpha = 0$, we get the trapezoidal rule and the numerical dissipation increases with $|\alpha|$.

The HHT scheme

From (Hughes, 1987) :

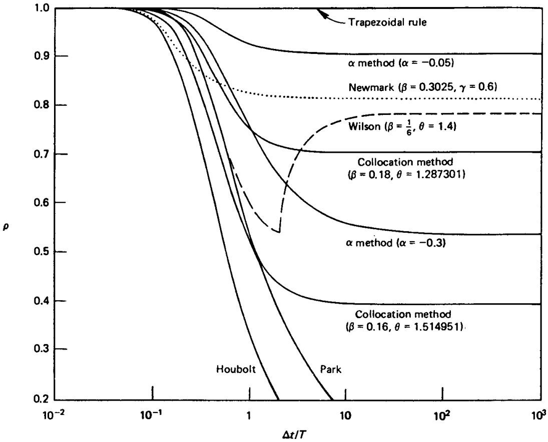


Figure 9.3.1 Spectral radii for α-methods, optimal collocation schemes, and Houbolt, Newmark, Park, and Wilson methods [22].

Generalized α-methods (Chung and Hulbert, 1993)

Principle

Given three parameters γ, β, α_m and α_f and the notation

$$M\ddot{q}_{k+1} = -(Kq_{k+1} + Cv_{k+1}) + F_{k+1} \tag{61}$$

$$\begin{cases} Ma_{k+1-\alpha_m} = M\ddot{q}_{k+1-\alpha_f} \\ v_{k+1} = v_k + ha_{k+\gamma} \\ q_{k+1} = q_k + hv_k + \dfrac{h^2}{2} a_{k+2\beta} \end{cases} \tag{62}$$

Standard parameters (Chung and Hulbert, 1993) are chosen as

$$\alpha_m = \frac{2\rho_\infty - 1}{\rho_\infty + 1}, \quad \alpha_f = \frac{\rho_\infty}{\rho_\infty + 1}, \quad \gamma = \frac{1}{2} + \alpha_f - \alpha_m \text{ and } \beta = \frac{1}{4}(\gamma + \frac{1}{2})^2 \tag{63}$$

where $\rho_\infty \in [0, 1]$ is the spectral radius of the algorithm at infinity.

Properties

- Two–step method in state.
- Unconditional stability and second order accuracy.
- Optimal combination of accuracy at low-frequency and numerical damping at high-frequency

A first naive approach

Direct Application of the HHT scheme to Linear Time "Invariant" Dynamics with contact

$$\begin{cases} M\dot{v}(t) + Kq(t) + Cv(t) = f(t) + r(t), \text{a.e} \\ \dot{q}(t) = v(t) \\ r(t) = G(q)\,\lambda(t) \\ g(t) = g(q(t)), \quad \dot{g}(t) = G^T(q(t))v(t), \\ 0 \leqslant g(t) \perp \lambda(t) \geqslant 0, \end{cases} \tag{64}$$

results in

$$\begin{cases} M\ddot{q}_{k+1} = -(Kq_{k+1} + Cv_{k+1}) + F_{k+1} + r_{k+1} \\ r_{k+1} = G_{k+1}\lambda_{k+1} \end{cases} \tag{65}$$

$$\begin{cases} Ma_{k+1} = M\ddot{q}_{k+1+\alpha} \\ v_{k+1} = v_k + ha_{k+\gamma} \\ q_{k+1} = q_k + hv_k + \dfrac{h^2}{2}a_{k+2\beta} \\ 0 \leqslant g_{k+1} \perp \lambda_{k+1} \geqslant 0, \end{cases} \tag{66}$$

A first naive approach

Direct Application of the HHT scheme to Linear Time "Invariant" Dynamics with contact

The scheme is not consistent for mainly two reasons:

- If an impact occur between rigid bodies, or if a restitution law is needed which is mandatory between semidiscrete structure, the impact law is not taken into account by the discrete constraint at position level
- Even if the constraint is discretized at the velocity level, i.e.

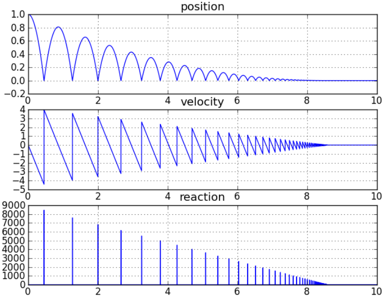
$$\text{if } \bar{g}_{k+1}, \text{ then } 0 \leqslant \dot{g}_{k+1} + e g_k \perp \lambda_{k+1} \geqslant 0 \tag{67}$$

the scheme is consistent only for $\gamma = 1$ and $\alpha = 0$ (first order approximation.)

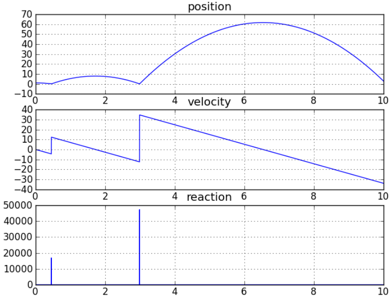
A first naive approach

Velocity based constraints with standard Newmark scheme ($\alpha = 0.0$)

Bouncing ball example. $m = 1$, $g = 9.81$, $x_0 = 1.0$ $v_0 = 0.0$, $e = 0.9$



$h = 0.001$, $\gamma = 1.0$, $\beta = \gamma/2$

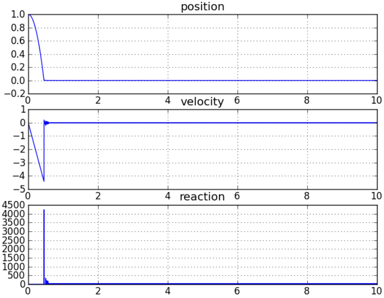


$h = 0.001$, $\gamma = 1/2$, $\beta = \gamma/2$

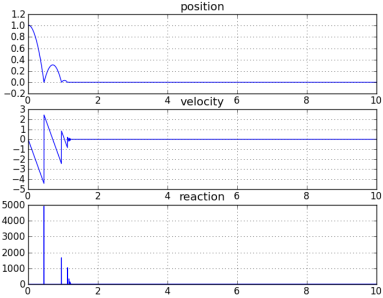
A first naive approach

Position based constraints with standard Newmark scheme ($\alpha = 0.0$)

Bouncing ball example. $m = 1$, $g = 9.81$, $v_0 = 0.0$, $e = 0.9$, $h = 0.001$, $\gamma = 1.0$, $\beta = \gamma/2$



$x_0 = 1.0$



$x_0 = 1.01$

The Nonsmooth Newmark and HHT scheme

Dynamics with contact and (possibly) impact

$$\begin{cases} M\,dv = F(t,q,v)\,dt + G(q)\;di \\ \dot{q}(t) = v^{+}(t), \\ g(t) = g(q(t)), \quad \dot{g}(t) = G^{T}(q(t))v(t), \\ \text{if } g(t) \leqslant 0, \quad 0 \leqslant g^{+}(t) + e\dot{g}^{-}(t) \perp \; di \geqslant 0, \end{cases} \tag{68}$$

The Nonsmooth Newmark and HHT scheme

Splitting the dynamics between smooth and nonsmooth part

$$M\,dv = Ma(t)\,dt + M\,dv^{\text{con}} \tag{69}$$

with

$$\begin{cases} Ma\,dt = F(t,q,v)\,dt \\ M\,dv^{\text{con}} = G(q)\;di \end{cases} \tag{70}$$

Different choices for the discrete approximation of the term $Ma\,dt$ and $M\,dv^{\text{con}}$

The Nonsmooth Newmark and HHT scheme

Principles

- As usual is the Newmark scheme, the smooth part of the dynamics $Ma\,dt = F(t,q,v)\,dt$ is collocated, i.e.

$$Ma_{k+1} = F_{k+1} \tag{71}$$

- the impulsive part a first order approximation is done over the time–step

$$M\Delta v^{con}_{k+1} = G_{k+1}\,\Lambda_{k+1} \tag{72}$$

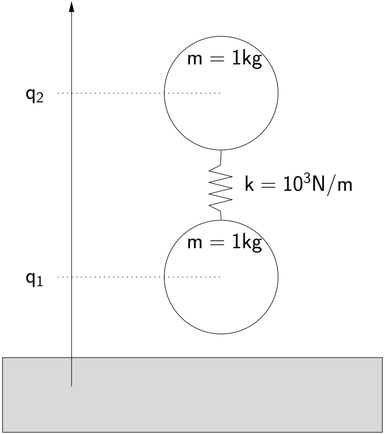
The Nonsmooth Newmark and HHT scheme

Principles

$$\begin{cases} Ma_{k+1} = F_{k+1+\alpha} \\ M\Delta v^{\text{con}}_{k+1} = G_{k+1}\ \Lambda_{k+1} \\ v_{k+1} = v_k + ha_{k+\gamma} + \Delta v^{\text{con}}_{k+1} \\ q_{k+1} = q_k + hv_k + \dfrac{h^2}{2} a_{k+2\beta} + \dfrac{1}{2} h\Delta v^{\text{con}}_{k+1} \end{cases} \tag{73}$$

