Higher order schemes for nonsmooth mechanical systems.

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NUMDIFF 12. September 14–18,2009

Introduction & Motivations

Numerical time–integration schemes

Adaptive time-step size time–stepping scheme

Time–stepping schemes of any order

Conclusions & Perspectives

NonSmooth Multibody Systems (NSMBS)

General definition

$$
\begin{cases}
M(q)\dot{v} = F(t,q,v) + G(t,q)\lambda & \text{(1a)}\\
\dot{q} = v & \text{(1b)}\\
w = g(t,q,v) & \text{(1c)}\\
0 \in S(w,\lambda,t) + T(w,\lambda,t) & \text{(1d)}\\
v^{+} = \mathcal{F}(v^{-},q,t) & \text{(1e)}
\end{cases}
$$

- $S : \mathbb{R}^{m\times m} \times \mathbb{R} \mapsto \mathbb{R}^{m\times m}$ continuously differentiable mapping
- $T : \mathbb{R}^{m\times m} \times \mathbb{R} \rightsquigarrow \mathbb{R}^{m\times m}$ multivalued mapping with a closed graph.

Various modelling

The definition includes the modelling of Mechanical systems with bilateral constraints, Coulomb's friction, impacts, cohesion, . . .

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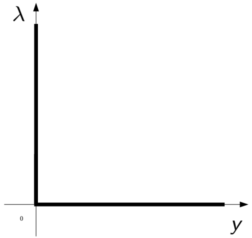
NonSmooth Multibody Systems (NSMBS)

Particular case: Scleronomous holonomic perfect unilateral constraints

$$\begin{cases} M(q)\dot{v} = F(t,q,v) + G(q)\,\lambda \\ \dot{q} = v \\ y = g(q) \\ 0 \leqslant y \perp \lambda \geqslant 0 \\ \dot{v}^{+} = -e\dot{y}^{-} \end{cases} \tag{2}$$

where $G(q) = \nabla g(q)$

Unilateral constraints (unilateral contact)



Mathematical aspects (in a nutshell).

Velocity level formulation. Index reduction

$$
\begin{array}{c}
0 \leqslant y \perp \lambda \geqslant 0 \\
\Updownarrow \\
-\lambda \in N_{\mathbb{R}^+}(y) \\
\Updownarrow \\
-\lambda \in N_{T_{\mathbb{R}^+}}(\dot{y}) \\
\Updownarrow \\
\text{if } y \leqslant 0 \text{ then } 0 \leqslant \dot{y} \perp \lambda \geqslant 0
\end{array}
\tag{3}
$$

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Mathematical aspects (in a nutshell)

Standard assumptions

- The proper mathematical formulation is a Measure Differential inclusion (MDI)
- The position $q(t)$ is absolutely continuous function of time t
- The velocity $v(t)$ is a function of Bounded Variations (BV) of time t
- The acceleration is defined as a differential measure dv associated with the BV function v
- The multiplier is also differential measure denoted by $d\lambda$

More details in (Glocker, 2001 ; A. and Brogliato, 2008).

Compact MDI formulation (Moreau, 1988)

$$M(q)dv = F(t, q, v^+)dt + G(q)d\lambda \tag{4}$$
$$y = q(q), \dot{y} = G(q)v \tag{5}$$
$$-d\lambda \in N_{T_{\mathbb{R}+}}(\dot{y}^+ + e\dot{y}^-) \tag{6}$$

Mathematical aspects

Comments

- *The inclusion concerns measures.* Therefore, it is necessary to define what is the inclusion of a measure into a cone.
- *The inclusion in terms of velocity v^+* rather than of the coordinates q.

