Nonsmooth dynamics of extrinsic cohesive models for fracture

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Outline

INRIA Tripop Team

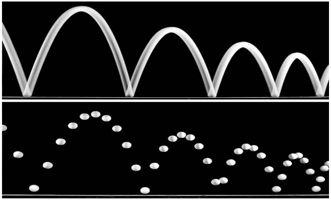
INRIA

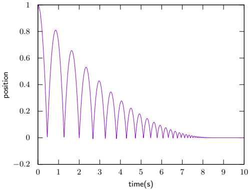
French national institute for computer sciences, applied mathematics and automatic control.

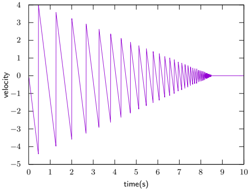
TRIPOP team-project

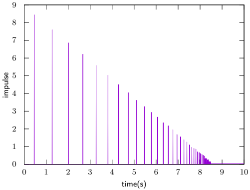
- Research object:
  Modeling, Simulation and Control of Nonsmooth Dynamics.
- Current main application: natural gravitational risks in mountains :
  - rockfall, rock slope stability, rock avalanche, landslides and debris flows
  - design of protection structures

nonsmooth = lack of continuity/differentiability





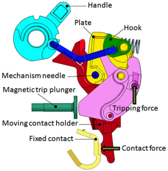


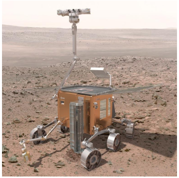


Where is nonsmoothness?

- nonsmooth solutions in time and space:
  - continuous, functions of bounded variations, measures and distributions.
- nonsmooth modeling of constitutive laws:
  - set–valued mapping, inequality constraints, complementarity, impact laws,
  - ODE with discontinuous r.h.s, differential inclusion, measure equation.

Application fields.





- Mechanical systems with unilateral contact, Coulomb friction and impacts : multi-body systems, robotic systems, frictional contact oscillators, granular materials, plasticity, fracture.
- Switched electrical circuits (diodes, transistors, switchs).
- Fluid Mechanics: cavitation, gas appearance multi-phasic fluid, permeability
- Hybrid and Cyber–physical systems
- Biology : gene regulatory networks
- Transportation networks with queues.

Nonsmooth approach is crucial for a correct modeling and an efficient simulation

Modeling and simulation in natural gravitational hazards

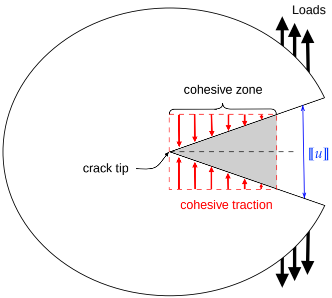
Fracture processes in gravitational hazards

- Rock fall trajectory simulation and rock mass flows
  - fracture of rocks in trajectory simulation
  - from instability to flow: understanding the emergence of large boulders, likely to have a long and dangerous trajectory from the collapse of the rock mass
- Stability of permafrost rock mass with global warning
- Concrete protection structures
  Fracture under impact of boulders

Modeling and simulation of fracture mechanics

- Linear Elastic Fracture Mechanics (LEFM) and Elasto-Plastic Fracture Mechanics (EPFM)
  - stress intensity factor, fracture toughness, Griffith and Irwin theory, …
- Phase-Field Models for Fracture
  - diffuse representation of the cracks without explicit crack tracking, handle complex crack patterns (branching, merging)
- Extended Finite Element Method (XFEM)
  - Incorporating discontinuities in the displacement field to avoid remeshing
- Cohesive Zone Models (CZM)
  - Introduces traction-separation laws to represent fracture

Cohesive zone models

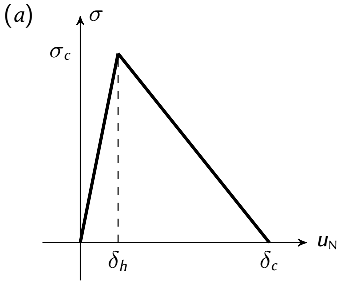


Why we chose CZM modeling?

- relatively “easy” to implement
- well suited for crack nucleation, branching and merging
- possible coarse-grain simulation for heterogeneous materials with uncertainties
- unilateral contact and Coulomb friction with impact as residual behavior of the interface

Extrinsic and intrinsic cohesive zone models

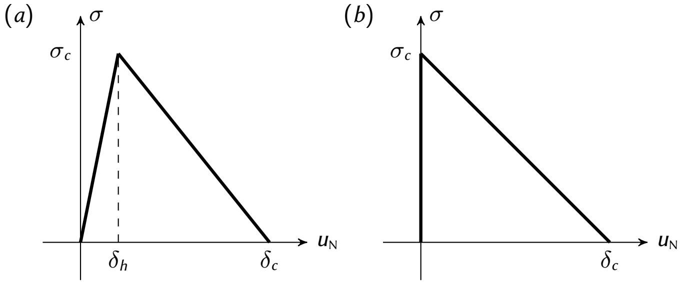
An intrinsic (a) and an extrinsic (b) cohesive zone model



- Intrinsic models : initial stiffness in the interface.
  - σ is a function of u_{N}
  - difficulty to give a value to the initial stiffness
  - modify the elasticity of the material prior to the crack
  - the effect is worse with a lot of interfaces (FEM applications)
  - need to use high initial stiffness value that implies numerical difficulties (stiff ODE systems)

Extrinsic and intrinsic cohesive zone models

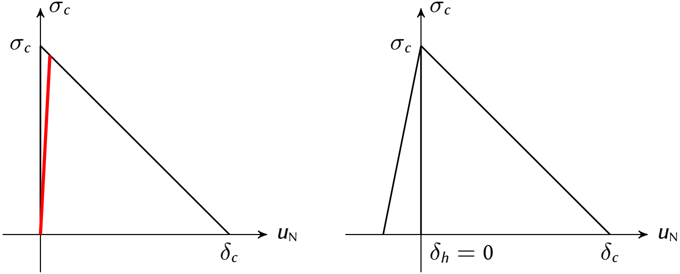
An intrinsic (a) and an extrinsic (b) cohesive zone model



- Extrinsic models : initially rigid, perfect bond, bilateral constraint.
  - the model is set-valued (like unilateral contact)
  - keep the original elasticity of the material.
  - bilateral constraints rather penalty (no stiff ODE)

Extrinsic and intrinsic cohesive zone models

Standard extrinsic and shifted intrinsic models



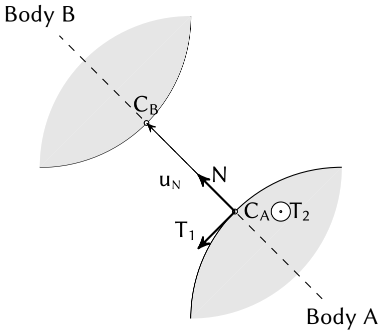
Standard extrinsic models

Shitfed intrinsic models

Motivations

- Design of an extrinsic set-valued CZM model:
  - with a residual behavior given by unilateral contact, Coulomb friction and an impact law
  - that satisfies thermodynamic principles (manly positive dissipation) in discrete time.
- A framework for nonsmooth fracture dynamics with an implicit time-stepping scheme for stability
- A time-stepping scheme that also satisfies thermodynamic principles (energy conservation and dissipation)
- A CZM model that can be solved completely implicitly using efficient tools from optimisation.
- Well-posed results for the one-step discrete problem (linear complementarity problem)