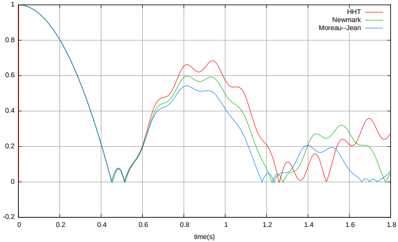
The Nonsmooth Newmark and HHT scheme

Example (Two balls oscillator with impact)

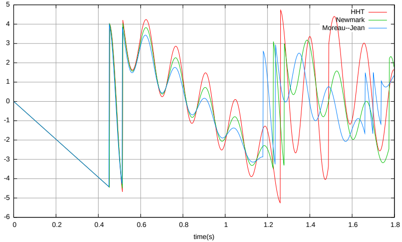


The Nonsmooth Newmark and HHT scheme

time–step : $h = 2e-3$. Moreau ($\theta = 1.0$). Newmark ($\gamma = 1.0, \beta = 0.5$). HHT ($\alpha = 0.1$)

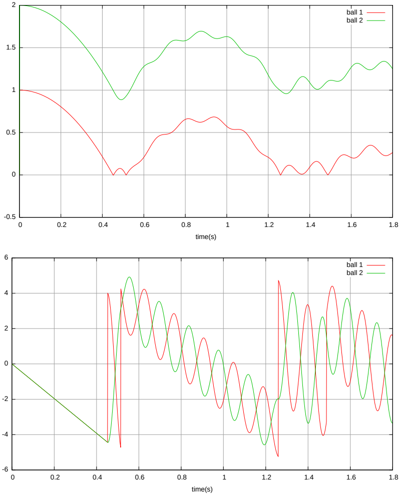


Position of the first ball

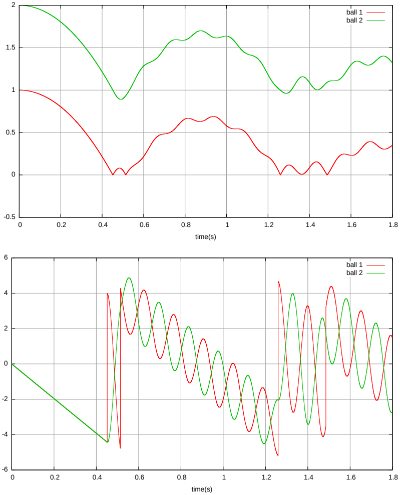


Velocity of the first ball

The Nonsmooth Newmark and HHT scheme



HHT $h = 1e-3, \alpha = 0.1$



Moreau time –step $h = 1e-5, \theta = 1.0$

The Nonsmooth Newmark and HHT scheme

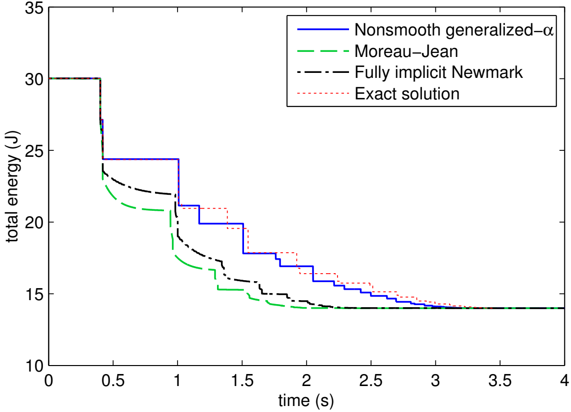


Figure 7. Numerical results for the total energy of the bouncing oscillator.

The Nonsmooth Newmark and HHT scheme

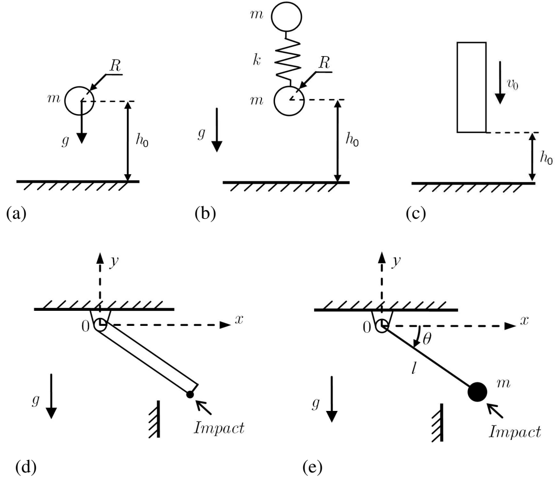


Figure 2. Examples: (a) bouncing ball; (b) linear vertical oscillator; (c) bouncing of an elastic bar; (d) bouncing of a nonlinear beam pendulum; (e)bouncing of a rigid pendulum

The Nonsmooth Newmark and HHT scheme

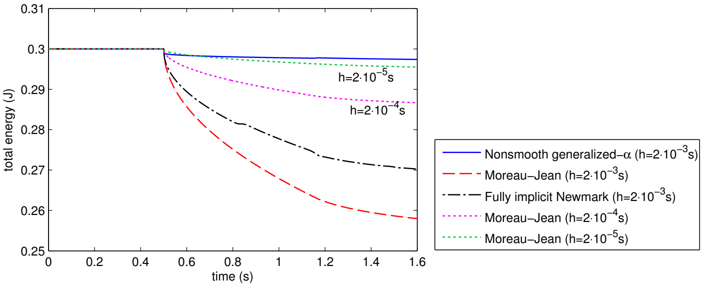


Figure 9. Numerical results for the total energy of the bouncing elastic bar

The Nonsmooth Newmark and HHT scheme

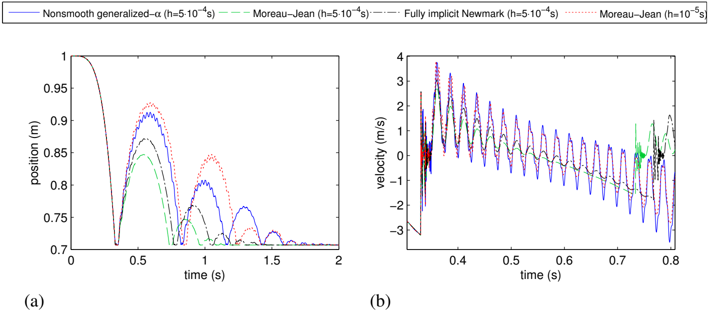


Figure 10. Numerical results for the impact of a flexible rotating beam: (a) position, (b) velocity.

The Nonsmooth Newmark and HHT scheme

Observed properties on examples

- the scheme is consistent and globally of order one.
- the scheme seems to share the stability property as the original HHT
- the scheme dissipates energy only in high-frequency oscillations (w.r.t the time–step.)

Conclusions & perspectives

- Extension to any multi–step schemes can be done in the same way.
- Improvements of the order by splitting.
- Recast into time–discontinuous Galerkin formulation.

Energy analysis

Time–continuous energy balance equations

Let us start with the "LTI" Dynamics

$$\begin{cases} M\,dv + (Kq + Cv)\;dt = F\;\;dt + \;\;di \\ dq = v^{\pm}\,dt \end{cases} \tag{74}$$

we get for the Energy Balance

$$d(v^\top Mv) + (v^+ + v^-)(Kq + Cv)\;dt \quad = \quad (v^+ + v^-)F\;\;dt + (v^+ + v^-)\;di \tag{75}$$

that is

$$2d\mathcal{E} := \;d(v^\top Mv) + 2q^\top Kdq \quad = \quad 2v^\top F\;\;dt - 2v^\top Cv\;\;dt + (v^+ + v^-)^\top\;\;di \tag{76}$$

with

$$\mathcal{E} := \frac{1}{2}v^\top Mv + \frac{1}{2}q^\top Kq. \tag{77}$$

Energy analysis

Time–continuous energy balance equations

If we split the differential measure in $di = \lambda \;dt + \sum_i p_i \delta_{t_i}$, we get

$$2d\mathcal{E} = \quad = \quad 2v^\top (F+\lambda) \;dt - 2v^\top C v \;dt + (v^+ + v^-)^\top p_i \delta_{t_i} \tag{78}$$

By integration over a time interval $[t_0, t_0]$ such that $t_i \in [t_0, t_1]$, we obtain an energy balance equation as

$$\begin{aligned} \Delta\mathcal{E} &:= \mathcal{E}(t_1) - \mathcal{E}(t_0) \\ &= \underbrace{\int_{t_0}^{t_1} v^\top F \;dt}_{W^{\text{ext}}} - \underbrace{\int_{t_0}^{t_1} v^\top C v \;dt}_{W^{\text{damping}}} + \underbrace{\int_{t_0}^{t_1} v^\top \lambda \;dt}_{W^{\text{con}}} + \underbrace{\frac{1}{2}\sum_i (v^+(t_i) + v^-(t_i))^\top p_i}_{W^{\text{impact}}} \end{aligned} \tag{79}$$

Energy analysis

Work performed by the reaction impulse *di*

- The term

$$W^{\text{con}} = \int_{t_0}^{t_1} v^\top \lambda \; dt \tag{80}$$

  is the work done by the contact forces within the time–step. If we consider perfect unilateral constraints, we have $W^{\text{con}} = 0$.