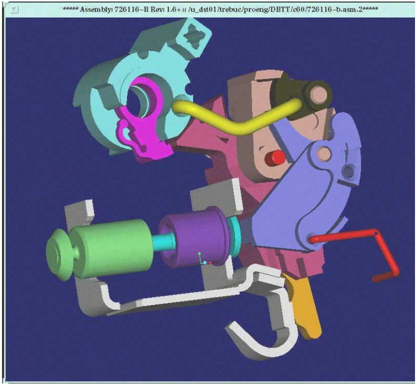
Interpretation

- Inclusion of measure, $-d\lambda \in K$
  - Case $d\lambda = \lambda' dt = f dt$.
$$-f \in K \tag{7}$$
  - Case $d\lambda = p_i \delta_i$.
$$-p_i \in K \tag{8}$$
- Inclusion in terms of the velocity. Viability Lemma
If $q(t_0) \in C(t_0)$, then
$$v^+ \in T_C(q), t \geqslant t_0 \Rightarrow q(t) \in C(t), t \geqslant t_0$$
➜ The unilateral constraints on q are satisfied. The equivalence needs at least an impact inelastic rule.

Applications

Mechanical systems with contact, impact and friction

Simulation of Circuit breakers (INRIA/Schneider Electric)



Applications

Mechanical systems with contact, impact and friction

Bipedal Robot INRIA BIPOP

Introduction & Motivations

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State–of–the–art

Numerical time–integration methods for Nonsmooth Multibody systems (NSMBS):

Nonsmooth event capturing methods (Time–stepping methods)

- $\oplus$ robust, stable and proof of convergence
- $\oplus$ low kinematic level for the constraints
- $\oplus$ able to deal with finite accumulation
- $\ominus$ very low order of accuracy even in free flight motions

Nonsmooth event tracking methods (Event–driven methods)

- $\oplus$ high level integration of free flight motions
- $\ominus$ no proof of convergence
- $\ominus$ sensibility to numerical thresholds
- $\ominus$ reformulation of constraints at higher kinematic levels.
- $\ominus$ unable to deal with finite accumulation

Objectives & means

Objectives

Design nonsmooth event capturing methods with

- same properties as standard methods (robustness, accumulation, ...)
- Higher resolution (ratio error/computational cost)
- Higher order of accuracy

Means

1. Adaptive time–step size and order strategies for standard methods
2. Mixed integrators with rough pre-detection of events
3. Splitting strategies
4. Ad hoc detection of discontinuity and order of discontinuity methods.

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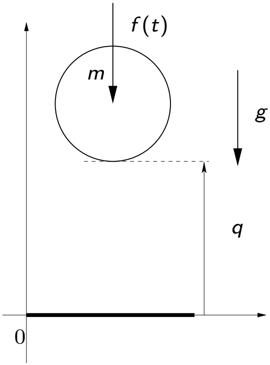
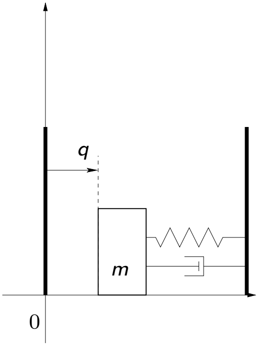
Academic examples I



(a) Bouncing ball example



(b) Linear Oscillator example

Figure: Academic test examples with analytical solutions

NonSmooth Multibody Systems (NSMBS)

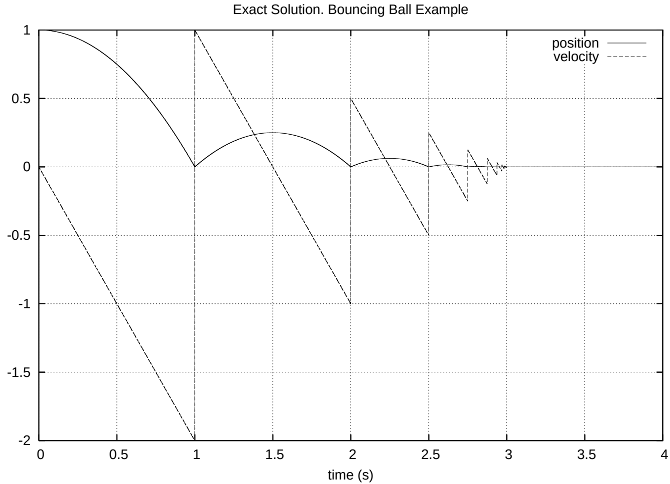


Figure: Analytical solutions. Bouncing ball example]