Signorini's condition and Coulomb's friction



- gap function (displacement jump)
  $u_{\mathrm{N}} = (C_B - C_A)\mathrm{N}.$
- reaction forces and velocities

$$r = r_{\mathrm{N}}\mathrm{N} + r_{\mathrm{T}}, \quad \text{with } r_{\mathrm{N}} \in \mathbb{R} \text{ and } r_{\mathrm{T}} \in \mathbb{R}^2.$$

$$v = v_{\mathrm{N}}\mathrm{N} + v_{\mathrm{T}}, \quad \text{with } v_{\mathrm{N}} \in \mathbb{R} \text{ and } v_{\mathrm{T}} \in \mathbb{R}^2.$$

- Signorini conditions

$$\text{position level} : 0 \leqslant u_{\mathrm{N}} \perp r_{\mathrm{N}} \geqslant 0.$$

$$\text{velocity level} : \begin{cases} 0 \leqslant v_{\mathrm{N}} \perp r_{\mathrm{N}} \geqslant 0 & \text{if } u_{\mathrm{N}} \leqslant 0 \\ r_{\mathrm{N}} = 0 & \text{otherwise} \end{cases}$$

Signorini's condition and Coulomb's friction

Modeling assumption

Let μ be the coefficient of friction. Let us define the Coulomb friction cone K which is chosen as the isotropic second order cone

$$K = \{r \in \mathbb{R}^3 \mid \|r_\mathrm{T}\| \leqslant \mu r_n\}.$$

Coulomb friction postulates

- for the sticking case that
$$v_\mathrm{T} = 0, \quad r \in K,$$
- and for the sliding case that
$$v_\mathrm{T} \neq 0, \quad r \in \partial K, \frac{r_\mathrm{T}}{\|r_\mathrm{T}\|} = -\frac{v_\mathrm{T}}{\|v_\mathrm{T}\|}.$$

Disjunctive formulation of the frictional contact behavior

$$\begin{cases} r = 0 & \text{if } u_\mathrm{N} > 0 \quad \text{(no contact)} \\ r = 0, v_\mathrm{N} \geqslant 0 & \text{if } u_\mathrm{N} \leqslant 0 \quad \text{(take–off)} \\ r \in K, v = 0 & \text{if } u_\mathrm{N} \leqslant 0 \quad \text{(sticking)} \\ r \in \partial K, v_\mathrm{N} = 0, \dfrac{r_\mathrm{T}}{\|r_\mathrm{T}\|} = -\dfrac{v_\mathrm{T}}{\|v_\mathrm{T}\|} & \text{if } u_\mathrm{N} \leqslant 0 \quad \text{(sliding)} \end{cases} \tag{1}$$

Thermodynamic framework

Unilateral contact (reversible process)

A nonsmooth free surface energy is postulated:

$$\Psi_S(u_{\mathrm{N}}, u_{\mathrm{T}}) = \mathcal{I}_{\mathbb{R}_+}(u_{\mathrm{N}}),$$

where $\mathcal{I}_C$ is the indicatrix function of a convex set C

$$\mathcal{I}_C(x) = \begin{cases} 0, & x \in C \\ +\infty, & x \notin C \end{cases}$$

The reversible reaction forces derives from this potential

$$-r^{\mathrm{r}}_{\mathrm{N}} \in \partial_{u_{\mathrm{N}}} \mathcal{I}_{\mathbb{R}_+}(u_{\mathrm{N}}) \Longleftrightarrow 0 \leqslant r^{\mathrm{r}}_{\mathrm{N}} \perp u_{\mathrm{N}} \geqslant 0 \qquad \text{(Signorini condition)},$$

where ∂ is the subdifferential of convex analysis.

$$-r^{\mathrm{r}}_{\mathrm{T}} \in \partial_{u_{\mathrm{T}}} \Psi_S(u_{\mathrm{N}}, u_{\mathrm{T}}) = 0$$

Thermodynamic framework of contact laws

Coulomb friction (irreversible process)

A nonsmooth pseudo-potential of dissipation is postulated

$$\Phi(v_\mathrm{N}, v_\mathrm{T}) = \mu r_\mathrm{N} |v_\mathrm{T}| \quad \text{(2D case)}.$$

The irreversible reaction forces derives from this pseudo-potential

$$-r_\mathrm{N}^{\mathrm{ir}} \in \partial_{v_\mathrm{N}} \Phi(v_\mathrm{N}, v_\mathrm{T}) = 0,$$

$$-r_\mathrm{T}^{\mathrm{ir}} \in \partial_{v_\mathrm{T}} \Phi(v_\mathrm{N}, v_\mathrm{T}) = \mu r_n \mathrm{sgn}(v_\mathrm{T}),$$

where sgn is the multivalued signum function.

Remark

The pseudo-potential Φ contains a dependence on r_N, due to the non associated character of Coulomb friction. The de Saxcé bi-potential framework would be more appropriate.

An extrinsic cohesive zone model

Frémond's approach is followed to develop a cohesive interface model that satisfies thermodynamic principles.

State variables of the interface

- $u_{\mathrm{N}}, u_{\mathrm{T}}$ displacement jumps
- $\beta \in [0, 1]$ cohesion state (damage-like variable)

$$\begin{aligned} \beta &= 1, && \text{completely intact interface} \\ \beta &= 0, && \text{completely fractured interface} \end{aligned}$$

Interface potentials

Constitutive laws derives from potentials to ensure thermodynamical principles (Clausius-Duhem inequality)

- $\Psi_S(u_{\mathrm{N}}, u_{\mathrm{T}}, \beta)$ free energy
- $\Phi(v_{\mathrm{N}}, v_{\mathrm{T}}, \dot{\beta})$ pseudo potential of dissipation

An extrinsic cohesive zone model

Free energy potential

$$\Psi_s(u_N, u_T, \beta) = \underbrace{\beta\sigma_c u_N + \beta\gamma\sigma_c|u_T|}_{\text{potential energy}} + \underbrace{wf(\beta)}_{\text{fracture energy}} + \underbrace{\mathcal{I}_{[0,1]}(\beta)}_{\text{constraints on } \beta} + \underbrace{\mathcal{I}_{\mathbb{R}^+}(u_N)}_{\text{unilateral contact}}$$

- w is the free energy released by the decohesion
- $f(\beta)$ is a function that describes the "shape" of the cohesive law,
- γ is the ratio of critical traction in mode II to mode I

