Concurrent multiple impact in rigid bodies: Formulation and simulation

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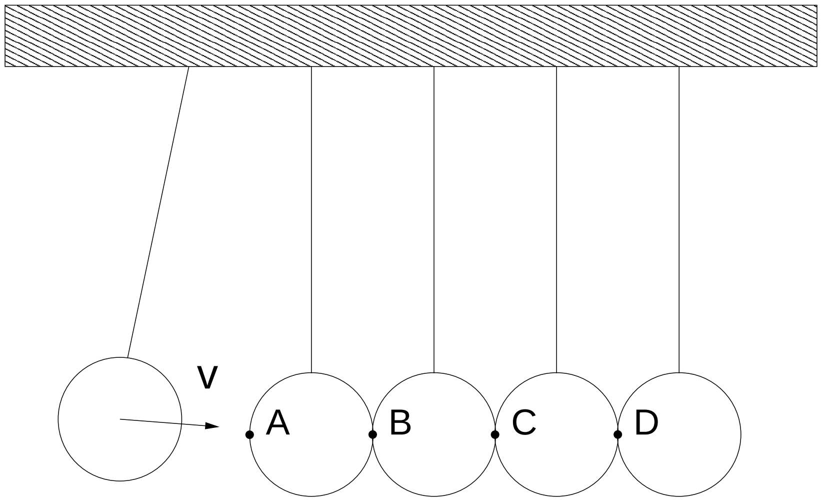
Enoc 2005, Eindhoven, Netherlands, 7-12 august 2005

Outline

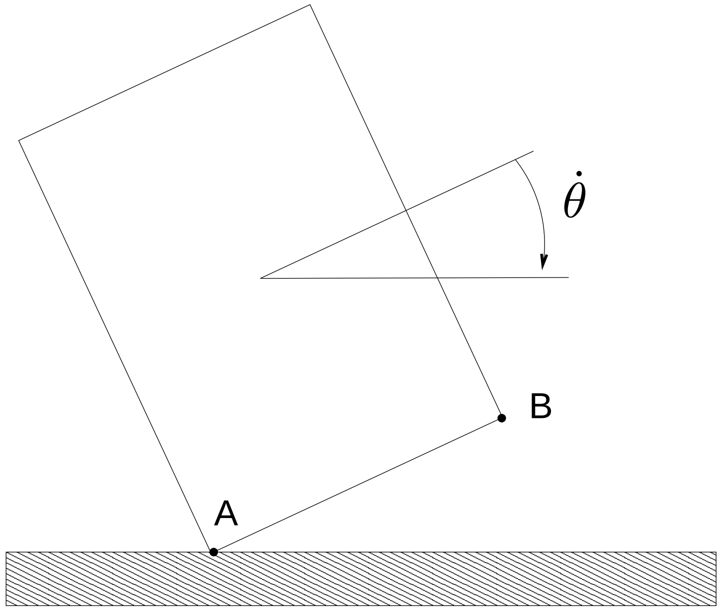
What is a concurrent multiple impact?

- A multiple impact in a multibody system may be defined as :
  The occurrence of several impacts at the same instant on a rigid body.
- Academical examples :

Newton's cradle

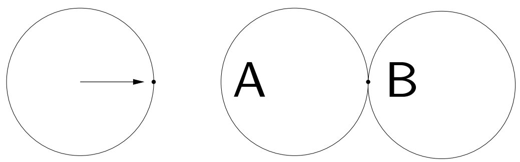
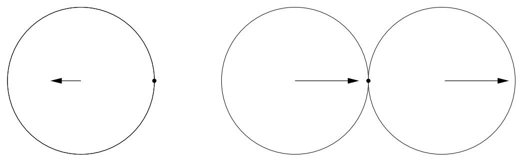
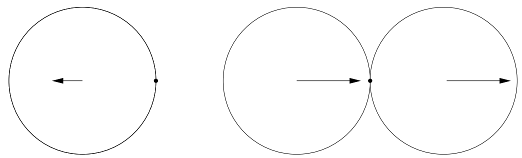
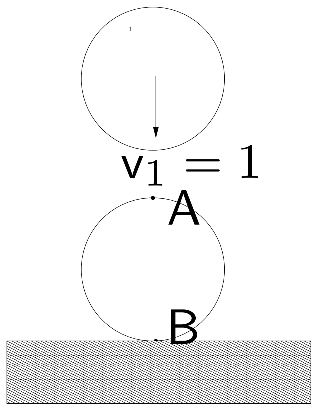
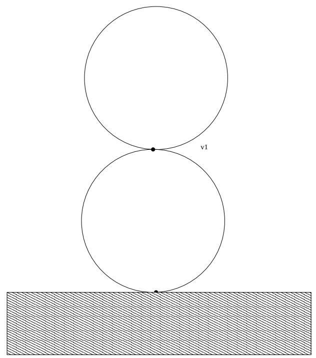
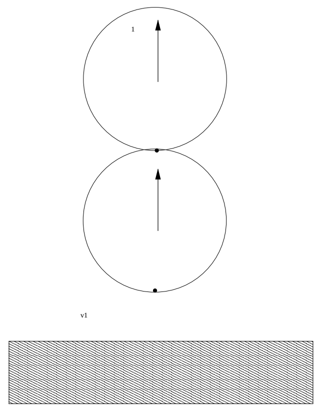