- The term

$$W^{\text{impact}} = \frac{1}{2} \sum_i (v^+(t_i) + v^-(t_i))^\top p_i \tag{81}$$

  represents the work done by the contact impulse p_i at the time of impact t_i. Since $p_i = G(t_i)P_i$ and if we consider the Newton impact law, we have

$$\begin{aligned} W^{\text{impact}} &= \frac{1}{2} \sum_i (v^+(t_i) + v^-(t_i))^\top G(t_i) P_i \\ &= \frac{1}{2} \sum_i (U^+(t_i) + U^-(t_i))^\top P_i \\ &= \frac{1}{2} \sum_i ((1-e)U^-(t_i))^\top P_i \leqslant 0 \text{ for } 0 \leqslant e \leqslant 1 \end{aligned} \tag{82}$$

  with the local relative velocity defines as $U(t) = G^\top(t) v(t)$

Energy analysis for Moreau–Jean scheme

Lemma

Let us assume that the dynamics is a LTI dynamics with $C = 0$. Let us define the discrete approximation of the work done by the external forces within the step (supply rate) by

$$\bar{W}^{\text{ext}}_{k+1} = h v^{\top}_{k+\theta} F_{k+\theta} \approx \int_{t_k}^{t_{k+1}} Fv \; dt \tag{83}$$

Then the variation of energy over a time–step performed by the Moreau–Jean is

$$\Delta\mathcal{E} - \bar{W}^{\text{ext}}_{k+1} \quad = \quad (\frac{1}{2} - \theta) \left[\|v_{k+1} - v_k\|^2_M + \|(q_{k+1} - q_k)\|^2_K\right] + U^{\top}_{k+\theta} P_{k+1} \tag{84}$$

Energy analysis for Moreau–Jean scheme

Proposition

Let us assume that the dynamics is a LTI dynamics. The Moreau–Jean scheme dissipates energy in the sense that

$$\mathcal{E}(t_{k+1}) - \mathcal{E}(t_k) - \bar{W}^{\text{ext}}_{k+1} \leqslant 0 \tag{85}$$

if

$$\frac{1}{2} \leqslant \theta \leqslant \frac{1}{1+e} \leqslant 1 \tag{86}$$

In particular, for $e = 0$*, we get* $\frac{1}{2} \leqslant \theta \leqslant 1$ *and for* $e = 1$*, we get* $\theta = \frac{1}{2}$.

Energy analysis for Moreau–Jean scheme

Variant of the Moreau scheme that always dissipates energy

Let us consider the variant of the Moreau scheme

$$
\begin{cases}
M(v_{k+1}-v_k)+hKq_{k+\theta}-hF_{k+\theta}=p_{k+1}=GP_{k+1}, & (87a)\\
q_{k+1}=q_k+hv_{k+1/2}, & (87b)\\
U_{k+1}=G^\top\, v_{k+1} & (87c)\\
\begin{array}{l}\text{if} \quad \bar{g}^\alpha_{k+1}\leqslant 0 \text{ then } 0\leqslant U^\alpha_{k+1}+eU^\alpha_k \perp P^\alpha_{k+1}\geqslant 0,\\ \text{otherwise } P^\alpha_{k+1}=0.\end{array} \quad ,\alpha\in\mathcal{I} & (87d)
\end{cases}
$$

Energy analysis for Moreau–Jean scheme

Lemma

Let us assume that the dynamics is a LTI dynamics with $C = 0$. Then the variation of energy performed by the variant scheme over a time–step is

$$\Delta\mathcal{E} - \bar{W}^{\text{ext}}_{k+1} = (\frac{1}{2} - \theta)\|(q_{k+1} - q_k)\|^2_K + U^\top_{k+1/2} P_{k+1} \tag{88}$$

The scheme dissipates energy in the sense that

$$\mathcal{E}(t_{k+1}) - \mathcal{E}(t_k) - \bar{W}^{\text{ext}}_{k+1} \leqslant 0 \tag{89}$$

if

$$\theta \geqslant \frac{1}{2} \tag{90}$$

Energy analysis for Newmark's scheme

Lemma

Let us assume that the dynamics is a LTI dynamics with $C = 0$. Let us define the discrete approximation of the work done by the external forces within the step by

$$\bar{W}^{\text{ext}}_{k+1} = (q_{k+1} - q_k)^\top F_{k+\gamma} \approx \int_{t_k}^{t_{k+1}} Fv \; dt \tag{91}$$

Then the variation of energy over a time–step performed by the scheme is

$$\begin{array}{rcl}\Delta\mathcal{E} - \bar{W}^{\text{ext}}_{k+1} & = & (\frac{1}{2} - \gamma)\|(q_{k+1} - q_k)\|^2_K \\ & + & \frac{h}{2}(2\beta - \gamma)\left[(q_{k+1} - q_k)^\top K(v_{k+1} - v_k) - (v_{k+1} - v_k)^\top [F_{k+1} - F_k]\right] \\ & & + \frac{1}{2}P^\top_{k+1}(U_{k+1} + U_k) + \frac{h}{2}(2\beta - \gamma)(a_{k+1} - a_k)^\top G P_{k+1}\end{array} \tag{92}$$

Energy analysis for Newmark's scheme

Define an discrete "algorithmic energy" (discrete storage function) of the form

$$\mathcal{K}(q, v, a) = \mathcal{E}(q, v) + \frac{h^2}{4}(2\beta - \gamma)a^\top Ma. \quad (93)$$

The following result can be given

Proposition

Let us assume that the dynamics is a LTI dynamics with $C = 0$. Let us define the discrete approximation of the work done by the external forces within the step by

$$\bar{W}^{\text{ext}}_{k+1} = (q_{k+1} - q_k)^\top F_{k+\gamma} \approx \int_{t_k}^{t_{k+1}} Fv \; dt \quad (94)$$

Then the variation of energy over a time–step performed by the nonsmooth Newmark scheme is

$$\Delta\mathcal{K} - \bar{W}^{\text{ext}}_{k+1} = -(\gamma - \frac{1}{2})\left[\|q_{k+1} - q_k\|^2_K + \frac{h}{2}(2\beta - \gamma)\|(a_{k+1} - a_k)\|^2_M\right] + U^\top_{k+1/2}P_{k+1} \quad (95)$$

Moreover, the nonsmooth Newmark scheme is stable in the following sense

$$\Delta\mathcal{K} - \bar{W}^{\text{ext}}_{k+1} \leqslant 0 \quad (96)$$

for

$$2\beta \geqslant \gamma \geqslant \frac{1}{2} \quad (97)$$

Energy analysis for HHT scheme

Augmented dynamics

Let us introduce the modified dynamics

$$Ma(t) + Cv(t) + Kq(t) = F(t) + \frac{\alpha}{\nu}[Kw(t) + Cx(t) - y(t)] \tag{98}$$

and the following auxiliary dynamics that filter the previous one

$$\begin{array}{rcl} \nu h\dot{w}(t) + w(t) & = & \nu h\dot{q}(t) \\ \nu h\dot{x}(t) + x(t) & = & \nu h\dot{v}(t) \\ \nu h\dot{y}(t) + y(t) & = & \nu h\dot{F}(t) \end{array} \tag{99}$$