Remarks

- unilateral contact is contained in the model
- β is constrained to be $[0, 1]$
- the potential energy related to u_N and u_T is piece linear to avoid the introduction of elasticity

An extrinsic cohesive zone model

State laws, constitutive laws for reversible processes

$$\begin{cases} -r^{\mathrm{r}}_{\mathrm{N}} & \in & \partial_{u_{\mathrm{N}}}\Psi_s(u_{\mathrm{N}}, u_{\mathrm{T}}, \beta) = \beta\sigma_c + \partial\mathcal{I}_{\mathbb{R}^+}(u_{\mathrm{N}}), \\ -r^{\mathrm{r}}_{\mathrm{T}} & \in & \partial_{u_{\mathrm{T}}}\Psi_s(u_{\mathrm{N}}, u_{\mathrm{T}}, \beta) = \beta\gamma\sigma_c \mathrm{sgn}(u_{\mathrm{T}}), \\ -A^{\mathrm{r}} & \in & \partial_{\beta}\Psi_s(u_{\mathrm{N}}, u_{\mathrm{T}}, \beta) = \sigma_c \left(u_{\mathrm{N}} + \gamma|u_{\mathrm{T}}|\right) + wf'(\beta) + \partial\mathcal{I}_{[0,1]}(\beta), \end{cases}$$

where A is the thermodynamic driving force associated with the cohesion state β.

- unilateral contact with cohesion

$$-(r^{\mathrm{r}}_{\mathrm{N}} + \beta\sigma_c) \in \partial\mathcal{I}_{\mathbb{R}^+}(u_{\mathrm{N}}) \Longleftrightarrow 0 \leqslant r^{\mathrm{r}}_{\mathrm{N}} + \beta\sigma_c \perp u_{\mathrm{N}} \geqslant 0$$

- set-valued tangential cohesion

$$-r^{\mathrm{r}}_{\mathrm{T}} \in \beta\gamma\sigma_c \mathrm{sgn}(u_{\mathrm{T}})$$

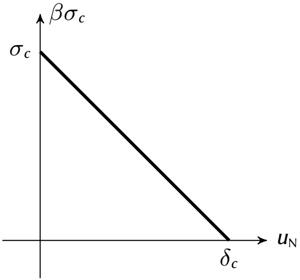
- cohesion state law

$$-(A^{\mathrm{r}} + \sigma_c \left(u_{\mathrm{N}} + \gamma|u_{\mathrm{T}}|\right) + wf'(\beta)) \in \partial\mathcal{I}_{[0,1]}(\beta)$$

An extrinsic cohesive zone model

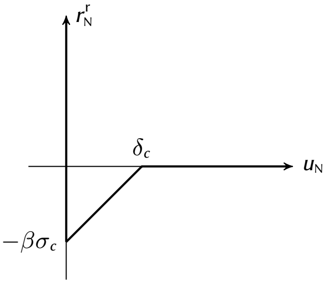
A simple triangle law as state cohesion law

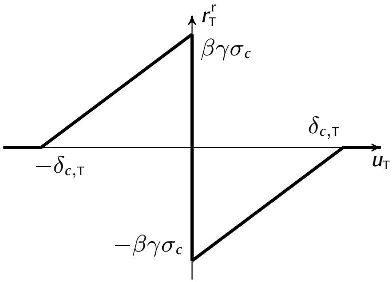
$$f(\beta) = (\beta^2 - 1), \qquad w = \frac{\sigma_c \delta_c}{2}$$



w area under the curve, free energy earns by the system.

An extrinsic cohesive zone model





An extrinsic cohesive zone model

Irreversible process (2D)

$$\Phi(v_{\mathrm{N}}, v_{\mathrm{T}}, \dot\beta) = \underbrace{\mathcal{I}_{\mathbb{R}^-}(\dot\beta)}_{\text{fracture irreversibility}} + \underbrace{\mu(r_{\mathrm{N}} + \beta\sigma_c)|v_{\mathrm{T}}|}_{\text{dissipation by friction}} \qquad (2)$$

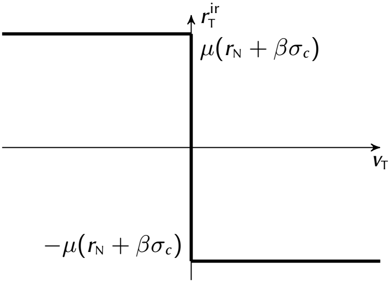
Comments

- the decohesion process is irreversible ($\dot\beta \leqslant 0$) but not dissipative and rate-independent.
- the friction threshold accounts for the cohesion force,

An extrinsic cohesive zone model

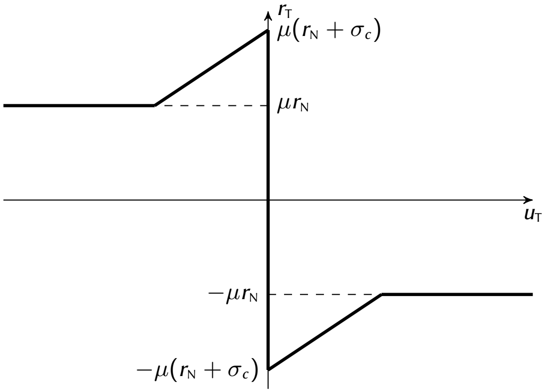
Irreversible process (2D). Constitutive laws

$$
\begin{aligned}
-r_{\mathrm{N}}^{\mathrm{ir}} &= \partial_{v_{\mathrm{N}}}\Phi(v_{\mathrm{N}}, v_{\mathrm{T}}, \dot{\beta}) = 0,\\
-r_{\mathrm{T}}^{\mathrm{ir}} &\in \partial_{v_{\mathrm{T}}}\Phi(v_{\mathrm{N}}, v_{\mathrm{T}}, \dot{\beta}) = \mu(r_{\mathrm{N}} + \beta\sigma_c)\mathrm{sgn}(v_{\mathrm{T}}),\\
-A^{\mathrm{ir}} &\in \partial_{\dot{\beta}}\Phi(v_{\mathrm{N}}, v_{\mathrm{T}}, \dot{\beta}) = \partial\mathcal{I}_{\mathbb{R}^-}(\dot{\beta}).
\end{aligned}
$$



An extrinsic cohesive zone model

The net tangential behaviour assuming $u_T = \pm v_T t$



Comments

The tangential depends on two separated terms: a cohesion forces that depends on displacement and a frictional forces that depends on the velocity.

