Investigations toward higher resolution time–stepping schemes for NonSmooth Multibody Systems (NSMBS)

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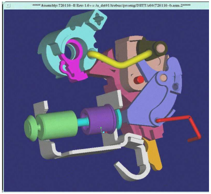
Applications

Mechanical systems with contact, impact and friction

Simulation of Circuit breakers (INRIA/Schneider Electric)



Applications

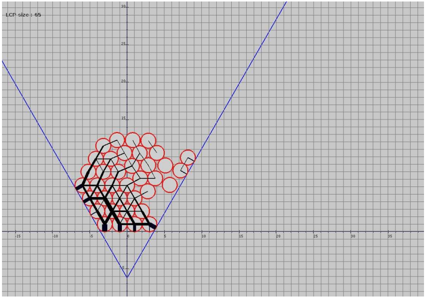
Mechanical systems with contact, impact and friction

Bipedal Robot INRIA BIPOP

Applications

Mechanical systems with contact, impact and friction

Stack of beads with perturbation



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State–of–the–art

Numerical time–integration methods for Nonsmooth Multibody systems (NSMBS):

Nonsmooth event capturing methods (Time–stepping methods)

- $\oplus$ robust, stable and proof of convergence
- $\oplus$ low kinematic level for the constraints
- $\oplus$ able to deal with finite accumulation
- $\ominus$ very low order of accuracy even in free flight motions

Nonsmooth event tracking methods (Event–driven methods)

- $\oplus$ high level integration of free flight motions
- $\ominus$ no proof of convergence
- $\ominus$ sensibility to numerical thresholds
- $\ominus$ reformulation of constraints at higher kinematic levels.
- $\ominus$ unable to deal with finite accumulation

Objectives & means

Objectives

Design nonsmooth event capturing methods with

- same properties as standard methods (robustness, accumulation, . . .)
- Higher resolution (ratio error/computational cost)
- Higher order of accuracy

Means

1. Adaptive time–step size and order strategies for standard methods
2. Mixed integrators with rough pre-detection of events
3. Splitting strategies
4. Ad hoc detection of discontinuity and order of discontinuity methods.

NonSmooth Multibody Systems (NSMBS)

General definition

$$
\begin{cases}
M(q)\dot{v} = F(t,q,v) + G(t,q)\lambda & \text{(1a)}\\
\dot{q} = v & \text{(1b)}\\
w = g(t,q,v) & \text{(1c)}\\
0 \in S(w,\lambda,t) + T(w,\lambda,t) & \text{(1d)}\\
v^+ = \mathcal{F}(v^-,q,t) & \text{(1e)}
\end{cases}
$$

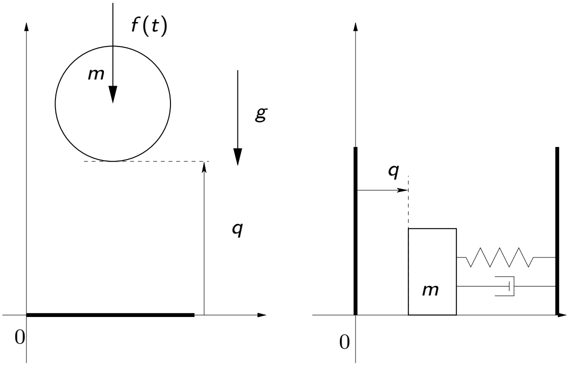
- $S : \mathbb{R}^{m\times m} \times \mathbb{R} \mapsto \mathbb{R}^{m\times m}$ continuously differentiable mapping
- $T : \mathbb{R}^{m\times m} \times \mathbb{R} \rightsquigarrow \mathbb{R}^{m\times m}$ multivalued mapping with a closed graph.

With scleronomous holonomic perfect unilateral constraints

$$
\begin{cases}
M(q)\dot{v} = F(t,q,v) + G(q)\,\lambda\\
\dot{q} = v\\
0 \leqslant y = g(q) \perp \lambda \geqslant 0\\
v^+ = \mathcal{F}(v^-,q,t)
\end{cases}
\tag{2}
$$

where $G(q) = \nabla g(q)$

Academic examples I



(a) Bouncing ball example (b) Linear Oscillator example

Figure: Academic test examples with analytical solutions

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NonSmooth Multibody Systems (NSMBS)

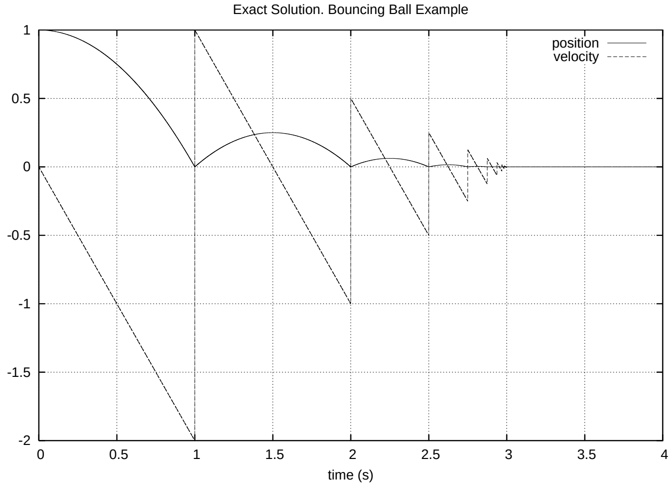


Figure: Analytical solutions. Bouncing ball example]