Energy analysis for HHT scheme

Discretized Augmented dynamics

The equation (99) are discretized as follows

$$\begin{aligned}
\nu(w_{k+1}-w_k)+\frac{1}{2}(w_{k+1}+w_k) &= \nu(q_{k+1}-q_k)\\
\nu(x_{k+1}-x_k)+\frac{1}{2}(x_{k+1}+x_k) &= \nu(v_{k+1}-v_k)\\
\nu(y_{k+1}-y_k)+\frac{1}{2}(y_{k+1}+y_k) &= \nu(F_{k+1}-F_k)
\end{aligned} \tag{100}$$

or rearranging the terms

$$\begin{aligned}
(\frac{1}{2}+\nu)w_{k+1}+(\frac{1}{2}-\nu)w_k &= \nu(q_{k+1}-q_k)\\
(\frac{1}{2}+\nu)x_{k+1}+(\frac{1}{2}-\nu)x_k &= \nu(v_{k+1}-v_k)\\
(\frac{1}{2}+\nu)y_{k+1}+(\frac{1}{2}-\nu)y_k &= \nu(F_{k+1}-F_k)
\end{aligned} \tag{101}$$

With the special choice $\nu=\frac{1}{2}$, we obtain the HHT scheme collocation that is

$$Ma_{k+1}+(1-\alpha)[Kq_{k+1}+Cv_{k+1}]+\alpha[Kq_k+Cv_k]=(1-\alpha)F_{k+1}+\alpha F_k \tag{102}$$

Energy analysis for HHT scheme

Discretized storage function

With

$$\mathcal{H}(q, v, a, w) = \mathcal{E}(q, v) + \frac{h^2}{4}(2\beta - \gamma)a^\top Ma + 2\alpha(1-\gamma)w^\top Kw. \tag{103}$$

we get

$$\begin{array}{rcl} 2\Delta\mathcal{H} & = & 2U_{k+1/2}^\top P_{k+1} \\ & - & h^2(\gamma - \frac{1}{2})(2\beta - \gamma)\|(a_{k+1} - a_k)\|_M^2 \\ & - & 2(\gamma - \frac{1}{2} - \alpha)\|q_{k+1} - q_k\|_K^2 \\ & - & 2\alpha(1 - 2(\gamma - \frac{1}{2}))\|w_{k+1} - w_k\|_K^2 \\ & + & 2(F_{k+\gamma-\alpha})^\top(q_{k+1} - q_k) + 2\alpha(1 - 2(\gamma - \frac{1}{2}))(q_{k+1} - q_k)^\top(y_{k+1} - y_k) \end{array}$$

Energy analysis for HHT scheme

Discretized storage function

With

$$\mathcal{H}(q, v, a, w) = \mathcal{E}(q, v) + \frac{h^2}{4}(2\beta - \gamma)a^\top Ma + 2\alpha(1-\gamma)w^\top Kw. \quad (103)$$

and with $\alpha = \gamma - \frac{1}{2}$, we obtain

$$\begin{array}{rcl} 2\Delta\mathcal{H} & = & 2U_{k+1/2}^\top P_{k+1} \\ & - & h^2(\alpha)(2\beta-\gamma)\|(a_{k+1} - a_k)\|^2_M \\ & - & 2\alpha(1-2\alpha)\|w_{k+1} - w_k\|^2_K \\ & + & 2(F_{k+\gamma-\alpha})^\top(q_{k+1} - q_k) + 2\alpha(1-2\alpha)(q_{k+1} - q_k)^\top(y_{k+1} - y_k) \end{array} \quad (104)$$

Energy analysis for HHT scheme

Conclusions

- For the Moreau–Jean, a simple variant allows us to obtain a scheme which always dissipates energy.
- For the Newmark and the HHT scheme with retrieve the dissipation properties as the smooth case. The term associated with impact is added is the balance.
- Open Problem: We get dissipation inequality for discrete with quadratic storage function and plausible supply rate. The nest step is to conclude to the stability of the scheme with this argument.

Adaptive time-step strategies for time–stepping schemes

Smooth ODEs

One–step numerical solvers for ODEs

Let us consider a ODE

$$\dot{x} = f(x,t), \tag{105}$$

where f is a mapping with sufficient regularity.
The one–step time–stepping method over the time–step $[t_k, t_{k+1} = t_k + h]$ is generically denoted by

$$x_{k+1} = x_k + h\Phi(t_k, h, x_k). \tag{106}$$

Order of consistency

The one–step time–stepping method is said to be consistent if $\Phi(t, 0, x, x) = f(x,t)$ and has a consistency order p if there exists a constant C such that

$$e_{k+1} = x(t_{k+1}) - x_{k+1} = Ch^{p+1} + \mathcal{O}(h^{p+2}), \tag{107}$$

assuming that $x_k = x(t_k)$.
If the time–stepping method has an order of consistency p and converges, then the global order of convergence is p,

Smooth ODEs

Basic practical error evaluation

1. Two "small" time steps of size $h/2 \implies x_{1/2}$.
2. One "big" time-step $h \implies x_1$.

$$
\begin{array}{l}
e_1 = x(t_0 + h) - x_1 = C\ h^{p+1} + \mathcal{O}(h^{p+2}),\\
e_{1/2} = x(t_0 + h) - x_{1/2} = 2C\ (h/2)^{p+1} + \mathcal{O}(h^{p+2}).
\end{array}
\tag{108}
$$

This procedure permits us to evaluate the constant C and to obtain and a local error estimate such that:

$$
e_2 = x(t_0 + h) - x_2 = \frac{x_{1/2} - x_1}{2^p - 1} + \mathcal{O}(h^{p+2}). \tag{109}
$$

Enhanced practical error evaluation

- Runge–Kutta Embedded pairs (Dormand-Price, Felhberg)
- Milne's device
- Nordsieck's method

Smooth ODEs

Automatic control of the time–step

$$\|e_k\| \leqslant etol = atol + rtol \circ \max(x_0, x_k) \tag{110}$$

The measure of the error is given by

$$\text{error} = \|e_k \circ invtol\| \tag{111}$$

with $invtol = [1/etol_i, i = 1 \dots n]$. The optima step size is then obtained by

$$h_{\text{opt}} = h(\frac{1}{\text{error}})^{1/(p+1)} \tag{112}$$

Usually, the step size is not allowed to decrease of to increase too fast, thanks to the following heuristic rule

$$h_{\text{new}} = h \min(\alpha_{max}, \max(\alpha_{min}, \alpha(\frac{1}{\text{error}})^{1/(p+1)})) \tag{113}$$

where α, α_{min} and α_{max} are some user parameters.

Local error estimates for the Moreau's time–stepping

Notation

$$e = x(t_k + h) - x_{k+1} = \begin{bmatrix} e_v \\ e_q \end{bmatrix} = \begin{bmatrix} v^+(t_k + h) - v_{k+1} \\ q(t_k + h) - q_{k+1} \end{bmatrix} \quad (114)$$

Local error estimates for the Moreau's time–stepping

Assumption 1 : Existence and uniqueness

A unique global solution over $[0, T]$ for Moreau's sweeping process is assumed such that $q(\cdot)$ is absolutely continuous and admits a right velocity $v^+(\cdot)$ at every instant t of $[0, T]$ and such that the function $v^+ \in LBV([0, T], \mathbb{R}^n)$.

➜ Assumption 92 is ensured in the framework introduced by Ballard (Ballard, 2000) who proves the existence and uniqueness of a solution in a general framework mainly based on the analyticity of data.

Assumption 2 : Smoothness of data

The following smoothness on the data will be assumed: a) the inertia operator $M(q)$ is assumed to be of class $\mathcal{C}^p$ and definite positive, b) the force mapping $F(t, q, v)$ is assumed to be of class $\mathcal{C}^p$, c) the constraint functions $g(q)$ are assumed to be of class $\mathcal{C}^{p+1}$ and d) the Jacobian matrix $G(q) = \nabla_q^T g(q)$ is assumed to have full-row rank.

Local error estimates for the Moreau's time–stepping

Lemma

Let $I = [t_k, t_{k+1}]$. Let us assume that the function $f \in BV(I, \mathbb{R}^n)$. Then we have the following inequality for the θ–method, $\theta \in [0,1]$,

$$\left\| \int_{t_k}^{t_{k+1}} f(s)\,ds - h(\theta f(t_{k+1}) + (1-\theta) f(t_k)) \right\| \leqslant C(\theta)(t_{k+1} - t_k)\, var(f, I), \qquad (115)$$

where $var(f, I) \in \mathbb{R}$ is the variation of f on I and $C(\theta) = \theta$ if $\theta \geqslant 1/2$ and $C(\theta) = 1 - \theta$ otherwise. Furthermore, the value of $C(\theta)$ yields a sharp bound in (115).

Local error estimates for the Moreau's time–stepping

Proposition

Under Assumptions 1 and 2, the local order of consistency of the Moreau time–stepping scheme for the generalized coordinates is

$$e_q = \mathcal{O}(h)$$

and at least for the velocities

$$e_v = \mathcal{O}(1)$$

.