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NonSmooth Multibody Systems (NSMBS)

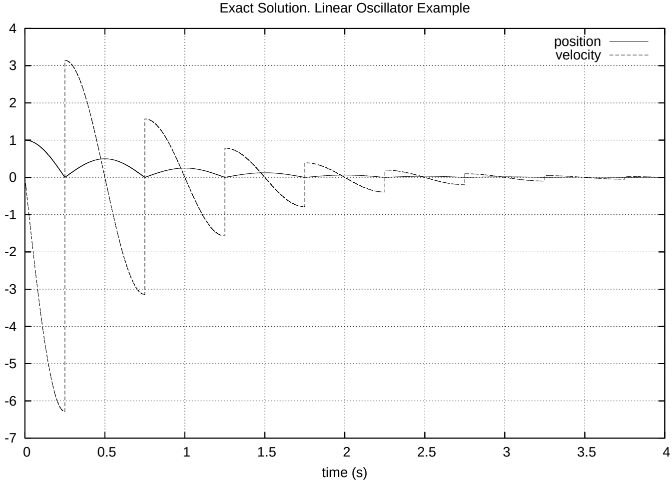
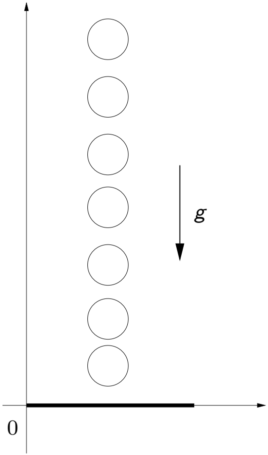


Figure: Analytical solutions. Linear Oscillator

NonSmooth Multibody Systems (NSMBS)

Academic examples II



(a) N Bouncing balls example

Figure: Academic test examples

Moreau's Time stepping scheme (Moreau, 1988)

Principle

$$\left\{\begin{aligned}
&M(q_{k+\theta})(v_{k+1}-v_k) - h\tilde{F}_{k+\theta} = G(q_{k+\theta})P_{k+1}, && (9a)\\
&q_{k+1} = q_k + hv_{k+\theta}, && (9b)\\
&U_{k+1} = G^T(q_{k+\theta})\,v_{k+1} && (9c)\\
&-P_{k+1} \in N_{T_{\mathbb{R}^m_+}(\tilde{y}_{k+\gamma})}(U_{k+1}+eU_k), && (9d)\\
&\tilde{y}_{k+\gamma} = y_k + h\gamma U_k, \quad \gamma\in[0,1]. && (9e)
\end{aligned}\right.$$

with $\theta \in [0,1], \gamma \geqslant 0$ and $x_{k+\alpha} = (1-\alpha)x_{k+1} + \alpha x_k$ and $\tilde{y}_{k+\gamma}$ is a prediction of the constraints.

Schatzman's Time stepping scheme (Paoli and Schatzman, 2002)

Principle

$$
\begin{cases}
M(q_k+1)(q_{k+1}-2q_k+q_{k-1}) - h^2 F(t_{k+\theta}, q_{k+\theta}, v_{k+\theta}) = p_{k+1}, & \text{(10a)} \\[2ex]
v_{k+1} = \dfrac{q_{k+1}-q_{k-1}}{2h}, & \text{(10b)} \\[2ex]
-p_{k+1} \in N_K\left(\dfrac{q_{k+1}+eq_{k-1}}{1+e}\right), & \text{(10c)}
\end{cases}
$$

where N_K defined the normal cone to K.
For $K = \{q \in \mathbb{R}^n, y = g(q) \geqslant 0\}$

$$
0 \leqslant g\left(\frac{q_{k+1}+eq_{k-1}}{1+e}\right) \perp \nabla g\left(\frac{q_{k+1}+eq_{k-1}}{1+e}\right) P_{k+1} \geqslant 0 \qquad (11)
$$

Comparison

Shared mathematical properties

- Convergence results for one constraints
- Convergence results for multiple constraints problems with acute kinetic angles
- No theoretical proof of order

Mechanical properties

- Position vs. velocity constraints
- Respect of the impact in one step (Moreau) vs. Two-steps(Schatzman)
- Linearized constraints rather than nonlinear.