An extrinsic cohesive zone model

From the principle of virtual power with no external power on β (pure internal variable), we have

$$\Theta = A^{\mathrm{r}} + A^{\mathrm{ir}} = 0,\, r_{\mathrm{N}} = r_{\mathrm{N}}^{\mathrm{r}},\, r_{\mathrm{T}} = r_{\mathrm{T}}^{\mathrm{r}} + r_{\mathrm{T}}^{\mathrm{ir}}.$$

Introducing the contact force,

$$r_{\mathrm{N}}^{\mathrm{con}} = r_{\mathrm{N}} + \beta\sigma_c \geqslant 0$$

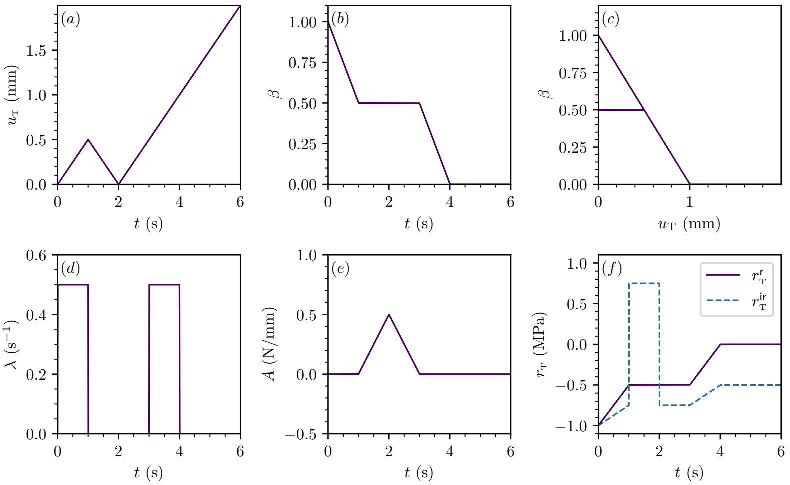
the extrinsic CZM can be written as

$$\begin{cases} \dot{\beta} = -\lambda, \\ A^{\mathrm{r}} + \sigma_c u_{\mathrm{N}} + \sigma_c \gamma |u_{\mathrm{T}}| + w f'(\beta) = \xi, \\ r_{\mathrm{N}}^{\mathrm{con}} = r_{\mathrm{N}} + \beta\sigma_c, \\ 0 \leqslant r_{\mathrm{N}}^{\mathrm{con}} \perp u_{\mathrm{N}} \geqslant 0, \\ 0 \leqslant \xi \perp \beta \geqslant 0, \\ 0 \leqslant \lambda \perp A^{\mathrm{r}} \geqslant 0, \\ -r_{\mathrm{T}}^{\mathrm{r}} \in \beta\sigma_c \mathrm{sgn}(u_{\mathrm{T}}) \\ -r_{\mathrm{T}}^{\mathrm{ir}} \in \mu r_{\mathrm{N}}^{\mathrm{con}} \mathrm{sgn}(u_{\mathrm{T}}) \end{cases} \tag{3}$$

where ξ and λ are slack variables (Lagrange multiplier) to enforce the constraint on β and $\dot{\beta}$.

An extrinsic cohesive zone model

Analytical solution for a simple shear test



An extrinsic cohesive zone model

- Other shape of the state law are possible
- Introduction of rate-dependent behavior is also possible,
- Other type of unloading behavior (see Curnier Talon for instance (vertical unloading))
- Direct elastic unloading is not directly possible due to the singularity for $\beta = 1$

Introduction & Motivations

Contact and friction modeling

An extrinsic cohesive zone model

Time–stepping scheme

Discrete energy principles

LCP solving

Numerical illustrations

Perspectives

Time-stepping schemes

Finite dimensional linear elasto-dynamics (after FEM for instance)

$$\begin{cases} M\dot{v} + Ku = F + H_{\mathrm{N}}^{\top} S r_{\mathrm{N}} + H_{\mathrm{T}}^{\top} S r_{\mathrm{T}}, \\ \dot{u} = v, \quad u_{\mathrm{N}} = H_{\mathrm{N}} u + b_{\mathrm{N}}, \quad H_{\mathrm{T}} u + b_{\mathrm{T}}, \end{cases} \tag{4}$$

where S is the matrix of cohesive area for each CZM points

Finite dimensional systems with unilateral constraints
$\implies$ velocity jumps and percussions

Nonsmooth dynamics

$$\begin{cases} M\,\mathrm{d}v + Ku\,\mathrm{d}t = F\,\mathrm{d}t + H_{\mathrm{N}}^{\top}\,\mathrm{d}i_{\mathrm{N}} + H_{\mathrm{T}}^{\top}\,\mathrm{d}i_{\mathrm{T}}, \\ \dot{u} = v, \end{cases} \tag{5}$$

where $\mathrm{d}v$ is the differential measure associated with v of bounded variations and $\mathrm{d}i$ is the interface impulses.

Time-stepping schemes

$\mathrm{d}t$ Lebesgue measure, $\mathrm{d}\nu$ discrete measure (a sum of Dirac atoms $\sum_i \delta_{t_i}$)

Measure decomposition w.r.t $\mathrm{d}t$

$$\frac{\mathrm{d}i_{\mathrm{N}}}{\mathrm{d}t} = -S\beta\sigma_c + \frac{\mathrm{d}i_{\mathrm{N}}^{\mathrm{con}}}{\mathrm{d}t}, \text{ and } \frac{\mathrm{d}i_{\mathrm{T}}}{\mathrm{d}t} = Sr_{\mathrm{T}}^{\mathrm{r}} + \frac{\mathrm{d}i_{\mathrm{T}}^{\mathrm{con}}}{\mathrm{d}t}, \quad \mathrm{d}t\text{–almost everywhere.} \tag{6}$$

with

$$\frac{\mathrm{d}i_{\mathrm{N}}^{\mathrm{con}}}{\mathrm{d}t} = Sr_{\mathrm{N}}^{\mathrm{con}}, \text{ and } \frac{\mathrm{d}i_{\mathrm{T}}^{\mathrm{con}}}{\mathrm{d}t} = Sr_{\mathrm{T}}^{\mathrm{ir}}, \quad \mathrm{d}t\text{–almost everywhere.} \tag{7}$$

Measure decomposition w.r.t $\mathrm{d}\nu$

$$p_{\mathrm{N}} = \frac{\mathrm{d}i_{\mathrm{N}}^{\mathrm{con}}}{\mathrm{d}\nu} = \frac{\mathrm{d}i_{\mathrm{N}}}{\mathrm{d}\nu}, \quad \text{and} \quad p_{\mathrm{T}} = \frac{\mathrm{d}i_{\mathrm{T}}^{\mathrm{con}}}{\mathrm{d}\nu} = \frac{\mathrm{d}i_{\mathrm{T}}}{\mathrm{d}\nu}, \quad \mathrm{d}\nu\text{–almost everywhere.} \tag{8}$$

Remarks

- Cohesive forces $-S\beta\sigma_c$ and $Sr_{\mathrm{T}}^{\mathrm{ir}}$ have no Dirac atom
- New variables p_{N} and p_{T} $\implies$ additional constitutive laws (impact laws).

