An extension of the Moreau–Jean scheme based on the generalized–α schemes for the numerical time integration of flexible dynamical systems with contact and friction

Vincent Acary
INRIA Rhône–Alpes, Grenoble.

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Joint work with O. Brüls, Q.Z. Chen and G. Virlez (Université de Liège)

Bio.

Team-Project BIPOP. INRIA. Centre de Grenoble Rhône–Alpes

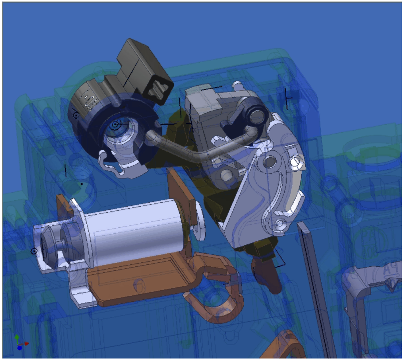
- Scientific leader : Bernard Brogliato
- 8 permanents, 5 PhD, 4 Post-docs, 3 Engineer,
- Nonsmooth dynamical systems : Modeling, analysis, simulation and Control.
- Nonsmooth Optimization : Analysis & algorithms.

Personal research themes

- Nonsmooth Dynamical systems. Higher order Moreau's sweeping process. Complementarity systems and Filippov systems
- Modeling and simulation of switched electrical circuits
- Discretization method for sliding mode control and Optimal control.
- Formulation and numerical solvers for Coulomb's friction and Signorini's problem. Second order cone programming.
- Time–integration techniques for nonsmooth mechanical systems : Mixed higher order schemes, Time–discontinuous Galerkin methods, Projected time–stepping schemes and generalized α–schemes.

Mechanical systems with contact, impact and friction

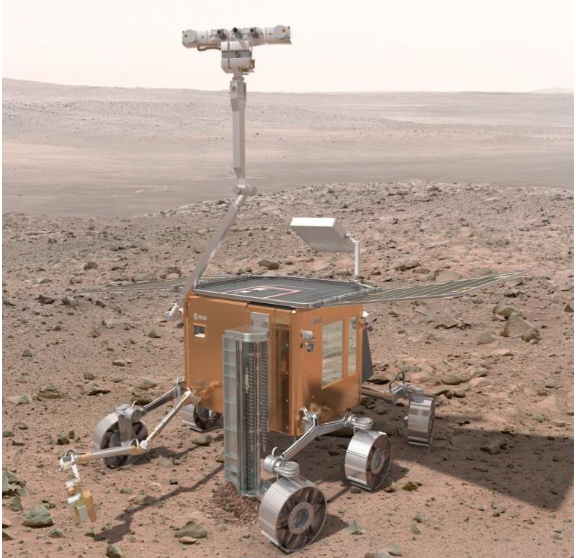
Simulation of Circuit breakers (INRIA/Schneider Electric)



Flexible multibody systems.

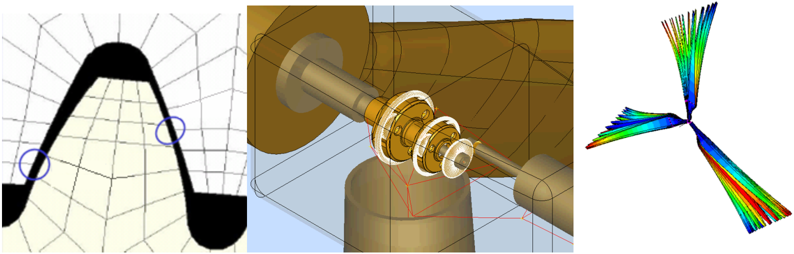
Mechanical systems with contact, impact and friction

Simulation of the ExoMars Rover (INRIA/Trasys Space/ESA)



Mechanical systems with contact, impact and friction

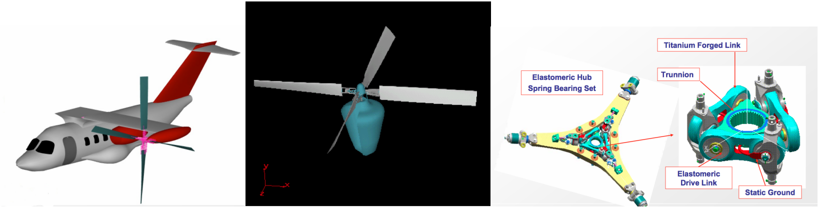
Simulation of wind turbines (DYNAWIND project)
Joint work with O. Brüls, Q.Z. Chen and G. Virlez (Université de Liège)



Flexible multibody systems.

Mechanical systems with contact, impact and friction

Simulation of Tilt rotor. (Politechnico di Milano, Masarati, P.)



Flexible multibody systems.

Objectives & Motivations

Outline

- Basic facts on nonsmooth dynamics and its time integration
  - Measure differential inclusion
  - Time–stepping schemes (Moreau–Jean and Schatzman–Paoli)
- Newmark based schemes for nonsmooth dynamics
  - Splitting impulsive and non impulsive forces
  - Velocity level constraints and impact law
- Simple Energy Analysis
- Impact in flexible structures
  - jump in velocity or standard impact ?
  - coefficient of restitution in flexible structure.

NonSmooth Multibody Systems

Scleronomous holonomic perfect unilateral constraints

$$
\begin{cases}
M(q(t))\dot{v} = F(t, q(t), v(t)) + G(q(t))\,\lambda(t), \ \text{a.e} \\
\dot{q}(t) = v(t), \\
g(t) = g(q(t)), \quad \dot{g}(t) = G^T(q(t))v(t), \\
0 \leqslant g(t) \perp \lambda(t) \geqslant 0, \\
\dot{g}^+(t) = -e\dot{g}^-(t),
\end{cases}
\tag{1}
$$

where $G(q) = \nabla g(q)$ and e is the coefficient of restitution.

Unilateral constraints as an inclusion

Definition (Perfect unilateral constraints on the smooth dynamics)

$$\begin{cases} \dot{q} = v \\ M(q)\dfrac{dv}{dt} + F(t,q,v) = r \\ -r \in N_{\mathcal{C}(t)}(q(t)) \end{cases} \tag{2}$$

where r it the generalized force or generalized reaction due to the constraints.

Remark

- The unilateral constraints are said to be perfect due to the normality condition.
- Notion of normal cones can be extended to more general sets. see (Clarke, 1975, 1983 ; Mordukhovich, 1994)
- When $\mathcal{C}(t) = \{q \in \boldsymbol{R}^n, g_\alpha(q,t) \geqslant 0, \alpha \in \{1 \ldots \nu\}\}$, the multipliers $\lambda \in \boldsymbol{R}^m$ such that $r = \nabla_q^T g(q,t)\,\lambda$.

Nonsmooth Lagrangian Dynamics

Fundamental assumptions.

- The velocity $v = \dot{q}$ is of Bounded Variations (B.V)
  ➜ The equation are written in terms of a right continuous B.V. (R.C.B.V.) function, v^+ such that
  $$v^+ = \dot{q}^+ \tag{3}$$
- q is related to this velocity by
  $$q(t) = q(t_0) + \int_{t_0}^{t} v^+(t)\, dt \tag{4}$$
- The acceleration, ($\ddot{q}$ in the usual sense) is hence a differential measure dv associated with v such that
  $$dv(]a,b]) = \int_{]a,b]} dv = v^+(b) - v^+(a) \tag{5}$$

Nonsmooth Lagrangian Dynamics

Definition (Nonsmooth Lagrangian Dynamics)

$$\begin{cases} M(q)dv + F(t,q,v^+)dt = di \\ \\ v^+ = \dot{q}^+ \end{cases} \tag{6}$$

where di is the reaction measure and dt is the Lebesgue measure.

Remarks

- The nonsmooth Dynamics contains the impact equations and the smooth evolution in a single equation.
- The formulation allows one to take into account very complex behaviors, especially, finite accumulation (Zeno-state).
- This formulation is sound from a mathematical Analysis point of view.

References

(Schatzman, 1973, 1978 ; Moreau, 1983, 1988)

Nonsmooth Lagrangian Dynamics

Measures Decomposition (for dummies)

$$\begin{cases} dv = & \gamma\, dt + & (v^+ - v^-)\, d\nu + & dv_s \\ di = & f\, dt + & p\, d\nu + & di_s \end{cases} \tag{7}$$

where

- $\gamma = \ddot{q}$ is the acceleration defined in the usual sense.
- f is the Lebesgue measurable force,
- $v^+ - v^-$ is the difference between the right continuous and the left continuous functions associated with the B.V. function $v = \dot{q}$,
- $d\nu$ is a purely atomic measure concentrated at the time t_i of discontinuities of v, i.e. where $(v^+ - v^-) \neq 0$,i.e. $d\nu = \sum_i \delta_{t_i}$
- p is the purely atomic impact percussions such that $pd\nu = \sum_i p_i \delta_{t_i}$
- dv_S and di_S are singular measures with the respect to $dt + d\eta$.