Comments

The bounds are reached if an impact is located within the time–step and the activation of the constraint is not correct.

One impact at time $t_* \in (t_k, t_{k+1}]$

Assumption

$$di = p\delta_{t_*}, \quad \text{or equivalently} \quad dI = P\delta_{t_*}, \text{with } P = G(t_*)p. \tag{116}$$

Notation

$$\mathcal{I} = \{\alpha, \alpha \in \{1..m\}\} \tag{117}$$

$$\mathcal{I}_* = \{\alpha \in \mathcal{I}, P^\alpha \geqslant 0, U^{\alpha,+}(t_*) - U^{\alpha,-}(t_*) = -(1+e)U^{\alpha,-}(t_*)\} \tag{118}$$

$$\mathcal{I}_p = \{\alpha \in \mathcal{I}, P^\alpha_{k+1} \geqslant 0, U^\alpha_{k+1} - U^\alpha_k = -(1+e)U^\alpha_k\} \tag{119}$$

Lemma

Let us assume that we have only one elastic impact at time $t_* \in (t_k, t_{k+1}]$ i.e. , $di = p\delta_{t_*} + r(t)dt$.

1. *If $\mathcal{I}_* = \mathcal{I}_p$, then the local order of consistency of the scheme is given by*

$$e_v = K_v h + \mathcal{O}(h^2) \tag{120}$$

2. *If $\mathcal{I}_* \neq \mathcal{I}_p$, then the local order of consistency of the scheme is given by*

$$e_v = K_v + \mathcal{O}(h) \tag{121}$$

Local error estimates for the Moreau's time–stepping

Example (The bouncing ball)

$$\begin{cases} \dot{v}(t) = f(t) + \lambda(t), & \dot{q}(t) = v(t), \\ 0 \leqslant q(t) \perp \lambda(t) \geqslant 0, & v^+(t) = -ev^-(t), \text{ if } q(t) = 0, \end{cases} \quad (122)$$

With chosen parameters as $f = -2$, $e = 1/2$ and the initial data as $t_0 = 0, q_0 = 1$ and $v_0 = 0$. The analytical solution reads as

- for $t \in [0, 1)$,

$$\begin{cases} q(t) = -t^2 + 1, \\ v(t) = -2t, \end{cases} \quad (123)$$

- for $t \in \left[3 - \frac{1}{2^{n-1}}, 3 - \frac{1}{2^n}\right)$,

$$\begin{cases} q(t) = -(t-3)^2 - \frac{3}{2^n}(t-1) + \frac{1}{2^{n-1}}\left(3 - \frac{1}{2^n}\right), \\ v(t) = -2(t-3) - \frac{3}{2^n}, \end{cases} \quad (124)$$

- and for $t \in [3, +\infty)$

$$\begin{cases} q(t) = 0, \\ v(t) = 0, \end{cases} \quad (125)$$

Local error estimates for the Moreau's time–stepping

Example (The bouncing ball (continued))

Let us consider a time interval such that the impacting time t_* belongs to $(t_k, t_{k+1}]$. The error is given by

$$
\begin{array}{ll}
\text{if} & p_{k+1} = 0 \\
& \left\{\begin{array}{l} e_v = -(1+e)[v_k + hf\sigma] \\ e_q = -q_k - h(e(1-\sigma+1))v_k - fh^2[e(1-\sigma)\sigma + \frac{1}{2}(1-\sigma)^2 + \theta] \end{array}\right. \\
\text{if} & p_{k+1} > 0 \\
& \left\{\begin{array}{l} e_v = -hf[1-\sigma-e\sigma] \\ e_q = -q_k - h((1+e)(1-\theta) - e\sigma)v_k - fh^2(e(1-\sigma)\sigma + \frac{1}{2}(1-\sigma)^2) \end{array}\right. ,
\end{array}
\tag{126}
$$

where $\sigma = (t_k - t_*)/h \in (0, 1]$.

The approximate solution of the Moreau scheme depends on the forecast of the active constraints, *i.e.* $\bar{g}_{k+1} = q_k + \gamma h v_k$.

Local error estimates for the Moreau's time–stepping

Example (The bouncing ball (continued))

Using the fact that $q(t_*) = q_k + v_k\sigma h + \frac{1}{2}(\sigma h)^2 = 0$, we obtain that $q_k = -\sigma v_k h - \frac{1}{2}f(\sigma h)^2$ and

$$\begin{array}{ll}
\text{if} & p_{k+1} = 0, \\
& \left\{\begin{array}{l} e_v = -(1+e)[v_k + hf\sigma] \\ e_q = -h(e(1-\sigma+1)-\sigma)v_k - fh^2[e(1-\sigma)\sigma + \dfrac{1}{2}(1-\sigma)^2 - \dfrac{1}{2}(\sigma)^2 + \theta] \end{array}\right. \\
\text{i.e.} & e_v = \mathcal{O}(1) \text{ and } e_q = \mathcal{O}(h) \\
\text{if} & p_{k+1} > 0, \\
& \left\{\begin{array}{l} e_v = -hf[1-\sigma-e\sigma] \\ e_q = -h((1+e)(1-\theta-\sigma))v_k - fh^2(e(1-\sigma)\sigma + \dfrac{1}{2}(1-\sigma)^2 - \dfrac{1}{2}(\sigma)^2) \end{array}\right. \\
\text{i.e.} & e_v = \mathcal{O}(h) \text{ and } e_q = \mathcal{O}(h)
\end{array} \tag{126}$$

Local error estimates for the Moreau's time–stepping

Example (The bouncing ball (continued))

Near the finite accumulation of impact at time $t = 3$.
Let us consider a time step such that $[t_k, t_{k+1}] = [3-h, 3+h]$ and n_0 such that $h \in [1/2^{n_0}, 1/2^{n_0-1}]$. The local error in velocity is given if the impact is detected $p_{k+1} > 0$ by

$$e_v = v(3+h) - v_{k+1} = -2h - \frac{3}{2^{n_0}}. \qquad (126)$$

As $h \to 0$, we have $n_0 \to \infty$, and $\frac{1}{2^{n_0}} = \mathcal{O}(h)$ and then $e_v = \mathcal{O}(h)$.

Local error estimates for the Moreau's time–stepping

To summarize

- In any case, we have $\mathcal{O}(h)$ in the error in coordinates and it cannot be improved if a jump occurs.
- The local error in velocity is at least $e_v = \mathcal{O}(1)$ if the impact is not well–forecast. In practice, this situation is usual. It illustrates the possible convergence problem that we can have in uniform norm
- *Finite accumulation* The order of the time–integration should be at least 0. Idea of the proof : use the fact that the velocity vanishes and is of bounded variations

Practical error estimates for the Moreau's time–stepping

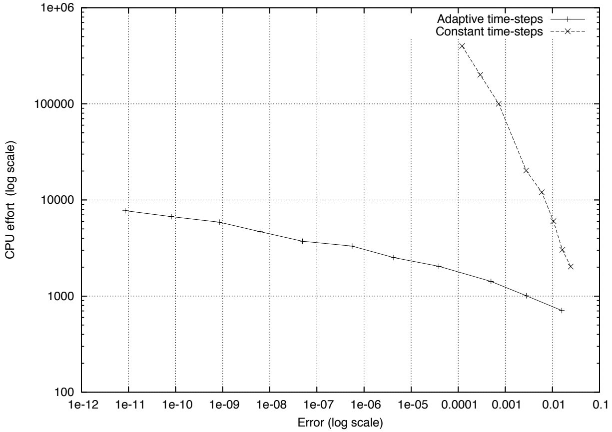
Order "0" case

Standard error estimates do not apply for Order 0.
We propose to extend it to the order 0 of consistency by assuming that the the local error estimate is given by

$$e_{1/2} = 2(x_{1/2} - x_1) + \mathcal{O}(h^2) \tag{127}$$

where x_1 is the result of the time integration with one time–step of length h and $x_{1/2}$ with two time-steps of length $h/2$.
The adaptive time–step control used for smooth ODE is then apply directlyHairer et al. (1993).