Time-stepping schemes

Additional constitutive laws (impact laws)

- Newton impact laws

$$0 \leqslant p_{\mathrm{N}} \perp v_{\mathrm{N}}^{+} + e v_{\mathrm{N}}^{-} \geqslant 0 \text{ if } u_{\mathrm{N}} \leqslant 0, \text{ else } p_{\mathrm{N}} = 0, \tag{9}$$

where e is a coefficient of restitution ($e = 0$ in FEM applications)

- Coulomb's friction at impact (Frémond impact with friction)

$$-p_{\mathrm{T}} \in \mu p_{\mathrm{N}} \operatorname{sgn}(\frac{1}{2}(v_{\mathrm{T}}^{+} + v_{\mathrm{T}}^{-})). \tag{10}$$

Measure formulation

$$0 \leqslant \mathrm{d}i_{\mathrm{N}}^{\mathrm{con}} \perp v_{\mathrm{N}}^{+} + e v_{\mathrm{N}}^{-} \geqslant 0 \text{ if } u_{\mathrm{N}} \leqslant 0, \text{ else } \mathrm{d}i_{\mathrm{N}}^{\mathrm{con}} = 0. \tag{11}$$

$$-\mathrm{d}i_{\mathrm{T}}^{\mathrm{con}} \in \mu\, \mathrm{d}i_{\mathrm{N}}^{\mathrm{con}} \operatorname{sgn}(\frac{1}{2}(v_{\mathrm{T}}^{+} + v_{\mathrm{T}}^{-})). \tag{12}$$

Time-stepping schemes

Nonsmooth dynamics to discretize in time

$$
\begin{cases}
M\,\mathrm{d}v + Ku\,\mathrm{d}t = F\,\mathrm{d}t + \bar{H}_{\mathrm{N}}^{\top}\,\mathrm{d}p_{\mathrm{N}} + \bar{H}_{\mathrm{T}}^{\top}\,\mathrm{d}p_{\mathrm{T}} - H_{\mathrm{N}}^{\top} S\beta\sigma_c\,\mathrm{d}t + H_{\mathrm{T}}^{\top} Sr_{\mathrm{T}}^{\mathrm{r}}\,\mathrm{d}t,\\
\dot{u} = v, \quad u_{\mathrm{N}} = H_{\mathrm{N}}u + b_{\mathrm{N}}, \quad u_{\mathrm{T}} = H_{\mathrm{T}}u + b_{\mathrm{T}}, \quad v_{\mathrm{N}} = \bar{H}_{\mathrm{N}}v, \quad v_{\mathrm{T}} = \bar{H}_{\mathrm{T}}v,\\
\dot{\beta} = -\lambda,\\
A^{\mathrm{r}} + \sigma_c u_{\mathrm{N}} + \sigma_c\gamma|u_{\mathrm{T}}| + \sigma_c\delta_{c,\mathrm{N}}(\beta - 1) = \xi,\\
0 \leqslant \xi \perp \beta \geqslant 0,\\
0 \leqslant \lambda \perp A^{\mathrm{r}} \geqslant 0,\\
-r_{\mathrm{T}}^{\mathrm{r}} = \beta\gamma\sigma_c \mathrm{sgn}(u_{\mathrm{T}})\\
0 \leqslant \mathrm{d}i_{\mathrm{N}}^{\mathrm{con}} \perp v_{\mathrm{N}}^{+} + ev_{\mathrm{N}}^{-} \geqslant 0\\
-\mathrm{d}i_{\mathrm{T}}^{\mathrm{con}} \in \mu\,\mathrm{d}i_{\mathrm{N}}^{\mathrm{con}}\mathrm{sgn}(\tfrac{1}{2}(v_{\mathrm{T}}^{+} + v_{\mathrm{T}}^{-})).
\end{cases}
\tag{13}
$$

Time-stepping schemes

Principles of Moreau–Jean scheme

Measure of interval $(k, k+1]$ as primary unknown

$$
\begin{array}{l}
p_{\mathrm{N},k,k+1} \approx \mathrm{d}i^{\mathrm{con}}_{\mathrm{N}}((k,k+1]) = \int_{(k,k+1]} \mathrm{d}i^{\mathrm{con}}_{\mathrm{N}} \\
p_{\mathrm{T},k,k+1} \approx \mathrm{d}i^{\mathrm{con}}_{\mathrm{T}}((k,k+1]) = \int_{(k,k+1]} \mathrm{d}i^{\mathrm{con}}_{\mathrm{T}}
\end{array} \tag{14}
$$

Approximation of Lebesgue integrable terms with a θ-method ($\theta \in (0, 1]$)

$$
\int_{t_k}^{t_{k+1}} x(t)\,\mathrm{d}t \approx h x_{k+\theta}
$$

For instance for the cohesion impulses.

$$
\int_{(k,k+1]} \mathrm{d}i_{\mathrm{N}} = \int_{(k,k+1]} \mathrm{d}i^{\mathrm{con}}_{\mathrm{N}} - S\sigma_c \int_{t_k}^{t_{k+1}} \beta\,\mathrm{d}t \approx p_{\mathrm{N},k,k+1} - hS\sigma_c\beta_{k+\theta}, \tag{15}
$$

and

$$
\int_{(k,k+1]} \mathrm{d}i_{\mathrm{T}} = \int_{(k,k+1]} \mathrm{d}i^{\mathrm{con}}_{\mathrm{T}} + S \int_{t_k}^{t_{k+1}} r^{\mathrm{r}}_{\mathrm{T}}\,\mathrm{d}t \approx p_{\mathrm{T},k,k+1} + hSr^{\mathrm{r}}_{\mathrm{T},k+\theta}, \tag{16}
$$

Time-stepping schemes

Time-stepping scheme for the full elasto-dynamic cohesive-frictional-contact problem

$$\begin{cases} M(v_{k+1} - v_k) + hKu_{k+\theta} = hF_{k+\theta} - h\sigma_c H_{\mathrm{N}}^{\top} S\beta_{k+\theta} + hH_{\mathrm{T}}^{\top} r^{\mathrm{r}}_{\mathrm{T},k+\theta} + \bar{H}_{\mathrm{N}}^{\top} p_{\mathrm{N},k,k+1} + \bar{H}_{\mathrm{T}}^{\top} p_{\mathrm{T},k,k+1} \\ u_{k+1} = u_k + hv_{k+\theta}, \\ u_{\mathrm{N},k+\theta} = H_{\mathrm{N}} u_{k+\theta} + b_{\mathrm{N},k+\theta}, \quad u_{\mathrm{T},k+\theta} = H_{\mathrm{T}} u_{k+\theta} + b_{\mathrm{T},k+\theta}, \\ v_{\mathrm{N},k+\theta} = \bar{H}_{\mathrm{N}} v_{k+\theta}, \quad v_{\mathrm{T},k+\theta} = \bar{H}_{\mathrm{T}} v_{k+\theta}, \\ \beta_{k+1} = \beta_k - h\lambda_{k+\theta}, \\ \sigma_c \delta_{c,\mathrm{N}}(\beta_{k+\theta} - 1) + \sigma_c u_{\mathrm{N},k+\theta} + \sigma_c \gamma |u_{\mathrm{T},k+\theta}| + A^{\mathrm{r}}_{k+1} = \xi_{k+\theta}, \\ 0 \leqslant A^{\mathrm{r}}_{k+\theta} \perp \lambda_{k+\theta} \geqslant 0, \\ 0 \leqslant \beta_{k+\theta} \perp \xi_{k+\theta} \geqslant 0, \\ -r^{\mathrm{r}}_{\mathrm{T},k+\theta} = \beta_{k+\theta}\gamma\sigma_c \mathrm{sgn}(u_{\mathrm{T},k+\theta}) \\ 0 \leqslant p_{\mathrm{N},k,k+1} \perp \theta v_{\mathrm{N},k+\theta} + (\theta(1+e) - 1) v_{\mathrm{N},k} \geqslant 0 \\ -p_{\mathrm{T},k,k+1} \in \mu p_{\mathrm{N},k,k+1} \mathrm{sgn}(v_{k+\theta}). \end{cases} \tag{17}$$

This problem is a finite-dimensional variational inequality.