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Measuring error and convergence

Convergence in the sense of filled-in graph (Moreau (1978))

$$gr^\star(f) = \{(t,x) \in [0,T] \times \mathbb{R}^n, 0 \leqslant t \leqslant T \text{ and } x \in [f(t^-), f(t^+)]\}. \quad (12)$$

Such graphs are closed bounded subsets of $[0,T] \times \mathbb{R}^n$, hence, we can use the Hausdorff distance between two such sets with a suitable metric:

$$d((t,x),(s,y)) = \max\{|t-s|, \|x-y\|\}. \quad (13)$$

Defining the excess of separation between two graphs by

$$e(gr^\star(f), gr^\star(g)) = \sup_{(t,x)\in gr^\star(f)} \inf_{(s,y)\in gr^\star(g)} d((t,x),(s,y)), \quad (14)$$

the Hausdorff distance between two filled-in graphs $h^\star$ is defined by

$$h^\star(gr^\star(f), gr^\star(g)) = \max\{e(gr^\star(f), gr^\star(g)), e(gr^\star(g), gr^\star(f))\}. \quad (15)$$

Measuring error and convergence

An equivalent grid-function norm to the function norm in $\mathcal{L}_1$

$$\|e\|_1 = h \sum_{i=0}^{N} |f_i - f(t_i)| \tag{16}$$

In the same way, the $p - norm$ can be defined by

$$\|e\|_p = \left(h \sum_{i=0}^{N} |f_i - f(t_i)|^p \right)^{1/p} \tag{17}$$

The computation of this two last norm is easier to implement for piecewise continuous analytical function than the Hausdorff distance.

Global order of convergence.

Definition

A one-step time–integration scheme is of order q for a given norm $\|\cdot\|$ if there exists a constant C such that

$$\|e\| = Ch^q + \mathcal{O}(h^{q+1}) \tag{18}$$

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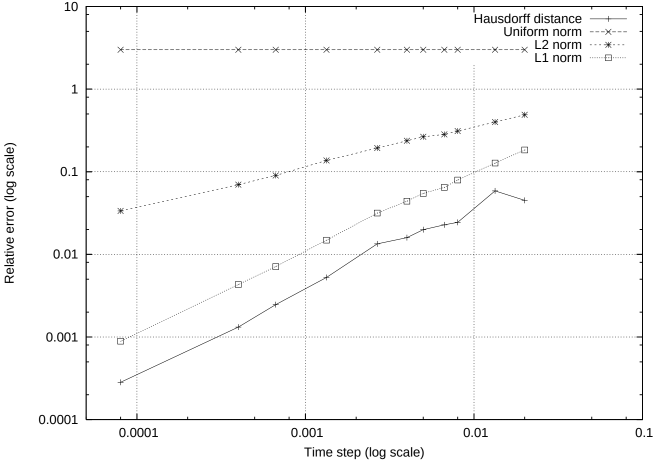
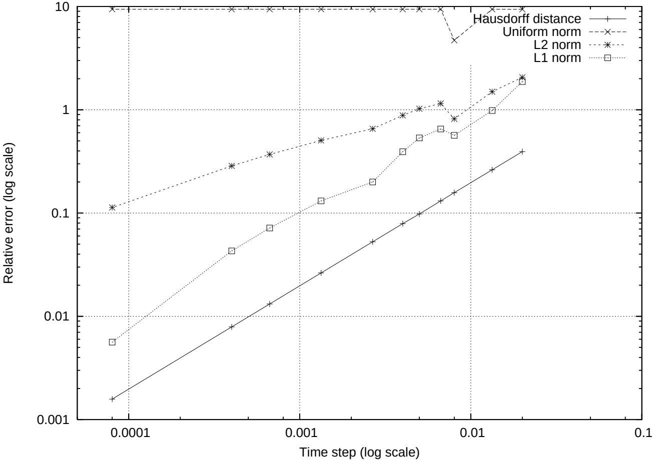
Empirical order of convergence. Moreau's time–stepping scheme



(a) The bouncing ball example

Figure: Empirical order of convergence of the Moreau's time-stepping scheme.