NonSmooth Multibody Systems (NSMBS)

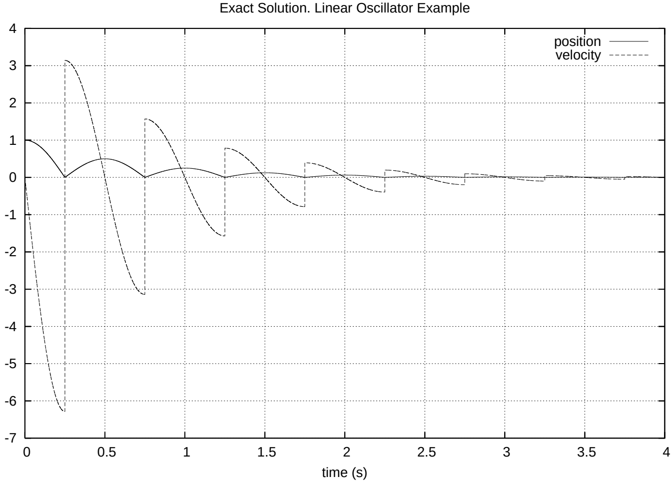
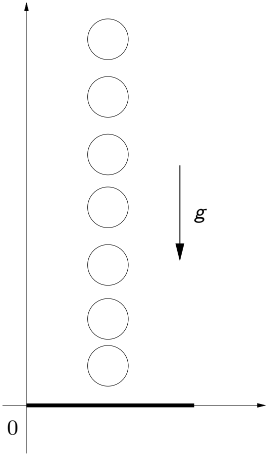


Figure: Analytical solutions. Linear Oscillator

NonSmooth Multibody Systems (NSMBS)

Academic examples II



(a) N Bouncing balls example

Figure: Academic test examples

Introduction & Motivations

Background time–stepping schemes
- Background

Adaptive time-step size time–stepping scheme
- Smooth ODE time integration
- Local error estimates for the Moreau's Time–stepping scheme

Adaptive time–step strategies
- An "Order 0" approach
- A control based on violation
- Variable order approach

Time–stepping schemes of any order
- Principle

Splitting based Time–stepping schemes
- Principle

Conclusions & Perspectives
- Conclusions
- Perspectives

Moreau's Time stepping scheme

Principle

$$
\begin{cases}
M(q_{k+\theta})(v_{k+1} - v_k) - h\tilde{F}_{k+\theta} = G(q_{k+\theta})P_{k+1}, & \text{(3a)} \\
q_{k+1} = q_k + hv_{k+\theta}, & \text{(3b)} \\
U_{k+1} = G^T(q_{k+\theta})\, v_{k+1} & \text{(3c)} \\
-P_{k+1} \in \partial\psi_{T_{\mathbb{R}^m_+}(\tilde{y}_{k+\gamma})}(U_{k+1} + eU_k), & \text{(3d)} \\
\tilde{y}_{k+\gamma} = y_k + h\gamma U_k, \quad \gamma \in [0,1]. & \text{(3e)}
\end{cases}
$$

with $\theta \in [0,1], \gamma \geqslant 0$ and $x_{k+\alpha} = (1-\alpha)x_{k+1} + \alpha x_k$ and $\tilde{y}_{k+\gamma}$ is a prediction of the constraints.

Properties

- Convergence results for one constraints
- Convergence results for multiple constraints problems with acute kinetic angles
- No theoretical proofs of order

Schatzman–Paoli's Time stepping scheme

Principle

$$\begin{cases} M(q_k+1)(q_{k+1}-2q_k+q_{k-1}) - h^2 F(t_{k+\theta}, q_{k+\theta}, v_{k+\theta}) = p_{k+1}, & \text{(4a)} \\ v_{k+1} = \dfrac{q_{k+1}-q_{k-1}}{2h}, & \text{(4b)} \\ -p_{k+1} \in N_K\left(\dfrac{q_{k+1}+eq_{k-1}}{1+e}\right), & \text{(4c)} \end{cases}$$

where N_K defined the normal cone to K.
For $K = \{q \in \mathbb{R}^n, y = g(q) \geqslant 0\}$

$$0 \leqslant g\left(\frac{q_{k+1}+eq_{k-1}}{1+e}\right) \perp \nabla g\left(\frac{q_{k+1}+eq_{k-1}}{1+e}\right) P_{k+1} \geqslant 0 \tag{5}$$

Properties

- Convergence results for one constraints
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Measuring error and convergence

Convergence in the sense of filled-in graph (Moreau (1978))

$$gr^\star(f) = \{(t,x) \in [0,T] \times \mathbb{R}^n, 0 \leqslant t \leqslant T \text{ and } x \in [f(t^-), f(t^+)]\}. \quad (6)$$

Such graphs are closed bounded subsets of $[0,T] \times \mathbb{R}^n$, hence, we can use the Hausdorff distance between two such sets with a suitable metric:

$$d((t,x),(s,y)) = \max\{|t-s|, \|x-y\|\}. \quad (7)$$

Defining the excess of separation between two graphs by

$$e(gr^\star(f), gr^\star(g)) = \sup_{(t,x)\in gr^\star(f)} \inf_{(s,y)\in gr^\star(g)} d((t,x),(s,y)), \quad (8)$$

the Hausdorff distance between two filled-in graphs $h^\star$ is defined by

$$h^\star(gr^\star(f), gr^\star(g)) = \max\{e(gr^\star(f), gr^\star(g)), e(gr^\star(g), gr^\star(f))\}. \quad (9)$$

Measuring error and convergence

An equivalent grid-function norm to the function norm in $\mathcal{L}_1$

$$\|e\|_1 = h\sum_{i=0}^{N} |f_i - f(t_i)| \tag{10}$$

In the same way, the $p-norm$ can be defined by

$$\|e\|_p = \left(h\sum_{i=0}^{N} |f_i - f(t_i)|^p\right)^{1/p} \tag{11}$$

The computation of this two last norm is easier to implement for piecewise continuous analytical function than the Hausdorff distance.