Discrete energy balance

Continuous space and time energy Balance

$$\Delta\mathcal{K} + \Delta\mathcal{U} + \Delta\mathcal{G} + \Delta\mathcal{F} = \int_{t_1}^{t_2} \mathcal{P}_{\text{ext}}\,\mathrm{d}t, \text{ and } \Delta\mathcal{E} = \Delta\mathcal{U} + \Delta\mathcal{G}$$

- Kinetic Energy $\mathcal{K} = \int_\Omega \rho v \cdot v \,\mathrm{d}x$
- Elastic potential energy $\mathcal{U} = \int_\Omega \boldsymbol{\varepsilon} : \boldsymbol{E} : \boldsymbol{\varepsilon} \,\mathrm{d}x$
- Fracture energy

$$\mathcal{G} = \int\int_\Gamma \beta\sigma_c v_{\mathrm{N}} - v_{\mathrm{T}} r_{\mathrm{T}}^{\mathrm{r}} \,\mathrm{d}x\,\mathrm{d}t = \int\int_\Gamma \dot{\psi}\,\mathrm{d}x\,\mathrm{d}t$$

- Dissipation energy by friction

$$\mathcal{F} = \int\int_\Gamma -v_{\mathrm{T}} r_{\mathrm{T}}^{\mathrm{ir}} \,\mathrm{d}x\,\mathrm{d}t$$

Discrete energy balance

Time continuous space discretized energy Balance

$$\mathrm{d}\mathsf{K} + \mathrm{d}\mathsf{U} + \mathrm{d}\mathsf{G} = v^\top F\,\mathrm{d}t + \frac{1}{2}\left(v_{\mathsf{N}}^{+} + v_{\mathsf{N}}^{-}\right)^\top \mathrm{d}p_{\mathsf{N}} + \frac{1}{2}\left(v_{\mathsf{T}}^{+} + v_{\mathsf{T}}^{-}\right)^\top \mathrm{d}p_{\mathsf{T}}.$$

- Kinetic energy $\mathsf{K} = \frac{1}{2} v^\top M v$
- Elastic potential energy $\mathsf{U} = \frac{1}{2} u^\top K u$
- Fracture Energy

$$\mathsf{G} = \int \sigma_c S \beta v_{\mathsf{N}} - v_{\mathsf{T}} S r_{\mathsf{T}}^{\mathsf{r}}\,\mathrm{d}t,$$

Discrete energy balance

Integrated form

$$\Delta(\mathsf{K}+\mathsf{U}+\mathsf{G}) = \Delta\mathsf{T} = \mathsf{T}^{+}(t_2) - \mathsf{T}^{-}(t_1) = \Delta W_{\text{ext}} + \Delta W_{\text{impact}} + \Delta W_{\text{friction}},$$

$$\begin{aligned}
\Delta W_{\text{ext}} &= \int_{t_1}^{t_2} v^\top F \, \mathrm{d}t, \\
\Delta W_{\text{impact}} &= \int_{(t_1,t_2]} \frac{1}{2}\left(v_{\mathrm{N}}^{+} + v_{\mathrm{N}}^{-}\right)^\top \mathrm{d}p_{\mathrm{N}}, \\
\Delta W_{\text{friction}} &= \int_{(t_1,t_2]} \frac{1}{2}\left(v_{\mathrm{T}}^{+} + v_{\mathrm{T}}^{-}\right)^\top \mathrm{d}p_{\mathrm{T}}.
\end{aligned}$$

Discrete energy balance

Discrete time energy balance

The scheme is dissipative

$$\Delta \mathrm{K}_{k,k+1} + \Delta \mathrm{U}_{k,k+1} + \Delta \mathrm{G}_{k,k+1} - \Delta \mathrm{W}_{\mathrm{ext},k,k+1} \leqslant \Delta \mathrm{W}_{\mathrm{impact},k,k+1} + \Delta \mathrm{W}_{\mathrm{friction},k,k+1} \leqslant 0$$

provided that

$$\frac{1}{2} \leqslant \theta \leqslant \frac{1}{1+e} \leqslant 1.$$

with

- $\Delta \mathrm{G}_{k,k+1} = h\left(v_{\mathrm{N},k+\theta}^\top \sigma_c S \beta_{k+\theta} - S v_{\mathrm{T},k+\theta}^\top r^{\mathrm{r}}_{\mathrm{T},k+\theta}\right) \approx \int \sigma_c S \beta v_{\mathrm{N}} - S v_{\mathrm{T}} r^{\mathrm{r}}_{\mathrm{T}} \,\mathrm{d}t$
- $\Delta \mathrm{W}_{\mathrm{ext},k,k+1} = h v_{k+\theta}^\top F_{k+\theta} \approx \int_{t_k}^{t_{k+1}} v^\top F \,\mathrm{d}t$
- $\Delta \mathrm{W}_{\mathrm{impact},k,k+1} = v_{\mathrm{N},k+\theta}^\top p_{\mathrm{N},k,k+1}$
- $\Delta \mathrm{W}_{\mathrm{friction},k,k+1} = v_{\mathrm{T},k+\theta}^\top p_{\mathrm{T},k,k+1},$

Discrete time energy balance for $\theta = 1/2$

$$\Delta \mathrm{K}_{k,k+1} + \Delta \mathrm{U}_{k,k+1} + \Delta \mathrm{G}_{k,k+1} - \Delta \mathrm{W}_{\mathrm{ext},k,k+1} = \Delta \mathrm{W}_{\mathrm{impact},k,k+1} + \Delta \mathrm{W}_{\mathrm{friction},k,k+1} \leqslant 0.$$

Linear complementarity formulation and solving

We chose to formulate our model in the form of a Linear Complementarity Problem(LCP)

Linear Complementarity Problem (LCP)