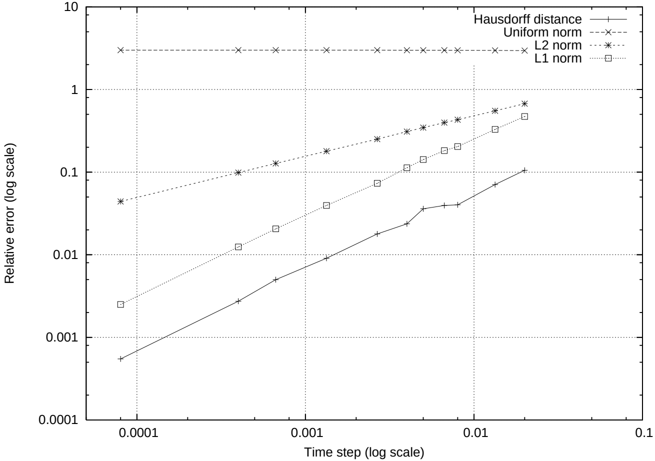
Empirical order of convergence. Moreau's time–stepping scheme



(a) The linear oscillator example

Figure: Empirical order of convergence of the Moreau's time-stepping scheme.

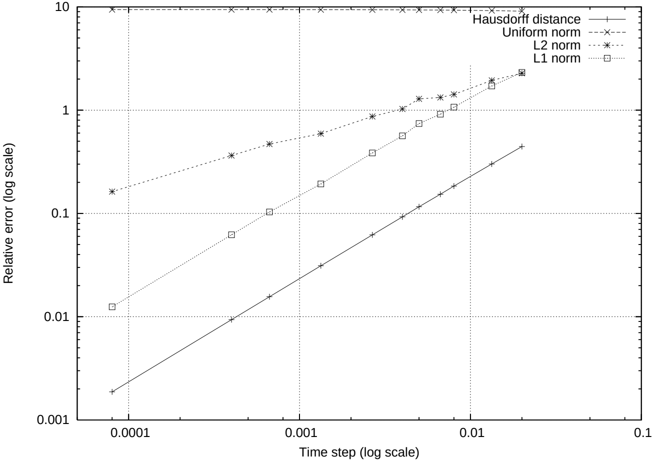
Empirical order of convergence. Schatzman–Paoli's time–stepping scheme



(a) The bouncing ball example

Figure: Empirical order of convergence of the Schatzman-Paoli's time-stepping scheme.

Empirical order of convergence. Schatzman–Paoli's time–stepping scheme



(a) The linear oscillator example

Figure: Empirical order of convergence of the Schatzman-Paoli's time-stepping scheme.

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Local error estimates for the Moreau's time–stepping

Notation

$$e = x(t_k + h) - x_{k+1} = \left[\begin{array}{c} e_v \\ e_q \end{array} \right] = \left[\begin{array}{c} v^+(t_k + h) - v_{k+1} \\ q(t_k + h) - q_{k+1} \end{array} \right] \qquad (19)$$

One impact at time $t_* \in (t_k, t_{k+1}]$

Assumption

$$di = p\delta_{t_*}, \quad \text{or equivalently} \quad dI = P\delta_{t_*}, \text{with } P = G(t_*)p. \tag{20}$$

Notation

$$\mathcal{I} = \{\alpha, \alpha \in \{1..m\}\} \tag{21}$$

$$\mathcal{I}_* = \{\alpha \in \mathcal{I}, P^\alpha \geqslant 0, U^{\alpha,+}(t_*) - U^{\alpha,-}(t_*) = -(1+e)U^{\alpha,-}(t_*)\} \tag{22}$$

$$\mathcal{I}_p = \{\alpha \in \mathcal{I}, P^\alpha_{k+1} \geqslant 0, U^\alpha_{k+1} - U^\alpha_k = -(1+e)U^\alpha_k\} \tag{23}$$

Lemma

Let us assume that we have only one elastic impact at time $t_* \in (t_k, t_{k+1}]$ without persistent contact, i.e. , $di = p\delta_{t_*}$.