Impact equations and Smooth Lagrangian dynamics

Substituting the decomposition of measures into the nonsmooth Lagrangian Dynamics, one obtains

Definition (Impact equations)

$$M(q)(v^+ - v^-)d\nu = pd\nu, \tag{8}$$

or

$$M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \tag{9}$$

Definition (Smooth Dynamics between impacts)

$$M(q)\gamma dt + F(t, q, v)dt = fdt \tag{10}$$

or

$$M(q)\gamma^+ + F(t, q, v^+) \quad = \quad f^+ \quad [dt - a.e.] \tag{11}$$

The Moreau's sweeping process of second order

Definition (Moreau (1983, 1988))

A key stone of this formulation is the inclusion in terms of velocity. Indeed, the inclusion (2) is "replaced" by the following inclusion

$$\begin{cases} M(q)dv + F(t,q,v^+)dt = di \\ v^+ = \dot{q}^+ \\ -di \in N_{T_C(q)}(v^+) \end{cases} \tag{12}$$

Comments

This formulation provides a common framework for the nonsmooth dynamics containing inelastic impacts without decomposition.

➜ Foundation for the Moreau–Jean time–stepping approach.

The Moreau's sweeping process of second order

Comments

- *The inclusion concerns measures.* Therefore, it is necessary to define what is the inclusion of a measure into a cone.
- *The inclusion in terms of velocity* v^+ rather than of the coordinates q.

Interpretation

- Inclusion of measure, $-di \in K$
  - Case $di = r' dt = f dt$.
$$-f \in K \tag{13}$$
  - Case $di = p_i \delta_i$.
$$-p_i \in K \tag{14}$$
- Inclusion in terms of the velocity. Viability Lemma
If $q(t_0) \in C(t_0)$, then
$$v^+ \in T_C(q), t \geqslant t_0 \Rightarrow q(t) \in C(t), t \geqslant t_0$$

➜ The unilateral constraints on q are satisfied. The equivalence needs at least an impact inelastic rule.

The Moreau's sweeping process of second order

The Newton-Moreau impact rule

$$-\,di \in N_{T_{C}(q(t))}(v^{+}(t) + e v^{-}(t)) \tag{15}$$

where e is a coefficient of restitution.

Velocity level formulation. Index reduction

$$\begin{array}{c} 0 \leqslant y \perp \lambda \geqslant 0 \\ \Updownarrow \\ -\lambda \in N_{\boldsymbol{R}^{+}}(y) \\ \Uparrow \\ -\lambda \in N_{T_{\boldsymbol{R}^{+}}(y)}(\dot{y}) \\ \Updownarrow \\ \text{if } y \leqslant 0 \text{ then } 0 \leqslant \dot{y} \perp \lambda \geqslant 0 \end{array} \tag{16}$$

The Moreau's sweeping process of second order

The case of C is finitely represented

$$\mathcal{C} = \{q \in \mathcal{M}(t), g_\alpha(q) \geqslant 0, \alpha \in \{1 \dots \nu\}\}. \tag{17}$$

Decomposition of di and v^+ onto the tangent and the normal cone.

$$di = \sum_\alpha \nabla_q^T g_\alpha(q)\, d\lambda_\alpha \tag{18}$$

$$U_\alpha^+ = \nabla_q g_\alpha(q)\, v^+, \alpha \in \{1 \dots \nu\} \tag{19}$$

Complementarity formulation (under constraints qualification condition)

$$-d\lambda_\alpha \in N_{T_{\mathbb{R}_+}(g_\alpha)}(U_\alpha^+) \Leftrightarrow \text{ if } g_\alpha(q) \leqslant 0, \text{ then } 0 \leqslant U_\alpha^+ \perp d\lambda_\alpha \geqslant 0 \tag{20}$$

The case of C is $\mathbb{R}_+$

$$-di \in N_C(q) \Leftrightarrow 0 \leqslant q \perp di \geqslant 0 \tag{21}$$

is replaced by

$$-di \in N_{T_C(q)}(v^+) \Leftrightarrow \text{ if } q \leqslant 0, \text{ then } 0 \leqslant v^+ \perp di \geqslant 0 \tag{22}$$

Principle of nonsmooth event capturing methods (Time–stepping schemes)

1. A unique formulation of the dynamics is considered. For instance, a dynamics in terms of measures.

$$\begin{cases} -mdu = dr \\ q = \dot{u}^+ \\ 0 \leqslant dr \perp \dot{u}^+ \geqslant 0 \text{ if } q \leqslant 0 \end{cases} \tag{23}$$

2. The time-integration is based on a consistent approximation of the equations in terms of measures. For instance,

$$\int_{]t_k,t_{k+1}]} du = \int_{]t_k,t_{k+1}]} du = (v^+(t_{k+1}) - v^+(t_k)) \approx (u_{k+1} - u_k) \tag{24}$$

3. Consistent approximation of measure inclusion.

$$-dr \in N_{K(t)}(u^+(t)) \tag{25}$$

$$\rightarrow \quad \begin{cases} p_{k+1} \approx \int_{]t_k,t_{k+1}]} dr \\ \\ p_{k+1} \in N_{K(t)}(u_{k+1}) \end{cases} \tag{26}$$

State–of–the–art

Numerical time–integration methods for Nonsmooth Multibody systems (NSMBS):

Nonsmooth event capturing methods (Time–stepping methods)

- ⊕ robust, stable and proof of convergence
- ⊕ low kinematic level for the constraints
- ⊕ able to deal with finite accumulation
- ⊖ very low order of accuracy even in free flight motions

Two main implementations

- Moreau–Jean time–stepping scheme
- Schatzman–Paoli time–stepping scheme

Moreau's Time stepping scheme (Moreau, 1988 ; Jean, 1999)

Principle

$$\begin{cases} M(q_{k+\theta})(v_{k+1} - v_k) - hF_{k+\theta} = p_{k+1} = G(q_{k+\theta})P_{k+1}, & \text{(27a)} \\ q_{k+1} = q_k + hv_{k+\theta}, & \text{(27b)} \\ U_{k+1} = G^T(q_{k+\theta})\, v_{k+1} & \text{(27c)} \\ \begin{array}{ll} 0 \leqslant U^{\alpha}_{k+1} + eU^{\alpha}_k \perp P^{\alpha}_{k+1} \geqslant 0 & \text{if} \quad \bar{g}^{\alpha}_{k,\gamma} \leqslant 0 \\ P^{\alpha}_{k+1} = 0 & \text{otherwise} \end{array} . & \text{(27d)} \end{cases}$$

with

- $\theta \in [0, 1]$
- $x_{k+\theta} = (1-\theta)x_{k+1} + \theta x_k$
- $F_{k+\theta} = F(t_{k\theta}, q_{k+\theta}, v_{k+\theta})$
- $\bar{g}_{k,\gamma} = g_k + \gamma h U_k, , \gamma \geqslant 0$ is a prediction of the constraints.

Schatzman's Time stepping scheme (Paoli and Schatzman, 2002)

Principle

$$\begin{cases} M(q_{k+1})(q_{k+1} - 2q_k + q_{k-1}) - h^2 F_{k+\theta} = p_{k+1}, & (28a) \\ v_{k+1} = \dfrac{q_{k+1} - q_{k-1}}{2h}, & (28b) \\ -p_{k+1} \in N_K\left(\dfrac{q_{k+1} + e q_{k-1}}{1+e}\right), & (28c) \end{cases}$$

where N_K defined the normal cone to K.
For $K = \{q \in \mathbb{R}^n, y = g(q) \geqslant 0\}$

$$0 \leqslant g\left(\frac{q_{k+1} + e q_{k-1}}{1+e}\right) \perp \nabla g\left(\frac{q_{k+1} + e q_{k-1}}{1+e}\right) P_{k+1} \geqslant 0 \qquad (29)$$

Comparison

Shared mathematical properties

- Convergence results for one constraints
- Convergence results for multiple constraints problems with acute kinetic angles
- No theoretical proof of order

Mechanical properties

- Position vs. velocity constraints
- Respect of the impact law in one step (Moreau) vs. Two-steps(Schatzman)
- Linearized constraints rather than nonlinear.