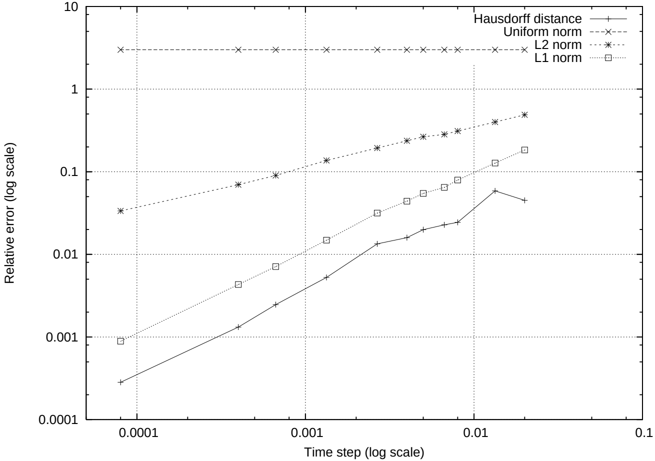
Global order of convergence.

Definition

A one-step time–integration scheme is of order q for a given norm $\|\cdot\|$ if there exists a constant C such that

$$\|e\| = Ch^q + \mathcal{O}(h^{q+1}) \tag{12}$$

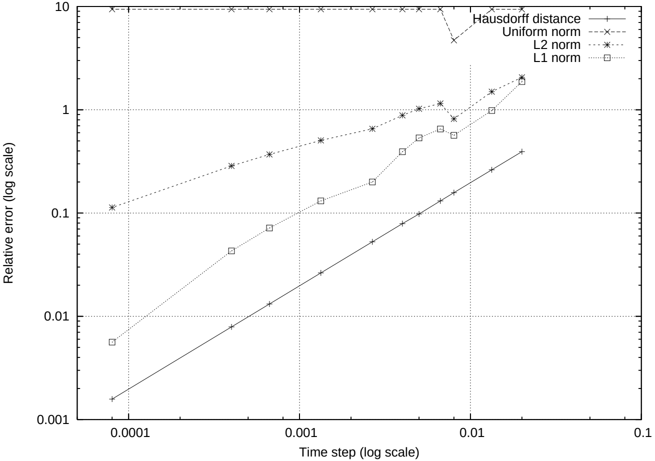
Empirical order of convergence. Moreau's time–stepping scheme



(a) The bouncing ball example

Figure: Empirical order of convergence of the Moreau's time-stepping scheme.

Empirical order of convergence. Moreau's time–stepping scheme



(a) The linear oscillator example

Figure: Empirical order of convergence of the Moreau's time-stepping scheme.

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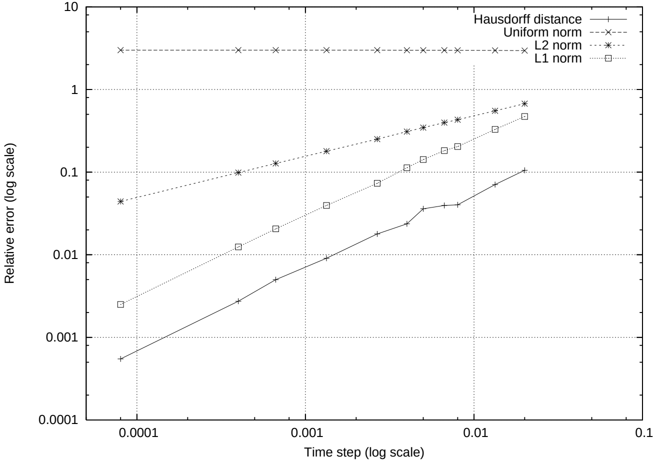
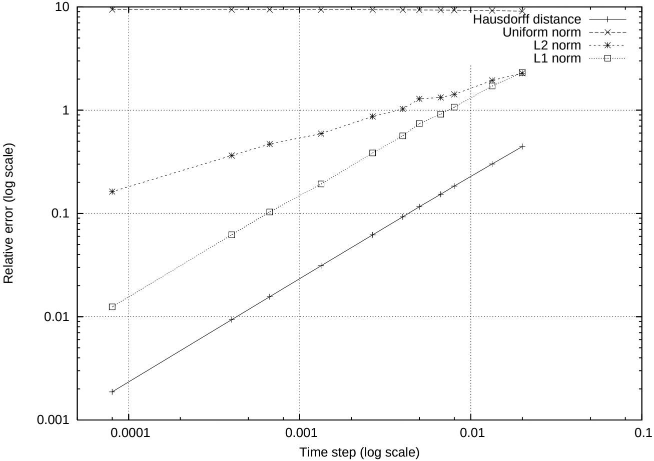
Empirical order of convergence. Schatzman–Paoli's time–stepping scheme



(a) The bouncing ball example

Figure: Empirical order of convergence of the Schatzman-Paoli's time-stepping scheme.