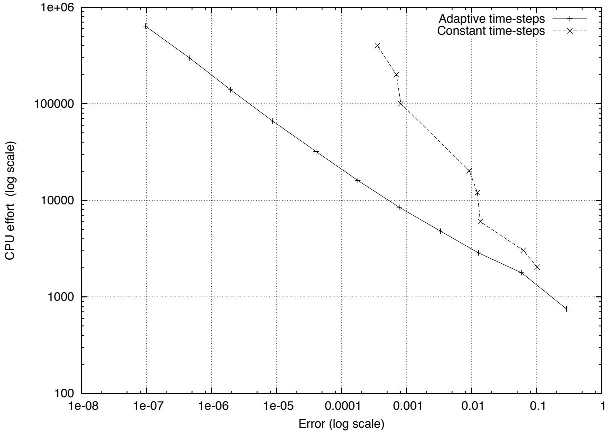
Order "0" time–step adjustment for the Moreau's time–stepping



(a) The bouncing ball example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 0

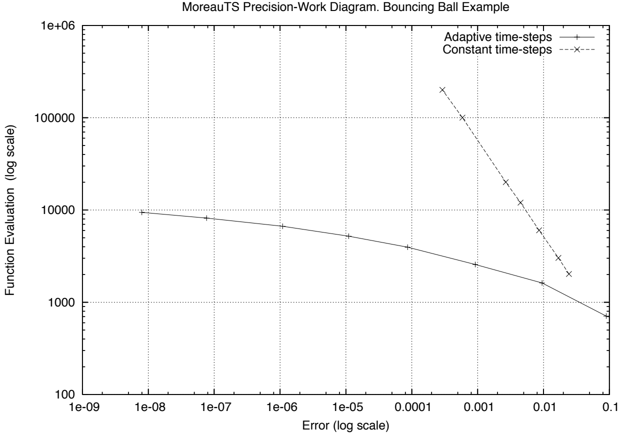
Order “0” time–step adjustment for the Moreau’s time–stepping



(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau’s time-stepping scheme. Order 0

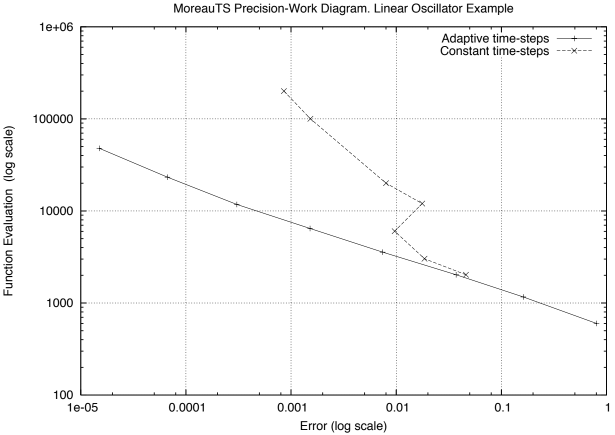
Order "1" time–step adjustment for the Moreau's time–stepping



(a) The bouncing ball example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 1

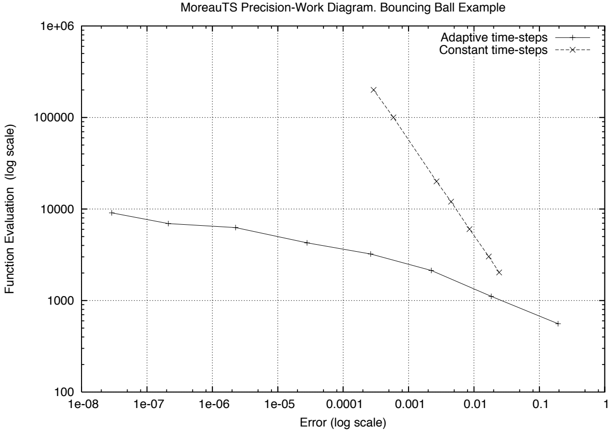
Order “1” time–step adjustment for the Moreau’s time–stepping



(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau’s time-stepping scheme. Order 1

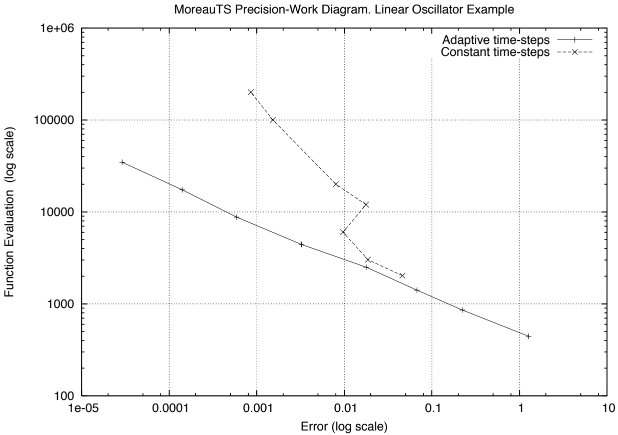
Order "2" time–step adjustment for the Moreau's time–stepping



(a) The bouncing ball example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 2

Order “2” time–step adjustment for the Moreau’s time–stepping



(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau’s time-stepping scheme. Order 2

Sizing the error in the violation of constraints

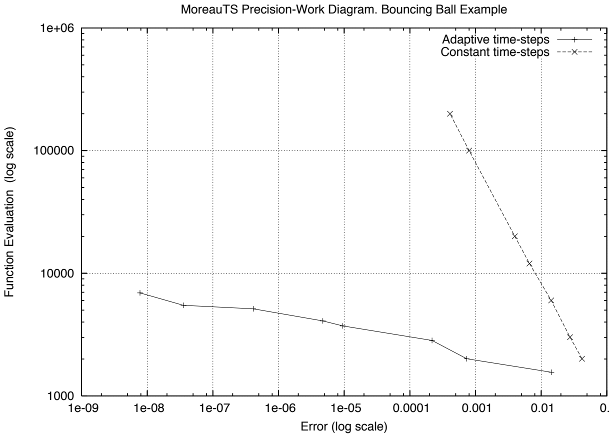
The violation of constraints is sized by the following rule:

$$e_{\mathrm{violation}} = \| \min(0, g(q)) \circ \mathit{invtol} \|_\infty \tag{128}$$

Assuming that the scheme is of order 1 almost everywhere in smooth phase and may be controlled by $e_{\mathrm{violation}}$ when an nonsmooth vent occurs, the step size adjustment is implemented by the means of the following error estimation

$$\mathrm{error} = \max(e_{\mathrm{violation}}, \|e_k \circ \mathit{invtol}\|_\infty) \tag{129}$$

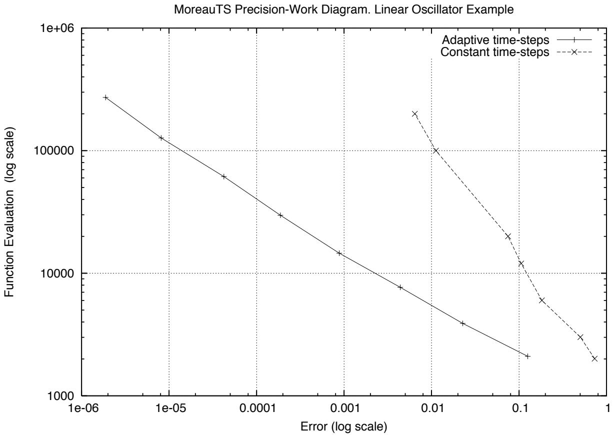
Results on two academic test examples



(a) The bouncing ball example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 0 + violation error

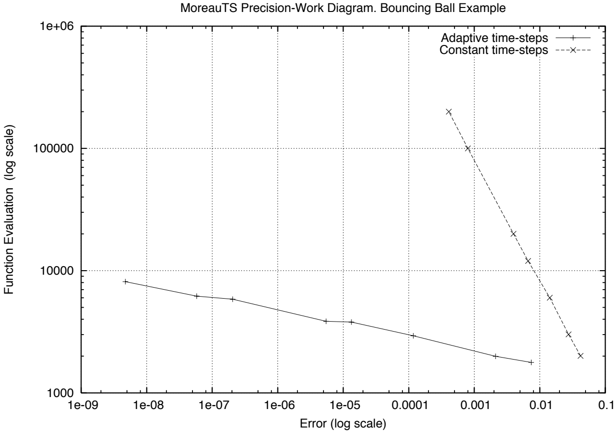
Results on two academic test examples



(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 0 + violation error

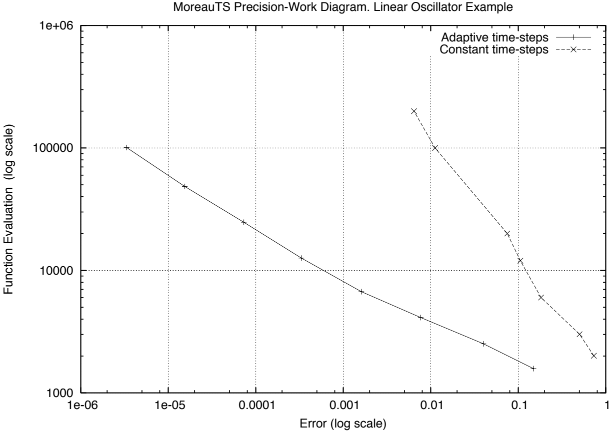
Results on two academic test examples



(a) The bouncing ball example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 1 + violation error