But

Both schemes do not are quite inaccurate and "dissipate" a lot of energy of vibrations.
This is a consequence of the first order approximation of the smooth forces term F

The Newmark scheme

Linear Time "Invariant" Dynamics without contact

$$\begin{cases} M\dot{v}(t) + Kq(t) + Cv(t) = f(t) \\ \dot{q}(t) = v(t) \end{cases} \tag{30}$$

The Newmark scheme (Newmark, 1959)

Principle

Given two parameters γ and β

$$\begin{cases} Ma_{k+1} = f_{k+1} - Kq_{k+1} - Cv_{k+1} \\ v_{k+1} = v_k + ha_{k+\gamma} \\ q_{k+1} = q_k + hv_k + \dfrac{h^2}{2} a_{k+2\beta} \end{cases} \tag{31}$$

Notations

$$f(t_{k+1}) = f_{k+1}, \quad x_{k+1} \approx x(t_{k+1}), \\ x_{k+\gamma} = (1-\gamma)x_k + \gamma x_{k+1} \tag{32}$$

The Newmark scheme

Implementation

Let us consider the following explicit prediction

$$\begin{cases} v_k^* = v_k + h(1-\gamma)a_k \\ q_k^* = q_k + hv_k + \frac{1}{2}(1-2\beta)h^2 a_k \end{cases} \tag{33}$$

The Newmark scheme may be written as

$$\begin{cases} a_{k+1} = \hat{M}^{-1}(-Kq_k^* - Cv_k^* + f_{k+1}) \\ v_{k+1} = v_k^* + h\gamma a_{k+1} \\ q_{k+1} = q_k^* + h^2\beta a_{k+1} \end{cases} \tag{34}$$

with the iteration matrix

$$\hat{M} = M + h^2\beta K + \gamma hC \tag{35}$$

The Newmark scheme

Properties

- One–step method in state. (Two steps in position)
- Second order accuracy if and only if $\gamma = \frac{1}{2}$
- Unconditional stability for $2\beta \geqslant \gamma \geqslant \frac{1}{2}$

Average acceleration (Trapezoidal rule)	implicit	$\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$
central difference	explicit	$\gamma = \frac{1}{2}$ and $\beta = 0$
linear acceleration	implicit	$\gamma = \frac{1}{2}$ and $\beta = \frac{1}{6}$
Fox–Goodwin (Royal Road)	implicit	$\gamma = \frac{1}{2}$ and $\beta = \frac{1}{12}$

Table : Standard values for Newmark scheme ((Hughes, 1987, p 493)Géradin and Rixen (1993))

The Newmark scheme

High frequencies dissipation

- In flexible multibody Dynamics or in standard structural dynamics discretized by FEM, high frequency oscillations are artifacts of the semi-discrete structures.
- In Newmark's scheme, maximum high frequency damping is obtained with

$$\gamma \gg \frac{1}{2}, \quad \beta = \frac{1}{4}(\gamma + \frac{1}{2})^2 \tag{36}$$

example for $\gamma = 0.9$, $\beta = 0.49$

The Newmark scheme

From (Hughes, 1987) :

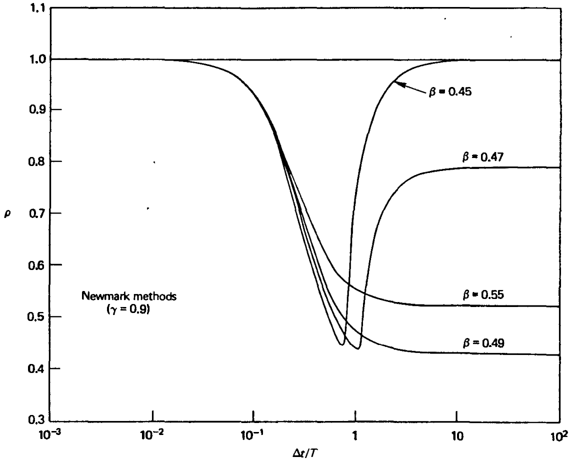


Figure 9.1.3 Spectral radii for Newmark methods for varying β [9].

The Hilber–Hughes–Taylor scheme. Hilber et al. (1977)

Objectives

- to introduce numerical damping without dropping the order to one.

Principle

Given three parameters γ, β and α and the notation

$$M\ddot{q}_{k+1} = -(Kq_{k+1} + Cv_{k+1}) + F_{k+1} \tag{37}$$

$$\begin{cases} Ma_{k+1} = M\ddot{q}_{k+1-\alpha} = -(Kq_{k+1-\alpha} + Cv_{k+1-\alpha}) + F_{k+1-\alpha} \\ v_{k+1} = v_k + ha_{k+\gamma} \\ q_{k+1} = q_k + hv_k + \dfrac{h^2}{2} a_{k+2\beta} \end{cases} \tag{38}$$

Standard parameters (Hughes, 1987, p532) are

$$\alpha \in [0, 1/3], \gamma = 1/2 + \alpha \text{ and } \beta = (1/2 + \alpha)^2/4 \tag{39}$$

Warning

The notation are abusive. a_{k+1} is not the approximation of the acceleration at t_{k+1}

The HHT scheme

Properties

- Two–step method in state. (Three–steps method in position)
- Unconditional stability and second order accuracy with the previous rule. (39)
- For $\alpha = 0$, we get the trapezoidal rule and the numerical dissipation increases with $|\alpha|$.

The HHT scheme

From (Hughes, 1987), with $\alpha \to -\alpha$:

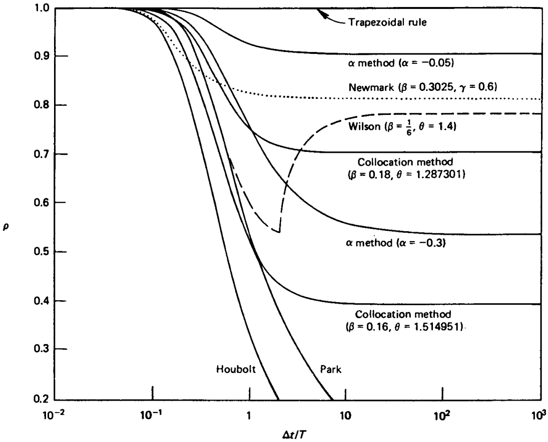


Figure 9.3.1 Spectral radii for α-methods, optimal collocation schemes, and Houbolt, Newmark, Park, and Wilson methods [22].

Generalized α-methods (Chung and Hulbert, 1993)

Principle

Given three parameters γ, β, α_m and α_f and the notation

$$M\ddot{q}_{k+1} = -(Kq_{k+1} + Cv_{k+1}) + F_{k+1} \tag{40}$$

$$\begin{cases} Ma_{k+1-\alpha_m} = M\ddot{q}_{k+1-\alpha_f} \\ v_{k+1} = v_k + ha_{k+\gamma} \\ q_{k+1} = q_k + hv_k + \dfrac{h^2}{2}a_{k+2\beta} \end{cases} \tag{41}$$

Standard parameters (Chung and Hulbert, 1993) are chosen as

$$\alpha_m = \frac{2\rho_\infty - 1}{\rho_\infty + 1}, \quad \alpha_f = \frac{\rho_\infty}{\rho_\infty + 1}, \quad \gamma = \frac{1}{2} + \alpha_f - \alpha_m \text{ and } \beta = \frac{1}{4}(\gamma + \frac{1}{2})^2 \tag{42}$$

where $\rho_\infty \in [0, 1]$ is the spectral radius of the algorithm at infinity.

Generalized α-methods (Chung and Hulbert, 1993)

Properties

- Two–step method in state.
- Unconditional stability and second order accuracy.
- Optimal combination of accuracy at low-frequency and numerical damping at high-frequency.

Special cases

- $\alpha_m = \alpha_f = 0.$ Newmark scheme
- $\alpha_m = 0$ and $\alpha_f = \alpha$ HHT scheme.