1. *If $\mathcal{I}_* = \mathcal{I}_p$, then the local order of consistency of the scheme is given by*

$$\begin{aligned} e_v &= K_v h + \mathcal{O}(h^2) \\ e_q &= K_q h + \mathcal{O}(h^2) \end{aligned} \tag{24}$$

2. *If $\mathcal{I}_* \neq \mathcal{I}_p$, then the local order of consistency of the scheme is given by*

$$\begin{aligned} e_v &= K_v + \mathcal{O}(h) \\ e_q &= K_q h + \mathcal{O}(h^2) \end{aligned} \tag{25}$$

Local error estimates for the Moreau's time–stepping

Other cases are treated in the same way

- *One impact and smooth Lagrange multiplier* The same result holds as in the first Lemma.
- *losing contact event (take–off) without impact* The order of the time–integration scheme depends on the regularity of the contact forces (at least continuous).
- *Finite accumulation* The order of the time–integration should be at least 0. Idea of the proof : use the fact that the velocity vanishes and is of bounded variations

Practical error estimates for the Moreau's time–stepping

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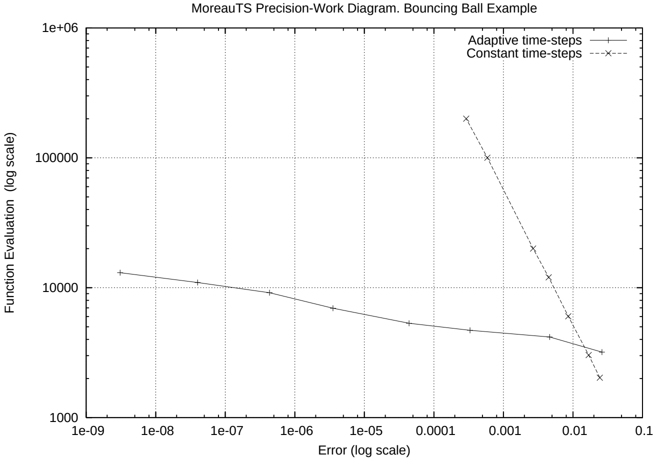
Order "0" case

Standard error estimates do not apply for Order 0.
We propose to extend it to the order 0 of consistency by assuming that the the local error estimate is given by

$$e_{1/2} = 2(x_{1/2} - x_1) + \mathcal{O}(h^2) \tag{26}$$

where x_1 is the result of the time integration with one time–step of length h and $x_{1/2}$ with two time-steps of length $h/2$.
The adaptive time–step control used for smooth ODE is then apply directlyHairer et al. (1993).

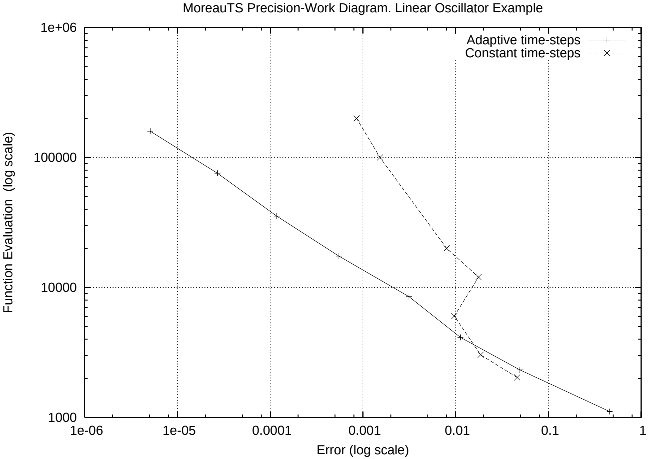
Order "0" time–step adjustment for the Moreau's time–stepping



(a) The bouncing ball example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 0