Empirical order of convergence. Schatzman–Paoli's time–stepping scheme



(a) The linear oscillator example

Figure: Empirical order of convergence of the Schatzman-Paoli's time-stepping scheme.

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Smooth ODEs

One–step numerical solvers for ODEs

Let us consider a ODE

$$\dot{x} = f(x,t), \tag{13}$$

where f is a mapping with sufficient regularity.
The one–step time–stepping method over the time–step $[t_k, t_{k+1} = t_k + h]$ is generically denoted by

$$x_{k+1} = x_k + h\Phi(t_k, h, x_k). \tag{14}$$

Order of consistency

The one–step time–stepping method is said to be consistent if $\Phi(t, 0, x, x) = f(x,t)$ and has a consistency order p if there exists a constant C such that

$$e_{k+1} = x(t_{k+1}) - x_{k+1} = Ch^{p+1} + \mathcal{O}(h^{p+2}), \tag{15}$$

assuming that $x_k = x(t_k)$.
If the time–stepping method has an order of consistency p and converges, then the global order of convergence is p,

Smooth ODEs

Basic practical error evaluation

1. Two "small" time steps of size $h/2$ $\Longrightarrow x_{1/2}$.
2. One "big" time-step h $\Longrightarrow x_1$.

$$\begin{array}{l} e_1 = x(t_0+h) - x_1 = C\ h^{p+1} + \mathcal{O}(h^{p+2}), \\ e_{1/2} = x(t_0+h) - x_{1/2} = 2C\ (h/2)^{p+1} + \mathcal{O}(h^{p+2}). \end{array} \tag{16}$$

This procedure permits us to evaluate the constant C and to obtain and a local error estimate such that:

$$e_2 = x(t_0+h) - x_2 = \frac{x_{1/2} - x_1}{2^p - 1} + \mathcal{O}(h^{p+2}). \tag{17}$$

Enhanced practical error evaluation

- Runge–Kutta Embedded pairs (Dormand-Price, Felhberg)
- Milne's device
- Nordsieck's method

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Smooth ODEs

Automatic control of the time–step

$$\|e_k\| \leqslant etol = atol + rtol \circ \max(x_0, x_k) \tag{18}$$

The measure of the error is given by

$$\mathrm{error} = \|e_k \circ invtol\| \tag{19}$$

with $invtol = [1/etol_i, i = 1 \dots n]$. The optima step size is then obtained by

$$h_{\mathrm{opt}} = h(\frac{1}{\mathrm{error}})^{1/(p+1)} \tag{20}$$

Usually, the step size is not allowed to decrease of to increase too fast, thanks to the following heuristic rule

$$h_{\mathrm{new}} = h \min(\alpha_{max}, \max(\alpha_{min}, \alpha(\frac{1}{\mathrm{error}})^{1/(p+1)})) \tag{21}$$

where α, α_{min} and α_{max} are some user parameters.

Local error estimates for the Moreau's time–stepping

Notation

$$e = x(t_k + h) - x_{k+1} = \begin{bmatrix} e_v \\ e_q \end{bmatrix} = \begin{bmatrix} v^+(t_k + h) - v_{k+1} \\ q(t_k + h) - q_{k+1} \end{bmatrix} \tag{22}$$

One impact at time $t_* \in (t_k, t_{k+1}]$

Assumption

$$di = p\delta_{t_*}, \quad \text{or equivalently} \quad dI = P\delta_{t_*}, \text{with } P = G(t_*)p. \qquad (23)$$

Notation

$$\mathcal{I} = \{\alpha, \alpha \in \{1..m\}\} \qquad (24)$$

$$\mathcal{I}_* = \{\alpha \in \mathcal{I}, P^\alpha \geqslant 0, U^{\alpha,+}(t_*) - U^{\alpha,-}(t_*) = -(1+e)U^{\alpha,-}(t_*)\} \qquad (25)$$

$$\mathcal{I}_p = \{\alpha \in \mathcal{I}, P^\alpha_{k+1} \geqslant 0, U^\alpha_{k+1} - U^\alpha_k = -(1+e)U^\alpha_k\} \qquad (26)$$

Lemma

Let us assume that we have only one elastic impact at time $t_* \in (t_k, t_{k+1}]$ without persistent contact, i.e. , $di = p\delta_{t_*}$.

1. *If $\mathcal{I}_* = \mathcal{I}_p$, then the local order of consistency of the scheme is given by*

$$\begin{array}{rcl} e_v & = & K_v h + \mathcal{O}(h^2) \\ e_q & = & K_q h + \mathcal{O}(h^2) \end{array} \qquad (27)$$

2. *If $\mathcal{I}_* \neq \mathcal{I}_p$, then the local order of consistency of the scheme is given by*

$$\begin{array}{rcl} e_v & = & K_v + \mathcal{O}(h) \\ e_q & = & K_q h + \mathcal{O}(h^2) \end{array} \qquad (28)$$

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Smooth Lagrange multiplier in persistent contact without impact in $(t_k, t_{k+1}]$