The Linear Complementarity Problem denoted by $\mathrm{LCP}(\mathrm{L}, \mathrm{q})$ is to find w and z such that

$$\begin{cases} w = Lz + q \\ 0 \leqslant w \perp z \geqslant 0 \end{cases}$$

Other formulation of the variational inequality are possible depending on the numerical solution procedure (projected fixed point, semi–smooth Newton method)

Linear complementarity formulation and solving

Lemma

The solution y_N, y_T, x_N, x_T of the following inclusion

$$\begin{cases} 0 \leqslant y_N \perp x_N \geqslant 0, \\ -y_T \in y_N sgn(x_T) = y_N \partial |x_T|, \end{cases} \tag{18}$$

is given by solving the following complementarity system

$$\begin{cases} 0 \leqslant y_N \perp x_N \geqslant 0, \\ 0 \leqslant \hat{y}_T \perp \mathbb{1}\lambda + D^\top x_T \geqslant 0, \\ 0 \leqslant \lambda \perp y_N - \mathbb{1}^\top \hat{y}_T \geqslant 0, \end{cases} \tag{19}$$

with $y_T = D\hat{y}_T$ and $D = [1, -1]$. Furthermore, we have $y_N|x_T| = y_N \lambda$.

Linear complementarity formulation and solving

Using Lemma 1, the system

$$
\begin{cases}
0 \leqslant \beta_{k+\theta} \perp \xi_{k+\theta} \geqslant 0, \\
-r^{\mathrm{r}}_{\mathrm{T},k+\theta} = \gamma \sigma_c \beta_{k+\theta} \mathrm{sgn}(u_{\mathrm{T},k+\theta}), \\
0 \leqslant p_{\mathrm{N},k,k+1} \perp \theta v_{\mathrm{N},k+\theta} + (\theta(1+e) - 1) v_{\mathrm{N},k} \geqslant 0, \\
-p_{\mathrm{T},k,k+1} \in \mu p_{\mathrm{N},k,k+1} \mathrm{sgn}(v_{k+\theta}).,
\end{cases} \tag{20}
$$

can be rewritten as

$$
\begin{cases}
r^{\mathrm{r}}_{\mathrm{T},k+\theta} = D \hat{r}^{\mathrm{r}}_{\mathrm{T},k+\theta}, \\
0 \leqslant S\beta_{k+\theta} \perp \xi_{k+\theta} \geqslant 0, \\
0 \leqslant \hat{r}^{\mathrm{r}}_{\mathrm{T},k+\theta} \perp \mathbb{1} \chi_{k+1} + D^\top u_{\mathrm{T},k+\theta} \geqslant 0, \\
0 \leqslant \chi_{k+\theta} \perp \sigma_c \gamma S \beta_{k+\theta} - \mathbb{1}^\top \hat{r}^{\mathrm{r}}_{\mathrm{T},k+\theta} \geqslant 0, \\
p^{\mathrm{r}}_{\mathrm{T},k+\theta} = D \hat{p}_{\mathrm{T},k+\theta}, \\
0 \leqslant p_{\mathrm{N},k,k+1} \perp v_{\mathrm{N},k+\theta} + (\theta(1+e) - 1) v_{\mathrm{N},k} \geqslant 0, \\
0 \leqslant \hat{p}_{\mathrm{T},k,k+1} \perp \mathbb{1} \zeta_{k+\theta} + D^\top v_{\mathrm{T},k+\theta} \geqslant 0, \\
0 \leqslant \zeta_{k+\theta} \perp \mu p_{\mathrm{N},k,k+1} - \mathbb{1}^\top \hat{p}_{\mathrm{T},k,k+1} \geqslant 0.
\end{cases} \tag{21}
$$

Linear complementarity formulation and solving

The complementarity variable vectors w and z are given by:

$$w = \begin{bmatrix} h\theta\lambda_{k+\theta} \\ \xi_{k+\theta} \\ \theta v_{\mathrm{N},k+\theta} + \theta(\theta(1+e)-1)v_{\mathrm{N},k} \\ \mathbb{1}\theta\zeta_{k+\theta} + \theta D^\top v_{\mathrm{T},k+\theta} \\ \mu p_{\mathrm{N},k,k+1} - \mathbb{1}^\top \hat{p}_{\mathrm{T},k,k+1} \\ \mathbb{1}\chi_{k+\theta} + D^\top u_{\mathrm{T},k+\theta} \\ \sigma_c\gamma S\beta_{k+\theta} - \mathbb{1}^\top \hat{r}^{\mathrm{r}}_{\mathrm{T},k+\theta} \end{bmatrix}, \quad z = \begin{bmatrix} SA^{\mathrm{r}}_{k+\theta} \\ S\beta_{k+\theta} \\ p_{\mathrm{N},k,k+1} \\ \hat{p}_{\mathrm{T},k,k+1} \\ \theta\zeta_{k+\theta} \\ \hat{r}^{\mathrm{r}}_{\mathrm{T},k+\theta} \\ \chi_{k+\theta} \end{bmatrix}, \tag{22}$$

Linear complementarity formulation and solving

After some simple (admittedly somewhat cumbersome) operations to substitute variables in linear equations, we obtain

$$L = \begin{bmatrix} \mathbf{0}^{m\times m} & -S^{-1} & \mathbf{0}^{m\times m} & \mathbf{0}^{m\times 2m} & \mathbf{0}^{m\times m} & \mathbf{0}^{m\times 2m} & \mathbf{0}^{m\times m} \\ S^{-1} & \sigma_c(\delta_{c,\mathrm{N}} S^{-1} - h^2\theta^2\sigma_c U_{\mathrm{NN}}) & h\theta^2\sigma_c V_{\mathrm{NN}}^\top & h\theta^2\sigma_c V_{\mathrm{TN}}^\top D & \mathbf{0}^{m\times m} & h^2\theta^2\sigma_c U_{\mathrm{NT}} D & \sigma_c\gamma I \\ \mathbf{0}^{m\times m} & -h\theta^2\sigma_c V_{\mathrm{NN}} & \theta^2 W_{\mathrm{NN}} & \theta^2 W_{\mathrm{NT}} D & \mathbf{0}^{m\times m} & h\theta^2 V_{\mathrm{NT}} D & \mathbf{0}^{m\times m} \\ \mathbf{0}^{2m\times m} & -h\theta^2\sigma_c D^\top V_{\mathrm{TN}} & \theta^2 D^\top W_{\mathrm{TN}} & \theta^2 D^\top W_{\mathrm{TT}} D & \mathbb{1} & h\theta^2 D^\top V_{\mathrm{TT}} D & \mathbf{0}^{2m\times m} \\ \mathbf{0}^{m\times m} & \mathbf{0}^{m\times m} & \mu I & -\mathbb{1}^\top & \mathbf{0}^{m\times m} & \mathbf{0}^{m\times 2m} & \mathbf{0}^{m\times m} \\ \mathbf{0}^{2m\times m} & -h^2\theta^2\sigma_c D^\top U_{\mathrm{TN}} & h\theta^2 D^\top V_{\mathrm{NT}}^\top & h\theta^2 D^\top V_{\mathrm{TT}}^\top D & \mathbf{0}^{2m\times m} & h^2\theta^2 D^\top U_{\mathrm{TT}} D & \mathbb{1} \\ \mathbf{0}^{m\times m} & \sigma_c\gamma I & \mathbf{0}^{m\times m} & \mathbf{0}^{m\times 2m} & \mathbf{0}^{m\times m} & -\mathbb{1}^\top & \mathbf{0}^{m\times m} \end{bmatrix} \tag{23}$$