A first naive approach

Direct Application of the HHT scheme to Linear Time "Invariant" Dynamics with contact

$$\begin{cases} M\dot{v}(t) + Kq(t) + Cv(t) = f(t) + r(t), \text{a.e} \\ \dot{q}(t) = v(t) \\ r(t) = G(q)\,\lambda(t) \\ g(t) = g(q(t)), \quad \dot{g}(t) = G^T(q(t))v(t), \\ 0 \leqslant g(t) \perp \lambda(t) \geqslant 0, \end{cases} \tag{43}$$

results in

$$\begin{cases} M\ddot{q}_{k+1} = -(Kq_{k+1} + Cv_{k+1}) + F_{k+1} + r_{k+1} \\ r_{k+1} = G_{k+1}\lambda_{k+1} \end{cases} \tag{44}$$

$$\begin{cases} Ma_{k+1} = M\ddot{q}_{k+1+\alpha} \\ v_{k+1} = v_k + ha_{k+\gamma} \\ q_{k+1} = q_k + hv_k + \dfrac{h^2}{2}a_{k+2\beta} \\ 0 \leqslant g_{k+1} \perp \lambda_{k+1} \geqslant 0, \end{cases} \tag{45}$$

A first naive approach

Direct Application of the HHT scheme to Linear Time "Invariant" Dynamics with contact

The scheme is not consistent for mainly two reasons:

- If an impact occur between rigid bodies, or if a restitution law is needed which is mandatory between semidiscrete structure, the impact law is not taken into account by the discrete constraint at position level
- Even if the constraint is discretized at the velocity level, i.e.

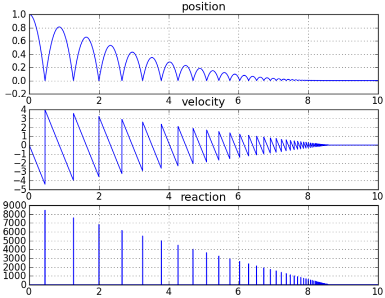
$$\text{if } \bar{g}_{k+1}, \text{ then } 0 \leqslant \dot{g}_{k+1} + e g_k \perp \lambda_{k+1} \geqslant 0 \tag{46}$$

the scheme is consistent only for $\gamma = 1$ and $\alpha = 0$ (first order approximation.)

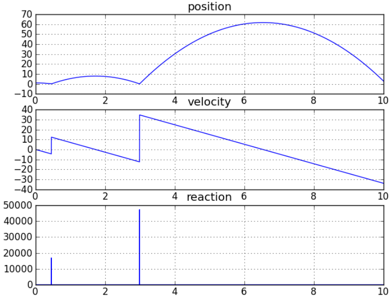
A first naive approach

Velocity based constraints with standard Newmark scheme ($\alpha = 0.0$)

Bouncing ball example. $m = 1$, $g = 9.81$, $x_0 = 1.0$ $v_0 = 0.0$, $e = 0.9$



$h = 0.001$, $\gamma = 1.0$, $\beta = \gamma/2$

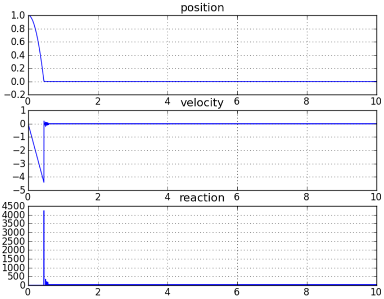


$h = 0.001$, $\gamma = 1/2$, $\beta = \gamma/2$

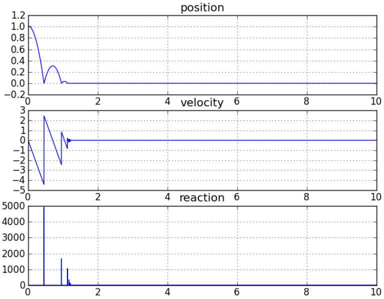
A first naive approach

Position based constraints with standard Newmark scheme ($\alpha = 0.0$)

Bouncing ball example. $m = 1$, $g = 9.81$, $v_0 = 0.0$, $e = 0.9$, $h = 0.001$, $\gamma = 1.0$, $\beta = \gamma/2$



$x_0 = 1.0$



$x_0 = 1.01$

The nonsmooth generalized α scheme

Dynamics with contact and (possibly) impact

$$\begin{cases} M\,dv = F(t,q,v)\,dt + G(q)\;di \\ \dot{q}(t) = v^+(t), \\ g(t) = g(q(t)), \quad \dot{g}(t) = G^T(q(t))v(t), \\ \text{if } g(t) \leqslant 0, \quad 0 \leqslant g^+(t) + e\dot{g}^-(t) \perp\; di \geqslant 0, \end{cases} \tag{47}$$

The nonsmooth generalized α scheme

Splitting the dynamics between smooth and nonsmooth part

$$M\,dv = M\dot{\tilde{v}}(t)\,dt + M\,dW \tag{48}$$

with

$$\begin{cases} M\dot{\tilde{v}}\,dt = F(t,q,v)\,dt \\ M\,dW = G(q)\;di \end{cases} \tag{49}$$

Different choices for the discrete approximation of the term $Ma\,dt$ and $M\,dv^{\text{con}}$

The nonsmooth generalized α scheme

Principles

- As usual is the Newmark scheme, the smooth part of the dynamics $Ma\,dt = F(t,q,v)\,dt$ is collocated, i.e.

$$Ma_{k+1} = F_{k+1} \tag{50}$$

- the impulsive part a first order approximation is done over the time–step

$$Mw_{k+1} = G_{k+1}\ P_{k+1} \tag{51}$$

$$P_{k+1} \approx \int_{(t_k,t_{k+1}]} di \tag{52}$$

The nonsmooth generalized α scheme

Principles

Given three parameters γ, β, α_m and α_f we define

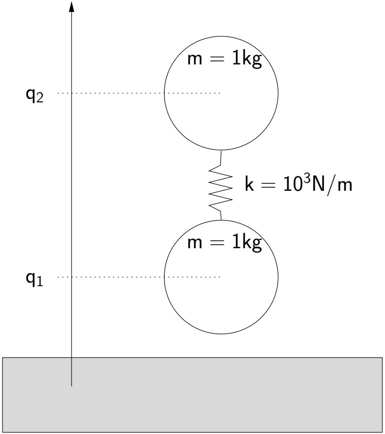
$$\begin{cases} Ma_{k+1-\alpha_m} = F_{k+1-\alpha_f} \\ Mw_{k+1} = G_{k+1}\ P_{k+1} \\ v_{k+1} = v_k + ha_{k+\gamma} + w_{k+1} \\ q_{k+1} = q_k + hv_k + \dfrac{h^2}{2} a_{k+2\beta} + \dfrac{1}{2} hw_{k+1} \end{cases} \tag{53}$$

Special cases

- $\alpha_m = \alpha_f = 0$. Nonsmooth Newmark scheme
- $\alpha_m = 0$ and $\alpha_f = \alpha$. Nonsmooth HHT scheme.

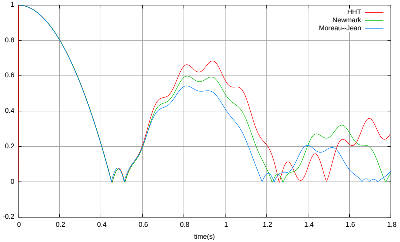
The nonsmooth generalized α scheme

Example (Two balls oscillator with impact)

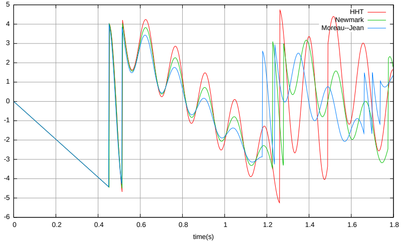


The nonsmooth generalized α scheme

time–step : $h = 2e-3$. Moreau ($\theta = 1.0$). Newmark ($\gamma = 1.0, \beta = 0.5$). HHT ($\alpha = 0.1$)

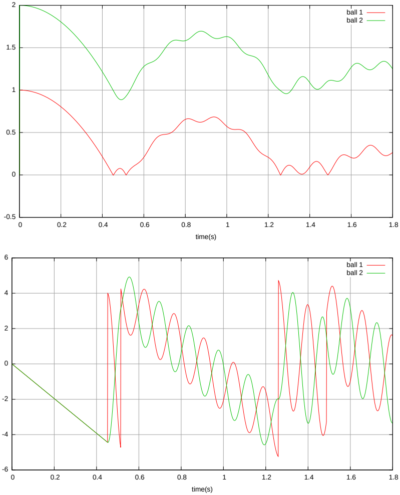


Position of the first ball

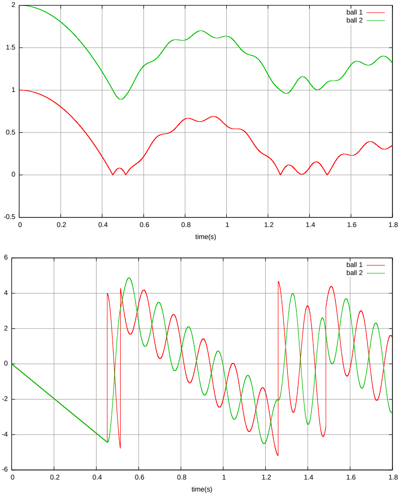


Velocity of the first ball

The nonsmooth generalized α scheme



HHT $h = 1e-3, \alpha = 0.1$



Moreau time –step $h = 1e-5, \theta = 1.0$

The nonsmooth generalized α scheme

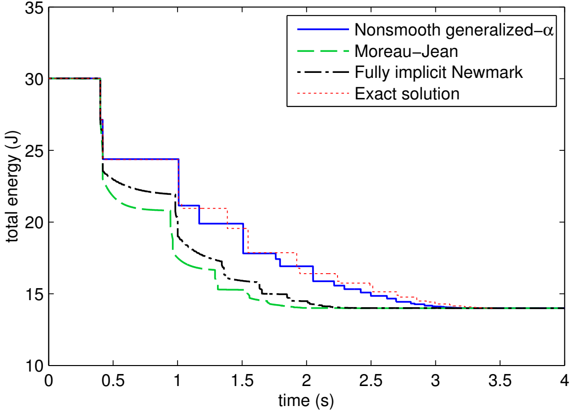


Figure 7. Numerical results for the total energy of the bouncing oscillator.