Assumption

$$di = \lambda(t)dt, \tag{29}$$

or equivalently

$$dI = \Lambda(t)dt, \text{with } \Lambda(t) = G(t)\lambda(t). \tag{30}$$

Notation

$$\mathcal{I}_\Lambda(t) = \{\alpha \in \mathcal{I}, \Lambda^\alpha(t) \geqslant 0, U^{\alpha,+}(t) = U^{\alpha,-}(t) = 0\} \tag{31}$$

$$\mathcal{I}_{\Lambda,k+1} = \{\alpha \in \mathcal{I}, \Lambda^\alpha_{k+1} \geqslant 0, U^\alpha_{k+1} = U^\alpha_k = 0\} \tag{32}$$

Lemma

Assuming that $\mathcal{I}_\Lambda(t) = \mathcal{I}_{\Lambda,k+1}$ for all $t \in (t_k, t_{k+1}]$. The local order of consistency of the scheme is one that is

$$\begin{array}{rcl} e_v & = & Kh^2 + \mathcal{O}(h^3) \\ e_q & = & K_q h^2 + \mathcal{O}(h^3) \end{array} \tag{33}$$

Local error estimates for the Moreau's time–stepping

Other cases

- *One impact and smooth Lagrange multiplier* The same result holds ad in first Lemma.
- *losing contact event (take–off) without impact* The order of the time–integration scheme depends on the regularity of the contact forces (at least continuous).
- *Finite accumulation* The order of the time–integration should be at least 0. Idea of the proof : use the fact that the velocity vanishes and is of bounded variations

Practical error estimates for the Moreau's time–stepping

Order "0" case

Standard error estimates do not apply for Order 0.
We propose to extend it to the order 0 of consistency by assuming that the constant can be evaluated by

$$C = \frac{2(e_1 - e_{1/2})}{h} \tag{34}$$

and the local error estimate by

$$e_{1/2} = 2(x_{1/2} - x_1) + \mathcal{O}(h^2) \tag{35}$$

The adaptive time–step control exposed for smooth ODE is then apply directly.

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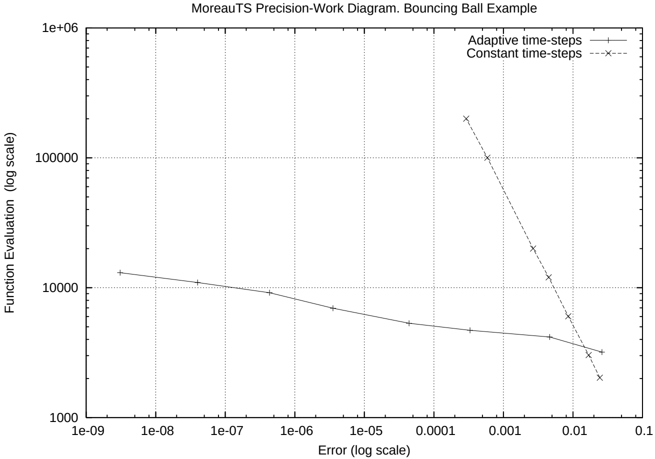
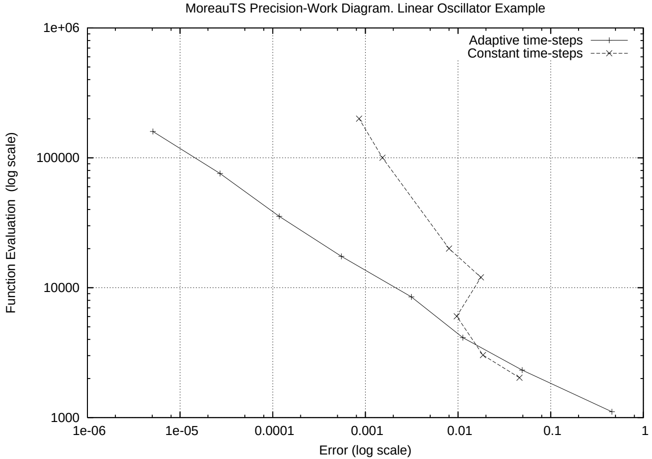
Order "0" time–step adjustment for the Moreau's time–stepping



(a) The bouncing ball example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 0

Order “0” time–step adjustment for the Moreau’s time–stepping



(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau’s time-stepping scheme. Order 0

Sizing the error in the violation of constraints

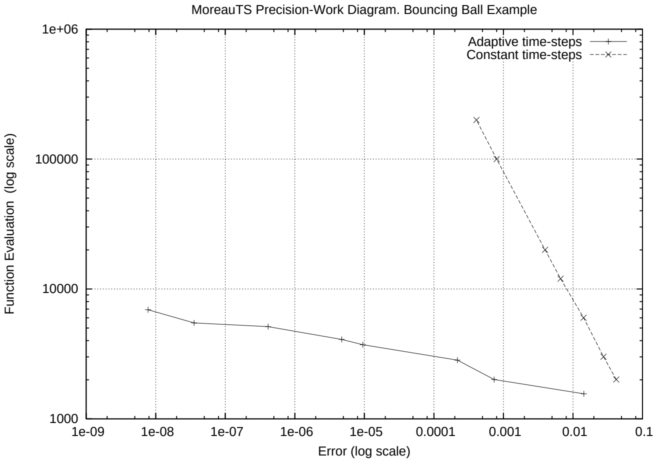
The violation of constraints is sized by the following rule:

$$e_{\mathrm{violation}} = \| \min(0, g(q)) \circ invtol\|_\infty \tag{36}$$

Assuming that the scheme is of order 1 almost everywhere in smooth phase and may be controlled by $e_{\mathrm{violation}}$ when an nonsmooth vent occurs, the step size adjustment is implemented by the means of the following error estimation

$$\mathrm{error} = \max(e_{\mathrm{violation}}, \|e_k \circ invtol\|_\infty) \tag{37}$$