and

$$q = \begin{bmatrix} \beta_k \\ \sigma_c\left(q_{u_\mathrm{N}} - \delta_{c,\mathrm{N}}\mathbf{1}\right) \\ \theta\bar{H}_\mathrm{N}\hat{M}^{-1}\hat{i}_{k,k+1} + \theta(\theta(1+e)-1)\bar{H}_\mathrm{N} v_k \\ \theta D^\top \bar{H}_\mathrm{T}\hat{M}^{-1}\hat{i}_{k,k+1} \\ \mathbf{0}^m \\ D^\top q_{u_\mathrm{T}} \\ \mathbf{0}^m \end{bmatrix}. \tag{24}$$

Linear complementarity formulation and solving

Assumption (1)

The time-step h is chosen small enough that $\sigma_c(\delta_{c,N}S^{-1} - h^2\theta^2\sigma_c U_{NN})$ is positive definite.

Lemma

Under Assumption 1, L is copositive on the positive orthant, i.e, $x^\top Lx \geqslant 0, \forall x \geqslant 0$.

Assumption (2)

The matrix H_T is surjective, i.e, $\forall b \in \mathbb{R}^m, \exists a \in \mathbb{R}^n$ such that $b = H_T a$.

Proposition

If Assumption (1) and (2) hold then the $\mathrm{LCP}(L, q)$ has a solution. Furthermore, the LEMKE algorithm with lexicographic ordering is able to compute a solution.

Comments

- Existence result for any value of μ
- Uniqueness is not ensured due to Coulomb's friction

Linear complementarity formulation and solving

In the frictionless case, we obtain better results.

Lemma

Under Assumption 1, L is positive semi-definite and the solution is unique.

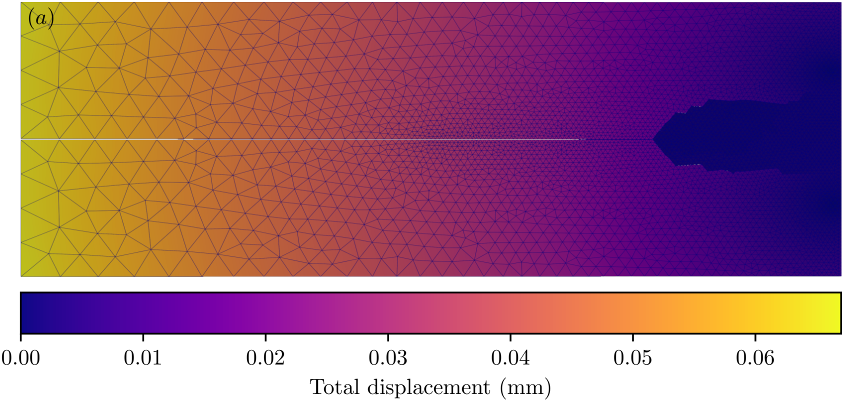
In the frictionless case, the system can be further formulated a convex quadratic programming problem with unique solution.

Discussion

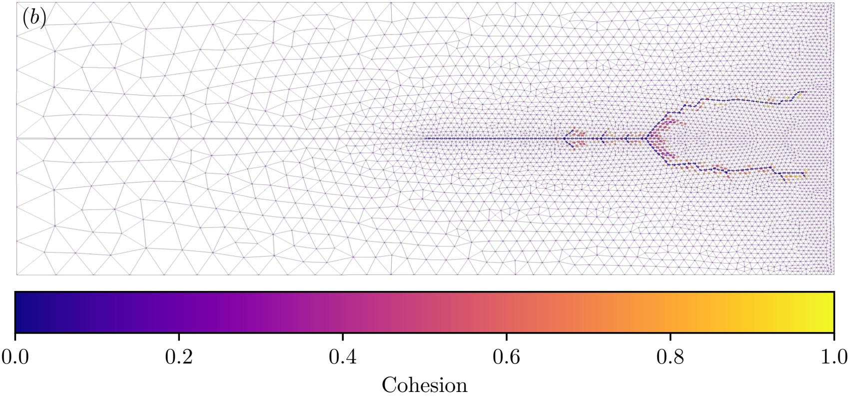
- The system we model contains non convex free energy and leads to softening behavior in the interface.
  A constitutive softening model with a well-defined solution, without any kind of regularization (viscosity, internal length, second gradient).
- A priori, no issue with mesh convergence and energetic behavior (this remains to be proved formally)
- Dynamics renders the system well posed and there is no snap-trough as in the quasi-static case
- What can be extended in the plasticity with softening? micro-inertia ?

The edge-cracked block

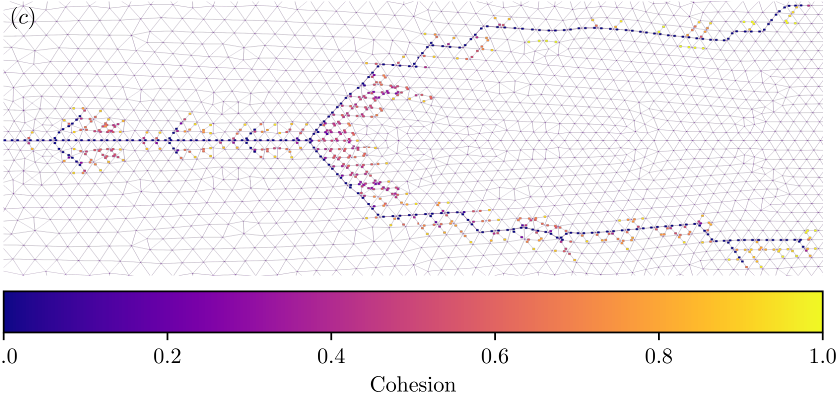
Coupling with Akantu (special thanks to Guillaume Anciaux and Nicolas Richart)



Numerical illustrations



Numerical illustrations



Branching is possible with extrinsic (initially rigid) model

Perspectives

- LEMKE algorithm is a pivoting algorithm : robust, solution at machine accuracy but slow for large systems (number of contact points > 5000)
  - Semi–smooth Newton method (à la Alart–Curnier)
  - Interior point methods
- Python prototyping is slow and prone to bug.
  Coding in serious HPC framework, C++ Petsc.
- Simpler implementation with explicit integration of β for faster simulations.
- PhD in progress (Chloé Gergely) on adding heat equations and temperature coupling for stability for the rock permafrost in high mountains.

Rockfall at Mel de la Niva. Evolène, Switzerland 18 October 2015. video on YouTube