The nonsmooth generalized α scheme

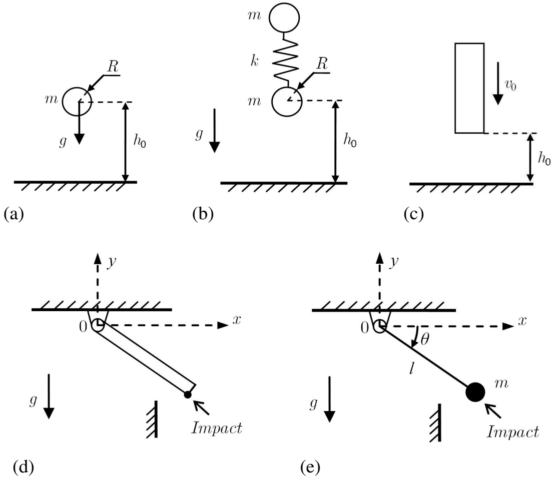


Figure 2. Examples: (a) bouncing ball; (b) linear vertical oscillator; (c) bouncing of an elastic bar; (d) bouncing of a nonlinear beam pendulum; (e)bouncing of a rigid pendulum

The nonsmooth generalized α scheme

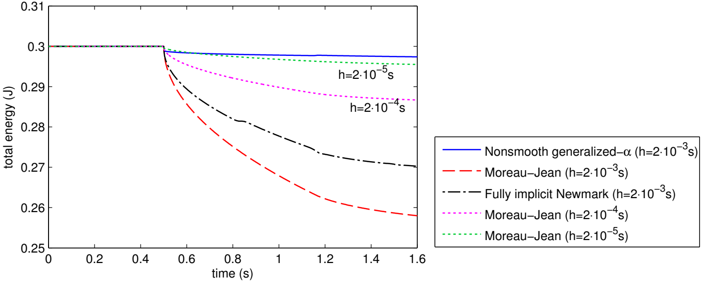


Figure 9. Numerical results for the total energy of the bouncing elastic bar

The nonsmooth generalized α scheme

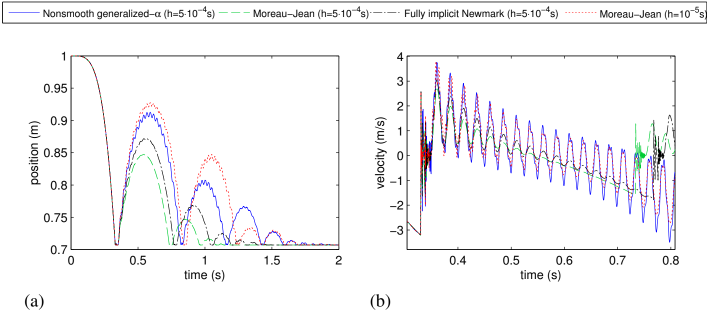


Figure 10. Numerical results for the impact of a flexible rotating beam: (a) position, (b) velocity.

The nonsmooth generalized α scheme

Observed properties on examples

- the scheme is consistent and globally of order one.
- the scheme seems to share the stability property as the original HHT
- the scheme dissipates energy only in high-frequency oscillations (w.r.t the time–step.)

Conclusions & perspectives

- Extension to any multi–step schemes can be done in the same way.
- Improvements of the order by splitting.
- Recast into time–discontinuous Galerkin formulation.

Energy analysis

Time–continuous energy balance equations

Let us start with the "LTI" Dynamics

$$\begin{cases} M\,dv + (Kq + Cv)\,\mathrm{d}t = F\,\mathrm{d}t + \mathrm{d}i \\ dq = v^{\pm}\mathrm{d}t \end{cases} \tag{54}$$

we get for the Energy Balance

$$d(v^\top Mv) + (v^+ + v^-)(Kq + Cv)\,\mathrm{d}t \quad = \quad (v^+ + v^-)F\,\mathrm{d}t + (v^+ + v^-)\,\mathrm{d}i \tag{55}$$

that is

$$2d\mathcal{E} := d(v^\top Mv) + 2q^\top Kdq \quad = \quad 2v^\top F\,\mathrm{d}t - 2v^\top Cv\,\mathrm{d}t + (v^+ + v^-)^\top\,\mathrm{d}i \tag{56}$$

with

$$\mathcal{E} := \frac{1}{2}v^\top Mv + \frac{1}{2}q^\top Kq. \tag{57}$$

Energy analysis

Time–continuous energy balance equations

If we split the differential measure in $\mathrm{d}i = \lambda\ \mathrm{d}t + \sum_i p_i \delta_{t_i}$, we get

$$2d\mathcal{E} = \quad = \quad 2v^\top (F+\lambda)\ \mathrm{d}t - 2v^\top Cv\ \mathrm{d}t + (v^+ + v^-)^\top p_i \delta_{t_i} \tag{58}$$

By integration over a time interval $[t_0, t_0]$ such that $t_i \in [t_0, t_1]$, we obtain an energy balance equation as

$$\begin{aligned} \Delta\mathcal{E} &:= \mathcal{E}(t_1) - \mathcal{E}(t_0) \\ &= \underbrace{\int_{t_0}^{t_1} v^\top F\,\mathrm{d}t}_{W^{\mathrm{ext}}} - \underbrace{\int_{t_0}^{t_1} v^\top Cv\ \mathrm{d}t}_{W^{\mathrm{damping}}} + \underbrace{\int_{t_0}^{t_1} v^\top \lambda\,\mathrm{d}t}_{W^{\mathrm{con}}} + \underbrace{\frac{1}{2}\sum_i (v^+(t_i) + v^-(t_i))^\top p_i}_{W^{\mathrm{impact}}} \end{aligned} \tag{59}$$

Energy analysis

Work performed by the reaction impulse $\mathrm{d}i$

- The term
$$W^{\text{con}} = \int_{t_0}^{t_1} v^\top \lambda \,\mathrm{d}t \tag{60}$$
is the work done by the contact forces within the time–step. If we consider perfect unilateral constraints, we have $W^{\text{con}} = 0$.
- The term
$$W^{\text{impact}} = \frac{1}{2}\sum_i (v^+(t_i) + v^-(t_i))^\top p_i \tag{61}$$
represents the work done by the contact impulse p_i at the time of impact t_i. Since $p_i = G(t_i)P_i$ and if we consider the Newton impact law, we have

$$\begin{aligned} W^{\text{impact}} &= \frac{1}{2}\textstyle\sum_i (v^+(t_i) + v^-(t_i))^\top G(t_i)P_i \\ &= \frac{1}{2}\textstyle\sum_i (U^+(t_i) + U^-(t_i))^\top P_i \\ &= \frac{1}{2}\textstyle\sum_i ((1-e)U^-(t_i))^\top P_i \leqslant 0 \text{ for } 0 \leqslant e \leqslant 1 \end{aligned} \tag{62}$$

with the local relative velocity defines as $U(t) = G^\top(t)v(t)$

Energy analysis for Moreau–Jean scheme

Let us define the discrete approximation of the work done by the external forces within the step by

$$W^{\text{ext}}_{k+1} = h v^{\top}_{k+\theta} F_{k+\theta} \approx \int_{t_k}^{t_{k+1}} F v \,\mathrm{d}t, \tag{63}$$

and the discrete approximation of the work done by the damping term by

$$W^{\text{damping}}_{k+1} = -h v^{\top}_{k+\theta} C v_{k+\theta} \approx -\int_{t_k}^{t_{k+1}} v^T C v \,\mathrm{d}t. \tag{64}$$

Lemma

The variation of the total mechanical energy over a time–step $[t_k, t_{k+1}]$ performed by the Moreau–Jean scheme (27) is

$$\Delta\mathcal{E} - W^{\text{ext}}_{k+1} - W^{\text{damping}}_{k+1} \quad = \quad (\frac{1}{2} - \theta)\left[\|v_{k+1} - v_k\|^2_M + \|(q_{k+1} - q_k)\|^2_K\right] + U^{\top}_{k+\theta} P_{k+1} \tag{65}$$

Energy analysis for Moreau–Jean scheme

Proposition

The Moreau–Jean scheme dissipates energy in the sense that

$$\mathcal{E}(t_{k+1}) - \mathcal{E}(t_k) \leqslant W^{\text{ext}}_{k+1} + W^{\text{damping}}_{k+1}, \tag{66}$$

if

$$\frac{1}{2} \leqslant \theta \leqslant \frac{1}{1+e} \leqslant 1. \tag{67}$$

where $e = \max e^{\alpha}, \alpha \in \mathcal{I}$. *In particular, for* $e = 0$, *we get* $\frac{1}{2} \leqslant \theta \leqslant 1$ *and or* $e = 1$, *we get* $\theta = \frac{1}{2}$.