Results on two academic test examples



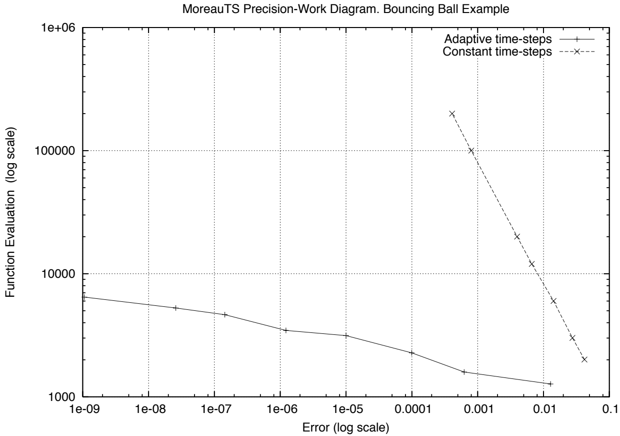
(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 1 + violation error

Variable order approach. Principle

Guess the order of consistency of the integration at each step.
Adapt the practical error estimation

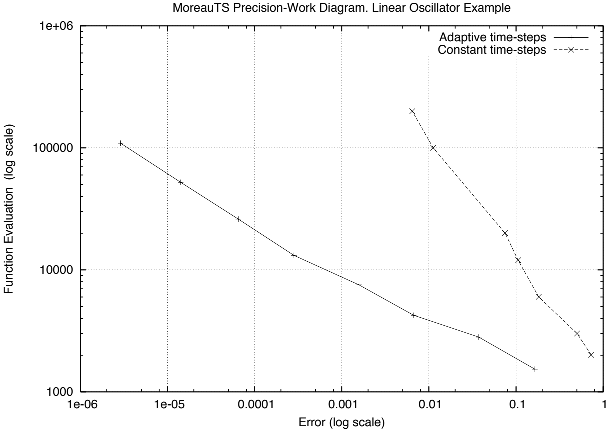
Results on two academic test examples



(a) The bouncing ball example

Figure: Precision Work diagram for the Moreau's time-stepping scheme.

Results on two academic test examples



(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau's time-stepping scheme.

Time–stepping schemes of any order

First attempt

In Studer et al. (2008) ; Studer (2009) the first attempt to increase the efficiency of Moreau's scheme by an extrapolation method has been published.

Higher Order Time–stepping schemes

Background

Work of Mannshardt (1978) on time–integration schemes of any order for ODE/DAEs with discontinuities (with tranversality assumption)

Principle

- Let us assume only one event per time–step at instants t_*.
- Choose any ODE/DAE solvers of order p
- Perform a rough location of the event inside the time step of length h
  Find an interval $[t_a, t_b]$ such that

$$t_* \in [t_a, t_b] \text{ and } |t_b - t_a| = Ch^{p+1} + \mathcal{O}(h^{p+2}) \tag{130}$$

  Dichotomy, Newton, Local Interpolants, Dense output,...
- Perform an integration on $[t_k, t_a]$ with the ODE solver of order p
- Perform an integration on $[t_a, t_b]$ with Moreau's time–stepping scheme
- Perform an integration on $[t_b, t_{k+1}]$ with the ODE solver of order p

Integration of the smooth dynamics

Mainly for the sake of simplicity, the numerical integration over a smooth period is made with a Runge–Kutta (RK) method on the following index-1 DAE,

$$\begin{cases} M(q(t))\dot{v}(t) = F(t, q(t), v(t)) + G(q)\lambda(t), \\ \dot{q}(t) = v(t), \\ \gamma(t) = G(q(t))\dot{v}(t) = 0. \end{cases} \tag{131}$$

In practice, the time–integration is performed for the following system

$$\begin{cases} M(q(t))\dot{v}(t) = F(t, q(t), v(t)) + G(q)\lambda(t), \\ \dot{q}(t) = v(t), \\ 0 \leqslant \gamma(t) = G(q(t))\dot{v}(t) \perp \lambda(t) \geqslant 0 \end{cases} \tag{132}$$

on the time–interval I where the index set $\mathcal{I}(t)$ of active constraints is assumed to be constant on I and $\lambda(t) > 0$ for all $t \in I$.

Integration of the smooth dynamics

Using the standard notation for the RK methods (see Hairer et al. (1993) for details), the complementarity problem that we have to solve at each time–step reads

$$\begin{cases} t_{ki} = t_k + c_i h, \\ v_{k+1} = v_k + h \sum_{i=1}^{s} b_i V'_{ki}, \\ q_{k+1} = q_k + h \sum_{i=1}^{s} b_i V_{ki}, \\ V'_{ki} = M^{-1}(Q_{ki}) \left[F(t_{ki}, Q_{ki}, V_{ki}) + G(Q_{ki})\lambda_{ki}\right], \\ V_{ki} = v_k + h \sum_{j=1}^{s} a_{ij} V'_{nj}, \\ Q_{ki} = q_k + h \sum_{j=1}^{s} a_{ij} V_{nj}, \\ 0 \leqslant \gamma_{ki} = G(Q_{ki}) V'_{ki} \perp \lambda_{ki} \geqslant 0. \end{cases} \tag{133}$$

Assumption 3

Let I a smooth period time–interval. We assume that

1. the local order of the RK method (133) is p that is

$$e_q = e_v = \mathcal{O}(h^{p+1}) \tag{134}$$

2. starting from inconsistent initial value $\tilde{q}_k$ such that $\tilde{q}_k - q_k = \mathcal{O}(h^{p+1})$, the error made by the RK method (133) is

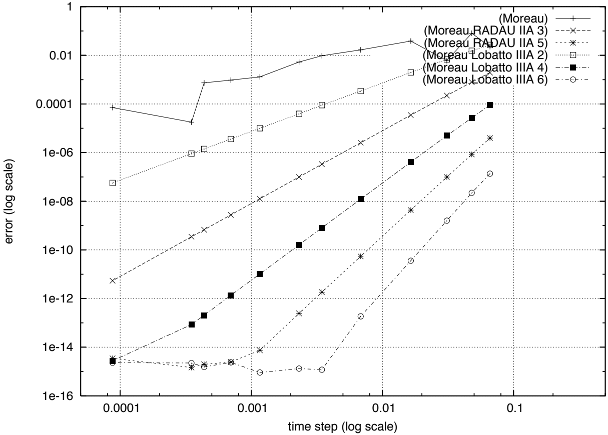
$$\tilde{q}_{k+1} - q_{k+1} = \mathcal{O}(h^{p+1}) \tag{135}$$

Theorem

Let us assume that Assumptions 1, 2 and 3 hold. The local error of consistency of the scheme is of order p in the generalized coordinates that is

$$e_q = \mathcal{O}(h^{p+1}). \tag{136}$$

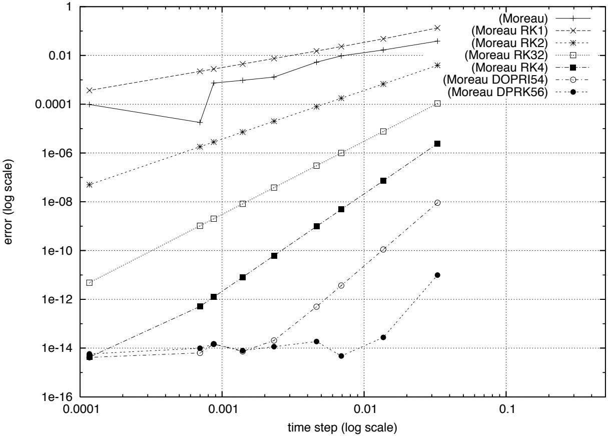
Results on the linear oscillator



(a) The linear oscillator example with implicit Runge Kutta Method

Figure: Precision Work diagram for the Moreau's time-stepping scheme coupled with Runge–Kutta method.

Results on the linear oscillator



(a) The linear oscillator example with half explicit Runge Kutta Method

Figure: Precision Work diagram for the Moreau's time-stepping scheme coupled with Runge–Kutta method.