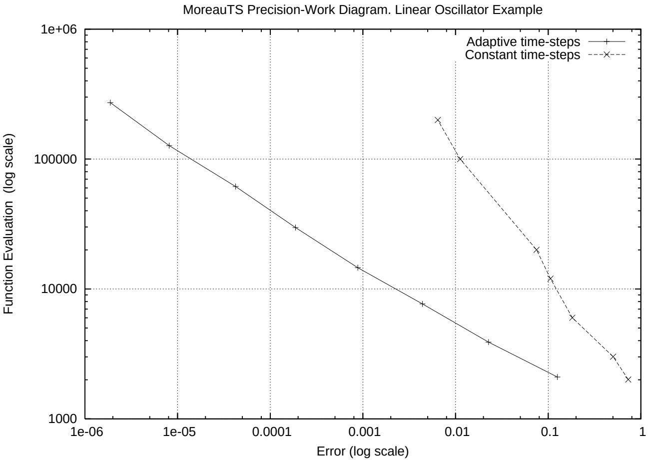
Results on two academic test examples



(a) The bouncing ball example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 0 + violation error

Results on two academic test examples



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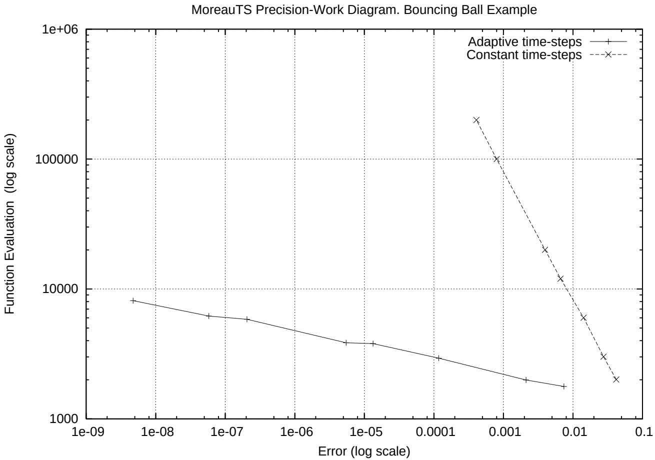

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Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 1 + violation error

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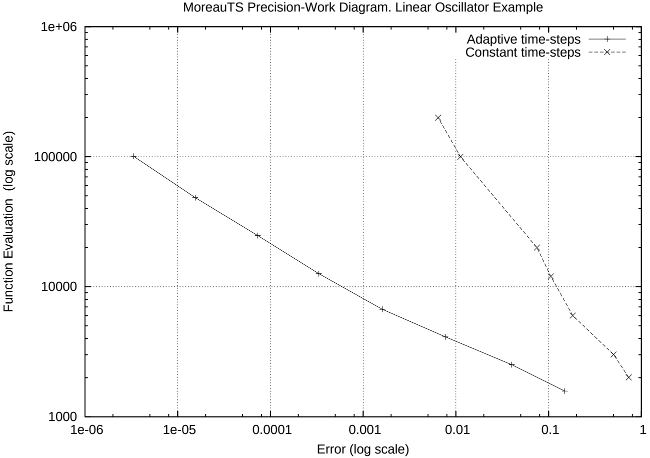

(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 1 + violation error

Variable order approach. Principle

Guess the order of consistency of the integration at each step.
Adapt the practical error estimation

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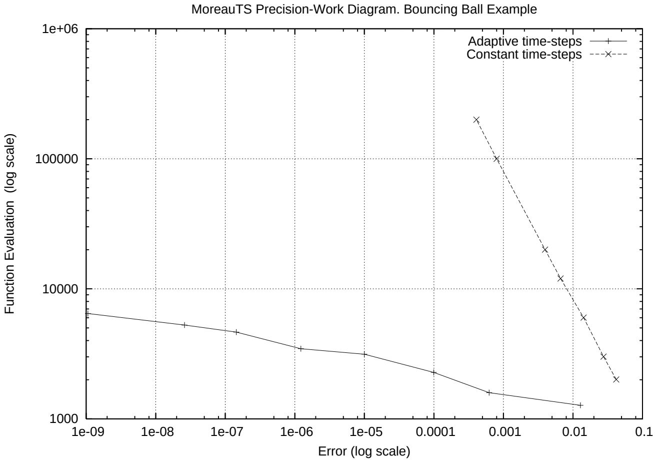
Results on two academic test examples



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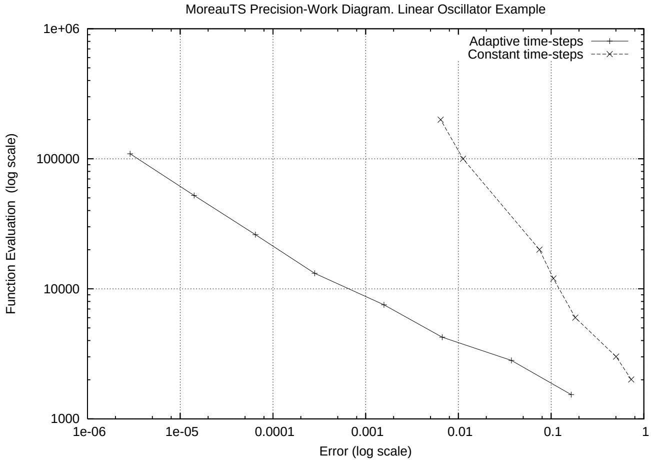