Order "0" time–step adjustment for the Moreau's time–stepping



(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 0

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Higher Order Time–stepping schemes

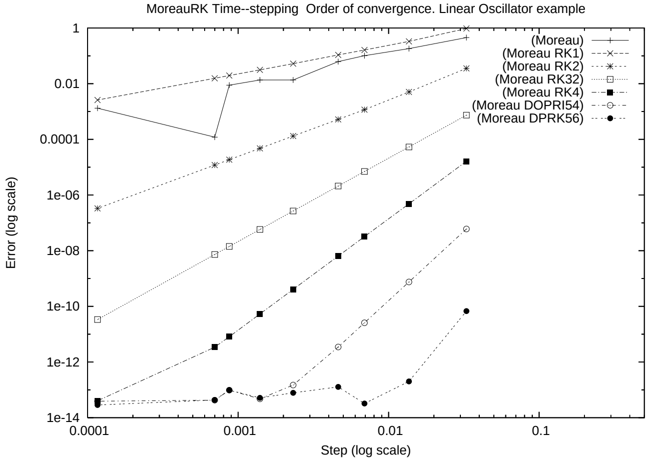
Background

Work of Mannshardt (1978) on time–integration schemes of any order for ODE/DAEs with discontinuities (with tranversality assumption)

Principle

- Let us assume only one event per time–step at instants t_*.
- Choose any ODE/DAE solvers of order p
- Perform a rough location of the event inside the time step of length h
  Find an interval $[t_a, t_b]$ such that
  $$t_* \in [t_a, t_b] \text{ and } |t_b - t_a| = Ch^{p+1} + \mathcal{O}(h^{p+2}) \qquad (27)$$
  Dichotomy, Newton, Local Interpolants, Dense output,. . .
- Perform an integration on $[t_k, t_a]$ with the ODE solver of order p
- Perform an integration on $[t_a, t_b]$ with Moreau's time–stepping scheme
- Perform an integration on $[t_b, t_{k+1}]$ with the ODE solver of order p

Results on the linear oscillator



(a) The linear oscillator example

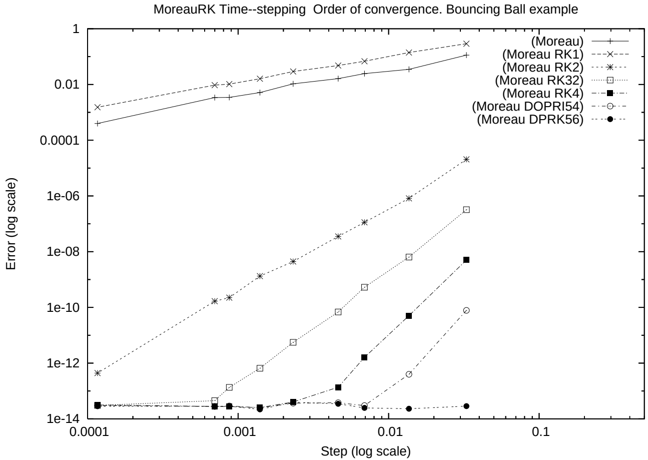
Figure: Precision Work diagram for the Moreau's time-stepping scheme.

Higher Order Time–stepping schemes

Finite accumulation

- Repeat the whole process on the remaining part of the interval $[t_b, t_k]$
- By induction, repeat this process up to the end of the original time step.

Results on the Bouncing Ball



(a) The Bouncing Ball example

Figure: Precision Work diagram for the Moreau's time-stepping scheme.

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Conclusions

Adaptive time–step strategies

- Higher resolution schemes
- Work with finite accumulation of events

Higher order schemes

- Schemes of any orders
- Work with finite accumulation of events

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Perspectives

- Theoretical works on orders and practical error estimations Collaborations with people the ODE/DAE communities are welcome.
- Adaptive time–step strategies on the higher order time–stepping schemes.
- Improve the pre–detection process of the event and the order of discontinuity
- Test on nonlinear mechanical systems.
- Other types of time–stepping schemes (Splitting and time discontinuous Galerkin methods)

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