Energy analysis for Moreau–Jean scheme

Variant of the Moreau scheme that always dissipates energy

Let us consider the variant of the Moreau scheme

$$\begin{cases} M(v_{k+1} - v_k) + hKq_{k+\theta} - hF_{k+\theta} = p_{k+1} = GP_{k+1}, & (68a) \\ q_{k+1} = q_k + hv_{k+1/2}, & (68b) \\ U_{k+1} = G^\top v_{k+1} & (68c) \\ \left.\begin{array}{l} \text{if} \quad \bar{g}^\alpha_{k+1} \leqslant 0 \text{ then } 0 \leqslant U^\alpha_{k+1} + eU^\alpha_k \perp P^\alpha_{k+1} \geqslant 0, \\ \text{otherwise } P^\alpha_{k+1} = 0. \end{array}\right. , \alpha \in \mathcal{I} & (68d) \end{cases}$$

Energy analysis for Moreau–Jean scheme

Lemma

The variation of the total mechanical energy performed by the scheme (68) over a time–step is

$$\Delta\mathcal{E} - \bar{W}^{\text{ext}}_{k+1} - W^{\text{damping}}_{k+1} \quad = \quad (\frac{1}{2} - \theta)\|(q_{k+1} - q_k)\|^2_K + U^\top_{k+1/2}P_{k+1} \tag{69}$$

If $\theta \geqslant 1/2$, then the scheme (68) dissipates energy in the sense that

$$\mathcal{E}(t_{k+1}) - \mathcal{E}(t_k) \leqslant \bar{W}^{\text{ext}}_{k+1} + W^{\text{damping}}_{k+1}. \tag{70}$$

Energy analysis for nonsmooth Newmark scheme

Let us define the discrete approximation of the work done by the external forces within the step by

$$W_{k+1}^{\text{ext}} = (q_{k+1} - q_k)^\top F_{k+\gamma} \approx \int_{t_k}^{t_{k+1}} Fv \,\mathrm{d}t \tag{71}$$

and the discrete approximation of the work done by the damping term by

$$W_{k+1}^{\text{damping}} = -(q_{k+1} - q_k)^\top C v_{k+\gamma} \approx -\int_{t_k}^{t_{k+1}} v^T C v \,\mathrm{d}t. \tag{72}$$

Lemma

The variation of energy over a time–step performed by the scheme is

$$\begin{aligned}\Delta\mathcal{E} - W_{k+1}^{\text{ext}} \quad - \quad & W_{k+1}^{\text{damping}} = (\frac{1}{2} - \gamma)\|(q_{k+1} - q_k)\|_K^2 + P_{k+1}^\top U_{k+1/2} \\ & + \frac{h}{2}(2\beta - \gamma)\left[(q_{k+1} - q_k)^\top K(v_{k+1} - v_k) + \|(v_{k+1} - v_k)\|_C^2\right] \\ & - \frac{h}{2}(2\beta - \gamma)\left[(v_{k+1} - v_k)^\top (F_{k+1} - F_k) - (a_{k+1} - a_k)^\top G P_{k+1}\right].\end{aligned} \tag{73}$$

Energy analysis for Newmark's scheme

Define an discrete "algorithmic energy" (discrete storage function) of the form

$$\mathcal{K}(q,v,a) = \mathcal{E}(q,v) + \frac{h^2}{4}(2\beta-\gamma)a^\top M a. \tag{74}$$

Proposition

The variation of the "algorithmic" energy $\Delta\mathcal{K}$ over a time–step performed by the nonsmooth Newmark scheme is

$$\begin{aligned}\Delta\mathcal{K} - W^{\text{ext}}_{k+1} - W^{\text{damping}}_{k+1} \quad = \quad & (\frac{1}{2}-\gamma)\left[\|q_{k+1}-q_k\|^2_K + \frac{h}{2}(2\beta-\gamma)\|(a_{k+1}-a_k)\|^2_M\right] \\ & + U^\top_{k+1/2}P_{k+1}.\end{aligned} \tag{75}$$

Moreover, the nonsmooth Newmark scheme dissipates the "algorithmic" energy $\mathcal{K}$ in the following sense

$$\Delta\mathcal{K} - W^{\text{ext}}_{k+1} - W^{\text{damping}}_{k+1} \leqslant 0, \tag{76}$$

for

$$2\beta \geqslant \gamma \geqslant \frac{1}{2}. \tag{77}$$

Energy analysis for HHT scheme

Augmented dynamics

Let us introduce the modified dynamics

$$Ma(t) + Cv(t) + Kq(t) = F(t) + \frac{\alpha}{\nu}[Kw(t) + Cx(t) - y(t)] \tag{78}$$

and the following auxiliary dynamics that filter the previous one

$$\begin{array}{rcl} \nu h\dot{w}(t) + w(t) & = & \nu h\dot{q}(t) \\ \nu h\dot{x}(t) + x(t) & = & \nu h\dot{v}(t) \\ \nu h\dot{y}(t) + y(t) & = & \nu h\dot{F}(t) \end{array} \tag{79}$$

Energy analysis for HHT scheme

Discretized Augmented dynamics

The equation (79) are discretized as follows

$$\begin{aligned}
\nu(w_{k+1} - w_k) + \frac{1}{2}(w_{k+1} + w_k) &= \nu(q_{k+1} - q_k) \\
\nu(x_{k+1} - x_k) + \frac{1}{2}(x_{k+1} + x_k) &= \nu(v_{k+1} - v_k) \\
\nu(y_{k+1} - y_k) + \frac{1}{2}(y_{k+1} + y_k) &= \nu(F_{k+1} - F_k)
\end{aligned} \tag{80}$$

or rearranging the terms

$$\begin{aligned}
(\frac{1}{2} + \nu)w_{k+1} + (\frac{1}{2} - \nu)w_k &= \nu(q_{k+1} - q_k) \\
(\frac{1}{2} + \nu)x_{k+1} + (\frac{1}{2} - \nu)x_k &= \nu(v_{k+1} - v_k) \\
(\frac{1}{2} + \nu)y_{k+1} + (\frac{1}{2} - \nu)y_k &= \nu(F_{k+1} - F_k)
\end{aligned} \tag{81}$$

With the special choice $\nu = \frac{1}{2}$, we obtain the HHT scheme collocation that is

$$Ma_{k+1} + (1-\alpha)[Kq_{k+1} + Cv_{k+1}] + \alpha[Kq_k + Cv_k] = (1-\alpha)F_{k+1} + \alpha F_k \tag{82}$$

Energy analysis for nonsmooth HHT scheme

Define an discrete "algorithmic energy" (discrete storage function) of the form

$$\mathcal{H}(q,v,a,w) = \mathcal{E}(q,v) + \frac{h^2}{4}(2\beta-\gamma)a^\top Ma + 2\alpha(1-\gamma)w^\top Kw. \tag{83}$$

Let us define the approximation of works as follows:

$$W^{\text{ext}}_{k+1} = (q_{k+1}-q_k)^\top \left[(1-\alpha)F_{k,\gamma} + \alpha F_{k-1,\gamma}\right] \approx \int_{t_k}^{t_{k+1}} Fv\,\mathrm{d}t \tag{84}$$

and

$$W^{\text{damping}}_{k+1} = -(q_{k+1}-q_k)^\top C\left[(1-\alpha)v_{k,\gamma} + \alpha v_{k-1,\gamma}\right] \approx -\int_{t_k}^{t_{k+1}} v^T Cv\,\mathrm{d}t. \tag{85}$$