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Time–stepping schemes of any order
- Principle

Splitting based Time–stepping schemes
- Principle

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- Conclusions
- Perspectives

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Higher Order Time–stepping schemes

Background

Work of Mannshardt (1978) on time–integration schemes of any order for ODEs with discontinuities (with tranversality assumption)

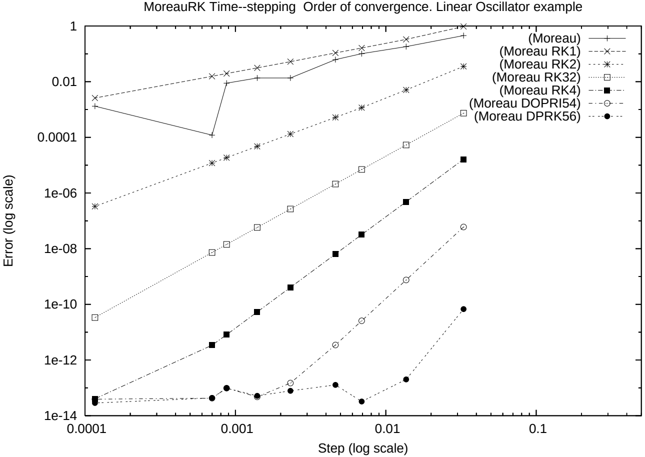
Principle

- Let us assume only one event per time–step at instants t_*.
- Choose any ODE solvers of order p
- Perform a rough location of the event inside the time step of length h Find an interval $[t_a, t_b]$ such that

$$t_* \in [t_a, t_b] \text{ and } |t_b - t_a| = Ch^{p+1} + \mathcal{O}(h^{p+2}) \tag{38}$$

  Dichotomy, Newton, Local Interpolants, Dense output,...
- Perform an integration on $[t_k, t_a]$ with the ODE solver of order p
- Perform an integration on $[t_a, t_b]$ with Moreau's time–stepping scheme
- Perform an integration on $[t_b, t_{k+1}]$ with the ODE solver of order p

Results on the linear oscillator



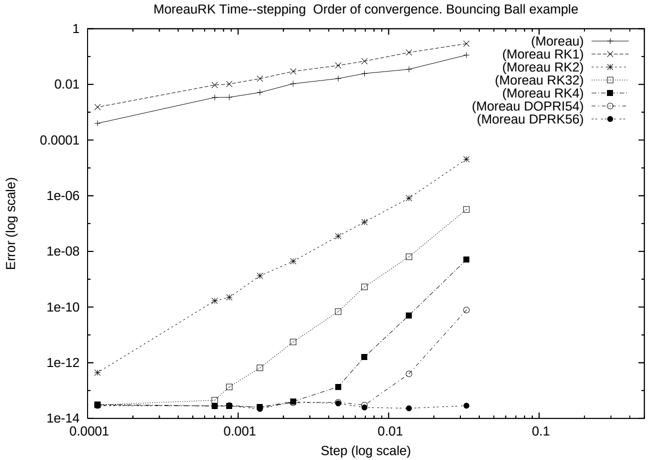
(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau's time-stepping scheme.

Finite accumulation

- Repeat the whole process on the remaining part of the interval $[t_b, t_k]$
- By induction, repeat this process up to the end of the original time step.

Results on the Bouncing Ball



(a) The Bouncing Ball example

Figure: Precision Work diagram for the Moreau's time-stepping scheme.

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Splitting–based methods.

Principle for smooth ODEs

Let us consider a smooth ODE which can be written as

$$\dot{x}(t) = f(x,t) + g(x,t) \tag{39}$$

A example of splitting–based method is given by the following procedure

1. Perform the integration of f on $[t_k, t_{k+1}]$ to obtain $\tilde{x}(t_{k+1})$ that is

$$\tilde{x}(t_{k+1}) = x(t_k) + \int_{t_k}^{t_{k+1}} f(x,t)\,dt \tag{40}$$

2. Perform the integration of g on $[t_k, t_{k+1}]$ with initial value $\tilde{x}(t_{k+1})$ to obtain $\hat{x}(t_{k+1})$ that is

$$\hat{x}(t_{k+1}) \quad = \quad \tilde{x}(t_{k+1}) + \int_{t_k}^{t_{k+1}} g(x,t)\,dt \tag{41}$$

Properties

- $x(t_k+1) \neq \hat{x}(t_{k+1})$ is the general case. (except special linear case, constant dynamics, . . .)
- $\hat{x}_(t_{k+1}) \longrightarrow x(t_{k+1})$ when $t_{k+1} \longrightarrow t_k$

Splitting–based methods.