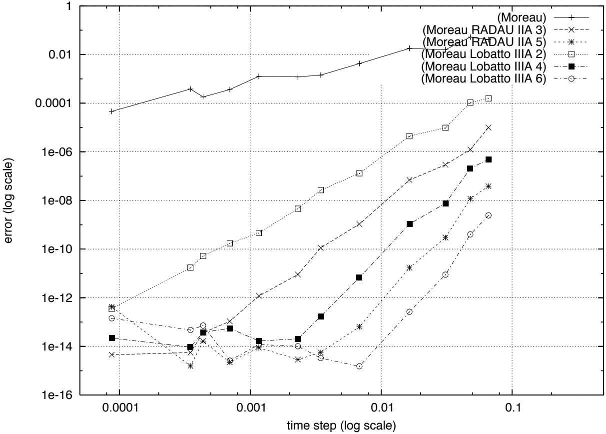
Higher Order Time–stepping schemes

Finite accumulation

- Repeat the whole process on the remaining part of the interval $[t_b, t_k]$
- By induction, repeat this process up to the end of the original time step.

Acary (2009)

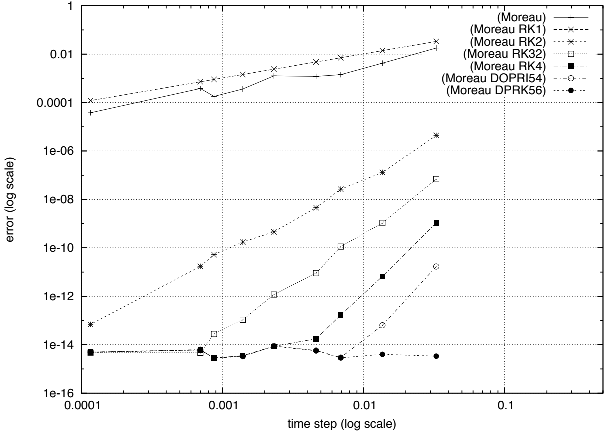
Results on the Bouncing Ball



(a) The Bouncing Ball example with implicit Runge Kutta Method

Figure: Precision Work diagram for the Moreau's time-stepping scheme.

Results on the Bouncing Ball



(a) The Bouncing Ball example half explicit Runge Kutta Method

Figure: Precision Work diagram for the Moreau's time-stepping scheme.

Splitting based Schemes

Splitting–based methods.

Principle for smooth ODEs

Let us consider a smooth ODE which can be written as

$$\dot{x}(t) = f(x,t) + g(x,t) \tag{137}$$

A example of splitting–based method is given by the following procedure

1. Perform the integration of f on $[t_k, t_{k+1}]$ to obtain $\tilde{x}(t_{k+1})$ that is

$$\tilde{x}(t_{k+1}) = x(t_k) + \int_{t_k}^{t_{k+1}} f(x,t)\,dt \tag{138}$$

2. Perform the integration of g on $[t_k, t_{k+1}]$ with initial value $\tilde{x}(t_{k+1})$ to obtain $\hat{x}(t_{k+1})$ that is

$$\hat{x}(t_{k+1}) \quad = \quad \tilde{x}(t_{k+1}) + \int_{t_k}^{t_{k+1}} g(x,t)\,dt \tag{139}$$

Properties

- $x(t_k+1) \neq \hat{x}(t_{k+1})$ is the general case. (except special linear case, constant dynamics, . . .)
- $\hat{x}(t_{k+1}) \to x(t_{k+1})$ when $t_{k+1} \to t_k$

Splitting–based methods.

Splitting–based for Moreau scheme without continuous contact forces

- The first part is

$$\begin{cases} M(q)\dot{v} = F(t,q,v), \\ \dot{q} = v, \\ q(t_k) = q_k, \qquad v(t_k) = v_k \end{cases} \tag{140}$$

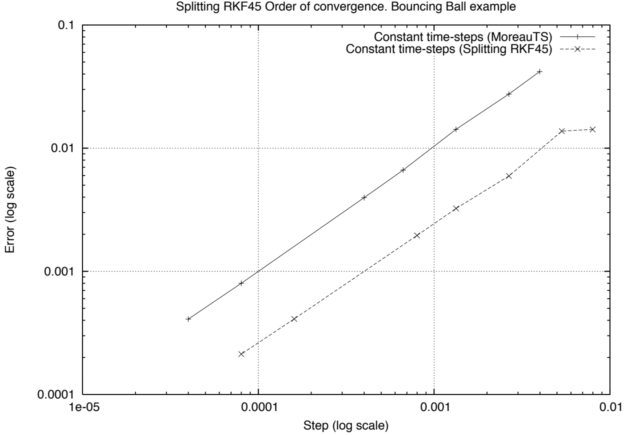
yielding to the approximations $q_1 = q(t_{k+1})$ and $v_1 = v(t_{k+1})$ which can integrated by any smooth ODE solvers.

- The second one is given by

$$\begin{cases} M(q)\dot{v} = G(q)\lambda, \\ \dot{q} = 0, \\ y = g(q) \\ -\lambda \in \partial\psi_{T_{\mathbb{R}_+}(y)}(\dot{y}(t^+) + e\dot{y}(t^-)) \\ q(t_k) = q_1; v(t_k) = v_1; \end{cases} \tag{141}$$

and leads to the approximation $q_{k+1} = q(t_{k+1})$ and$q_{k+1} = q(t_{k+1})$.

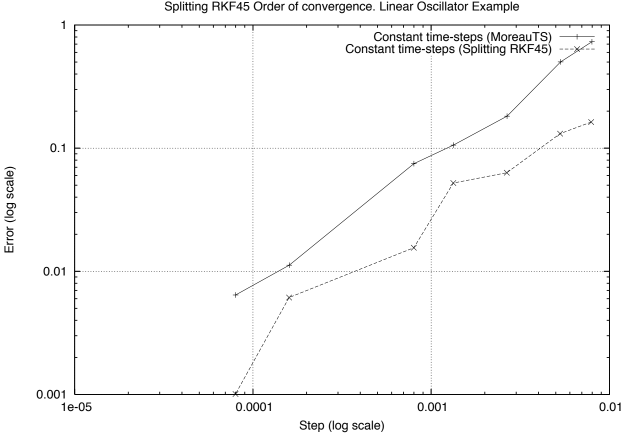
Splitting–based methods with constants time–step.



(a) The bouncing ball example

Figure: Empirical order of convergence of the Splitting RKF45 time-stepping scheme

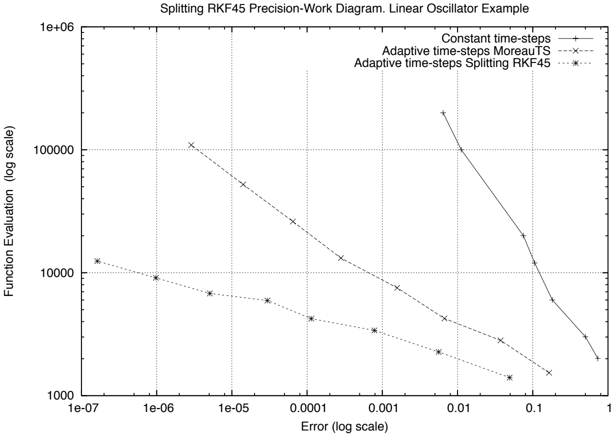
Splitting–based methods with constants time–step.



(a) The linear oscillator example

Figure: Empirical order of convergence of the Splitting RKF45 time-stepping scheme

Splitting–based methods with adaptive time–step.



(a) The linear oscillator example

Figure: Empirical order of convergence of the Splitting RKF45 time-stepping scheme

Splitting–based methods.

Splitting–based for Moreau scheme with continuous contact forces

- The first part is

$$\begin{cases} M(q)\dot{v} = F(t,q,v) + r(t), \\ \dot{q} = v, \\ y = g(q) \\ -r(t) \in \partial\psi_{T_{\mathbb{R}_+}(y)}(\dot{y}(t)) \\ q(t_k) = q_k, \qquad v(t_k) = v_k \end{cases} \tag{142}$$

  yielding to the approximations $q_1 = q(t_{k+1})$ and $v_1 = v(t_{k+1})$ which can integrated by any smooth ODE solvers.

- The second one is given by

$$\begin{cases} M(q)\dot{v} = G(q)\lambda, \\ \dot{q} = 0, \\ y = g(q) \\ -\lambda \in \partial\psi_{T_{\mathbb{R}_+}(y)}(\dot{y}(t^+) + e\dot{y}(t^-)) \\ q(t_k) = q_1; v(t_k) = v_1; \end{cases} \tag{143}$$

  and leads to the approximation $q_{k+1} = q(t_{k+1})$ and $q_{k+1} = q(t_{k+1})$.

Time–discontinuous Galerkin Method

Principle

Schindler and Acary (2011)

V. Acary. Toward higher order event–capturing schemes and adaptive time–step strategies for nonsmooth multibody systems. Research Report RR-7151, INRIA, 2009. URL http://hal.inria.fr/inria-00440771/en.

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