Energy analysis for nonsmooth HHT scheme

Proposition

The variation of the "algorithmic" energy $\Delta\mathcal{H}$ over a time–step performed by the nonsmooth HHT scheme is

$$\begin{aligned}\Delta\mathcal{H} - W^{\text{ext}}_{k+1} - W^{\text{damping}}_{k+1} \quad = \quad & U^{\top}_{k+1/2}P_{k+1} - \frac{1}{2}h^2(\gamma - \frac{1}{2})(2\beta-\gamma)\|(a_{k+1}-a_k)\|^2_M \\ & -(\gamma - \frac{1}{2} - \alpha)\|q_{k+1}-q_k\|^2_K - 2\alpha(1-\gamma)\|z_{k+1}-z_k\|^2_K. \end{aligned} \tag{86}$$

Moreover, the nonsmooth HHT scheme dissipates the "algorithmic" energy $\mathcal{H}$ in the following sense

$$\Delta\mathcal{H} - W^{\text{ext}}_{k+1} - W^{\text{damping}}_{k+1} \leqslant 0, \tag{87}$$

if

$$2\beta \geqslant \gamma \geqslant \frac{1}{2} \quad \textit{and} \quad 0 \leqslant \alpha \leqslant \gamma - \frac{1}{2} \leqslant \frac{1}{2}. \tag{88}$$

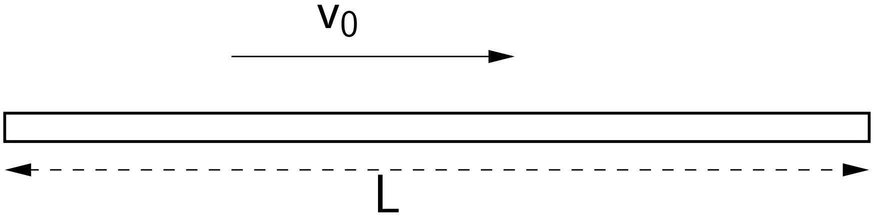
Energy analysis for nonsmooth schemes

Conclusions

- For the Moreau–Jean, a simple variant allows us to obtain a scheme which always dissipates energy.
- For the Newmark and the HHT scheme with retrieve the dissipation properties as the smooth case. The term associated with impact is added is the balance.
- For the generalized–α, similar analysis can be performed but some issues in the interpretation of results. New variant of the generalized–α scheme has been proposed
- Open Problem: We get dissipation inequality for discrete with quadratic storage function and plausible supply rate. The rest step is to conclude to the stability of the scheme with this argument. At least, we can bound discrete variable and conclude to the convergence of the scheme.

Impact in flexible structure

Example (The impacting bar)



Impact in flexible structure

Brief Literature

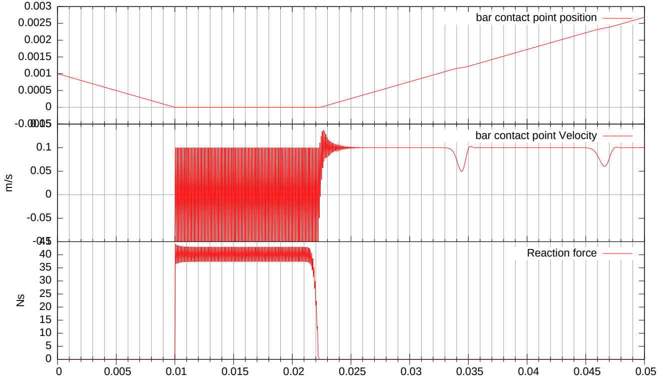
- (Hughes et al., 1976) Impact of two elastic bars. Standard Newmark in position and specific release and contact
- (Laursen and Love, 2002, 2003) Implicit treatment of contact reaction with a position level constraints
- (Chawla and Laursen, 1998 ; Laursen and Chawla, 1997) Implicit treatment of contact reaction with a pseudo velocity level constraints (algorithmic gap rate)
- (Vola et al., 1998) Comparison of Moreau–Jean scheme and standard Newmark scheme
- (Dumont and Paoli, 2006) Central–difference scheme with
- (Deuflhard et al., 2007) Contact stabilized Newmark scheme. Position level Newmark scheme with pre-projection of the velocity.
- (Doyen et al., 2011) Comparison of various position level schemes.

Although artifacts and oscillations are commonly observed, the question of nonsmoothness of the solution, the velocity based formulation and then a possible impact law in never addressed.

Impact in flexible structure

Position based constraints

1000 nodes. $v_0 = -0.1$. $h = 5.10^{-5}$ Nonsmooth Newmark scheme $\gamma = 0.6, \beta = \gamma/2$

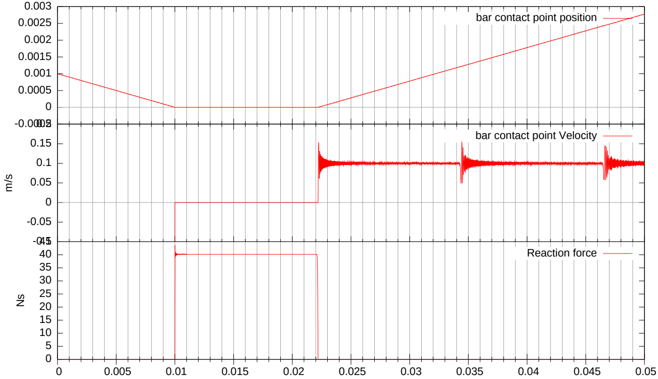


index 3 DAE problem: oscillations at the velocity level.$\Longrightarrow$ reduce the index.

Impact in flexible structure

Influence of high frequencies dissipation

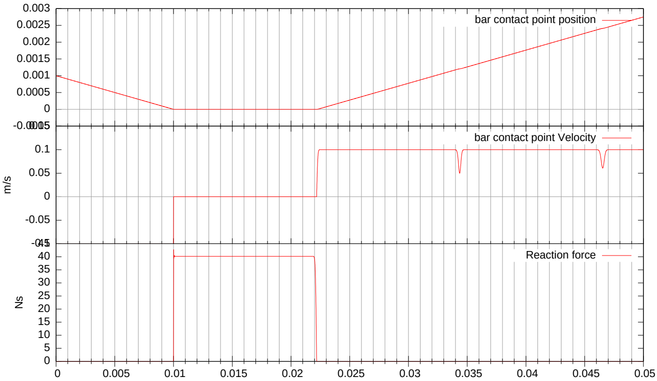
1000 nodes. $v_0 = -0.1$. $h = 5.10^{-6}$ $e = 0.0$ Nonsmooth Newmark scheme $\gamma = 0.5, \beta = \gamma/2$.



Impact in flexible structure

Influence of high frequencies dissipation

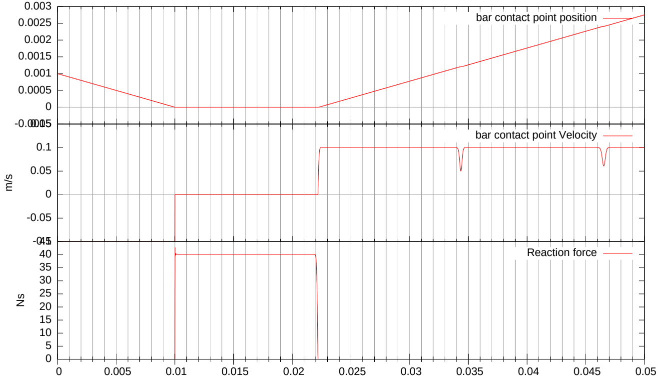
1000 nodes. $v_0 = -0.1$. $h = 5.10^{-6}$ $e = 0.0$ Nonsmooth Newmark scheme $\gamma = 0.6, \beta = \gamma/2$.



Impact in flexible structure

Influence of mesh discretization

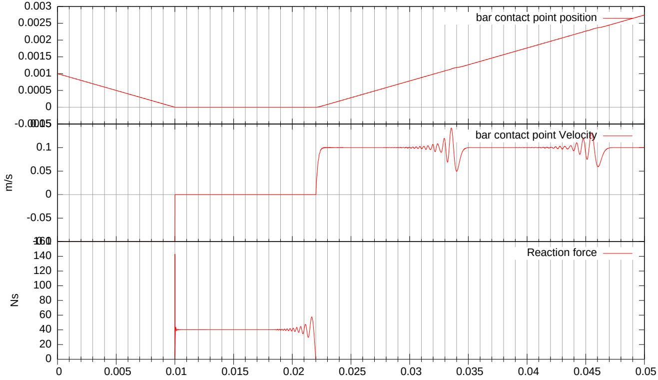
1000 nodes. $v_0 = -0.1$. $h = 5.10^{-6}$ $e = 0.0$ Nonsmooth Newmark scheme $\gamma = 0.6, \beta = \gamma/2$.