Splitting–based for Moreau scheme without continuous contact forces

- The first part is

$$\begin{cases} M(q)\dot{v} = F(t,q,v), \\ \dot{q} = v, \\ q(t_k) = q_k, \qquad v(t_k) = v_k \end{cases} \tag{42}$$

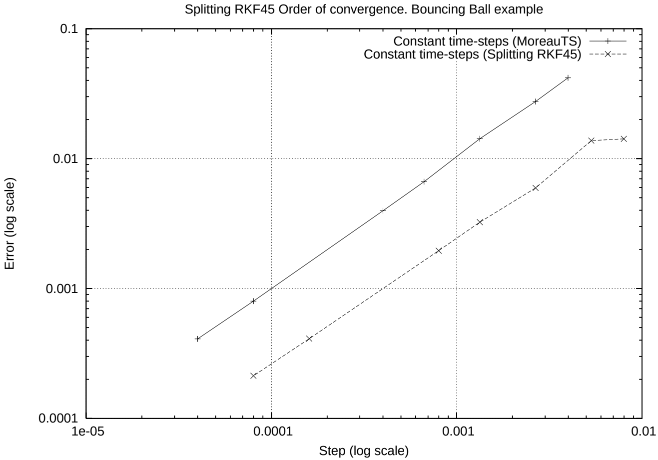
  yielding to the approximations $q_1 = q(t_{k+1})$ and $v_1 = v(t_{k+1})$ which can integrated by any smooth ODE solvers.

- The second one is given by

$$\begin{cases} M(q)\dot{v} = G(q)\lambda, \\ \dot{q} = 0, \\ y = g(q) \\ -\lambda \in \partial\psi_{T_{\mathbb{R}_+}(y)}(\dot{y}(t^+) + e\dot{y}(t^-)) \\ q(t_k) = q_1; v(t_k) = v_1; \end{cases} \tag{43}$$

  and leads to the approximation $q_{k+1} = q(t_{k+1})$ and$q_{k+1} = q(t_{k+1})$.

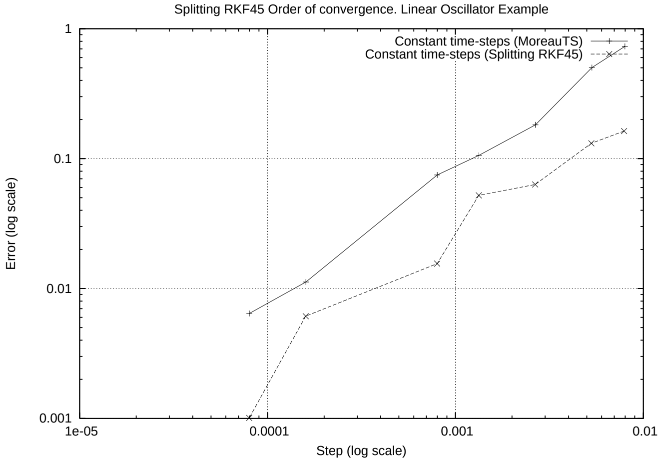
Splitting–based methods with constants time–step.



(a) The bouncing ball example

Figure: Empirical order of convergence of the Splitting RKF45 time-stepping scheme

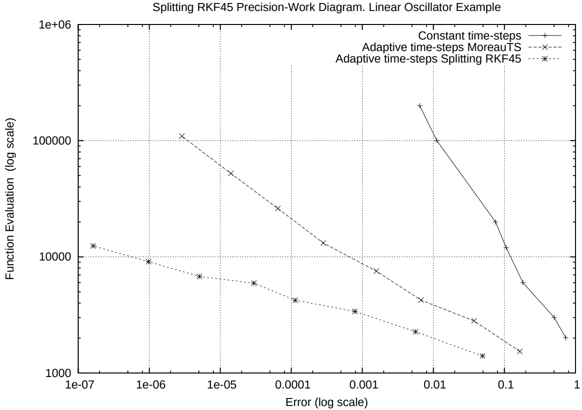
Splitting–based methods with constants time–step.



(a) The linear oscillator example

Figure: Empirical order of convergence of the Splitting RKF45 time-stepping scheme

Splitting–based methods with adaptive time–step.



(a) The linear oscillator example

Figure: Empirical order of convergence of the Splitting RKF45 time-stepping scheme

Splitting–based methods.

Splitting–based for Moreau scheme with continuous contact forces

- The first part is

$$\begin{cases} M(q)\dot{v} = F(t,q,v) + r(t), \\ \dot{q} = v, \\ y = g(q) \\ -r(t) \in \partial\psi_{T_{\mathbb{R}_+}(y)}(\dot{y}(t)) \\ q(t_k) = q_k, \qquad v(t_k) = v_k \end{cases} \tag{44}$$

yielding to the approximations $q_1 = q(t_{k+1})$ and $v_1 = v(t_{k+1})$ which can integrated by any smooth ODE solvers.

- The second one is given by

$$\begin{cases} M(q)\dot{v} = G(q)\lambda, \\ \dot{q} = 0, \\ y = g(q) \\ -\lambda \in \partial\psi_{T_{\mathbb{R}_+}(y)}(\dot{y}(t^+) + e\dot{y}(t^-)) \\ q(t_k) = q_1; v(t_k) = v_1; \end{cases} \tag{45}$$

and leads to the approximation $q_{k+1} = q(t_{k+1})$ and$q_{k+1} = q(t_{k+1})$.

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Conclusions

Adaptive time–step strategies

- Higher resolution schemes
- Work with finite accumulation of events

Higher order schemes

- Schemes of any orders
- Work with finite accumulation of events

Splitting based methods

- Higher resolution schemes
- Work with finite accumulation of events

Perspectives

- Theoretical works on orders and practical error estimations
- Adaptive time–step strategies on the higher order time–stepping schemes.
- Improve the pre–detection process of the event and the order of discontinuity
- Test on nonsmooth and nonlinear mechanical systems.
- Adapt the schemes with a step without external forces when the Moreau's scheme is used
- Other types of time–stepping schemes . . .

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Thank you for your attention.

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