Impact in flexible structure

Influence of mesh discretization

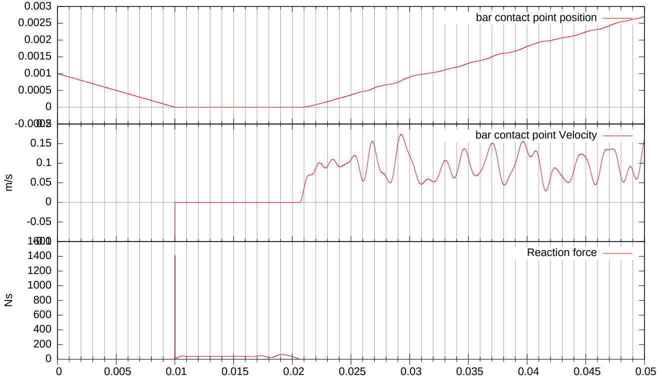
100 nodes. $v_0 = -0.1$. $h = 5.10^{-6}$ $e = 0.0$ Nonsmooth Newmark scheme $\gamma = 0.6, \beta = \gamma/2$.



Impact in flexible structure

Influence of mesh discretization

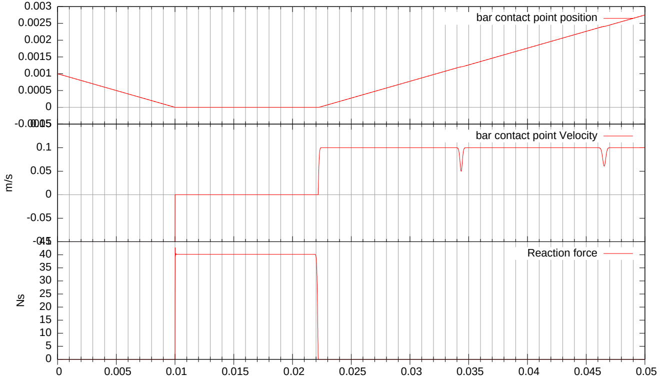
10 nodes. $v_0 = -0.1$. $h = 5.10^{-6}$ $e = 0.0$ Nonsmooth Newmark scheme $\gamma = 0.6, \beta = \gamma/2$.



Impact in flexible structure

Influence of time–step

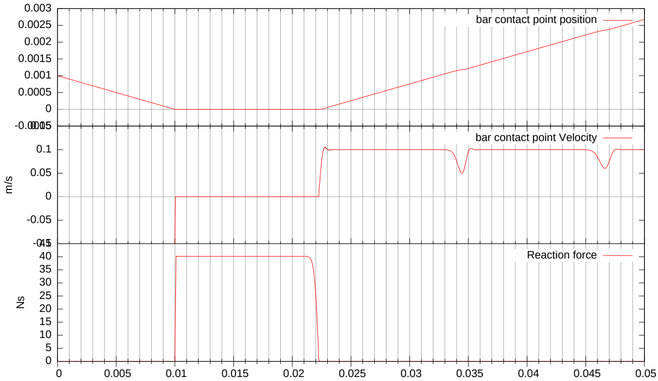
1000 nodes. $v_0 = -0.1$. $h = 5.10^{-6}$ $e = 0.0$ Nonsmooth Newmark scheme $\gamma = 0.6, \beta = \gamma/2$.



Impact in flexible structure

Influence of time–step

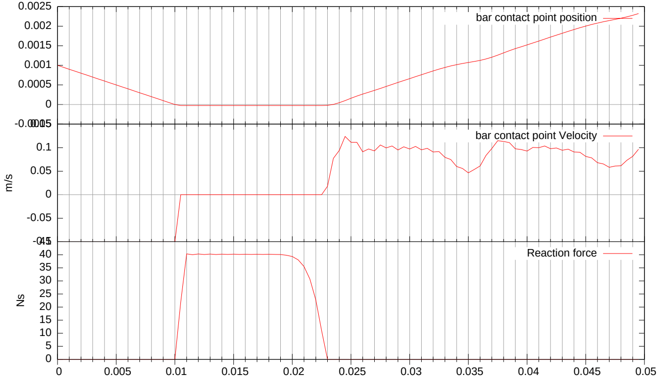
1000 nodes. $v_0 = -0.1$. $h = 5.10^{-5}$ $e = 0.0$ Nonsmooth Newmark scheme $\gamma = 0.6, \beta = \gamma/2$.



Impact in flexible structure

Influence of time–step

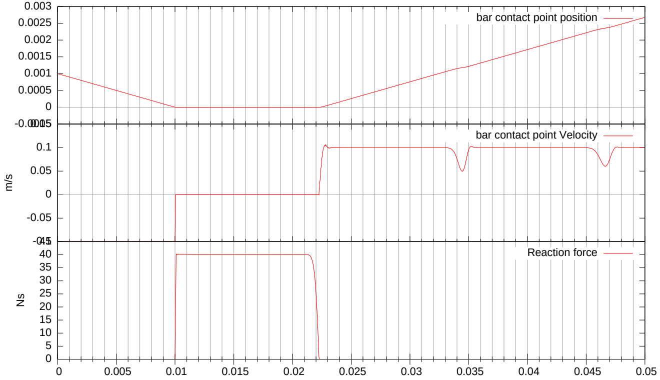
1000 nodes. $v_0 = -0.1$. $h = 5.10^{-4}$ $e = 0.0$ Nonsmooth Newmark scheme $\gamma = 0.6, \beta = \gamma/2$.



Impact in flexible structure

Influence of the coefficient of restitution

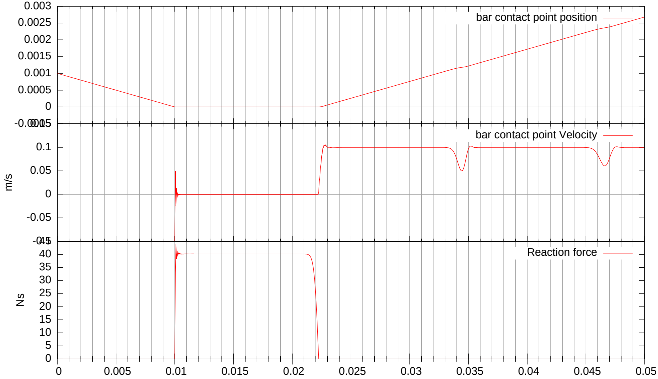
1000 nodes. $v_0 = -0.1$. $h = 5.10^{-5}$ $e = 0.0$ Nonsmooth Newmark scheme $\gamma = 0.6, \beta = \gamma/2$.



Impact in flexible structure

Influence of the coefficient of restitution

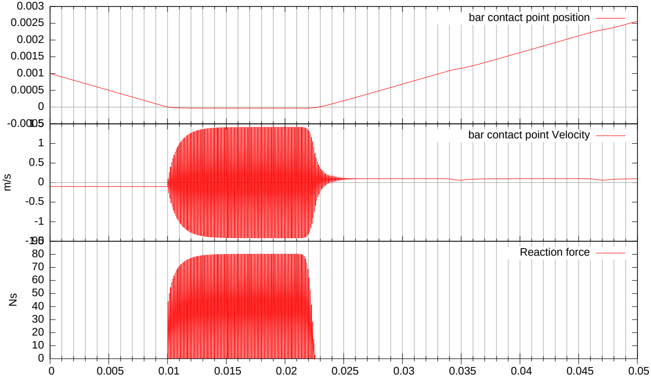
1000 nodes. $v_0 = -0.1$. $h = 5.10^{-5}$ $e = 0.5$ Nonsmooth Newmark scheme $\gamma = 0.6, \beta = \gamma/2$.



Impact in flexible structure

Influence of the coefficient of restitution

1000 nodes. $v_0 = -0.1$. $h = 5.10^{-5}$ $e = 1.0$ Nonsmooth Newmark scheme $\gamma = 0.6, \beta = \gamma/2$.



Impact in flexible structure

Discussion

- Reduction of order needs to write the constraints at the velocity level. Even in GGL approach.
- How to known if we need an impact law ? For a finite–freedom mechanical systems, we have to precise one. At the limit, the concept of coefficient of restitution can be a problem. Work of Michelle Schatzman.

Thank you for your attention.

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