The rocking block



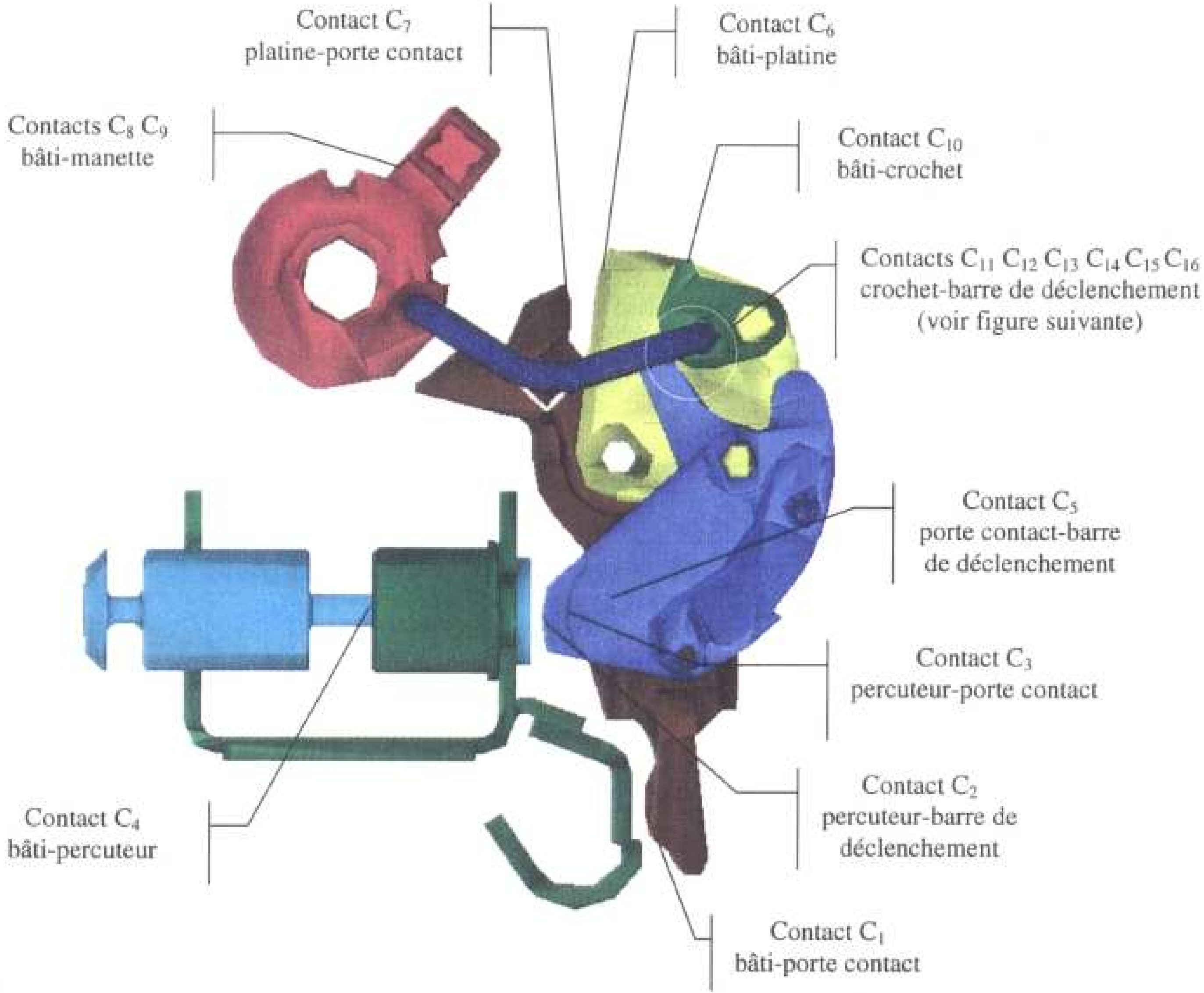
- Major difficulties:
  - The standard laws (Newton, Poisson, . . .) do no longer apply correctly
  - The continuity with the respect to initial conditions is usually lost.

What is a concurrent multiple impact? Continued ...

Initial conditions	Newton	Poisson
 $v_1 = 1$	$e_A = e_B = 1$  $v_1^+ = -1/3$ $v_2^+ = v_3^+ = 2/3$	$e_A = e_B = 1$  $v_1^+ = -1/3$ $v_2^+ = v_3^+ = 2/3$
	$e_A = 0, e_B = 1$  $v_1^+ = v_2 + = 0$	$e_A = 0, e_B = 1$  $v_1^+ = v_2 + \neq 0$

Industrial motivations

Long term collaboration with Schneider Electric (M. Abadie)
Study of the stability of circuits breakers w.r.t. external impact excitations



Main objectives

Find a multiple impact law, i.e.

$$\dot{q}^{+} = \mathcal{F}(\dot{q}^{-}, q, t)$$

with the following properties :

- single-valued mapping
- Ensure the fundamental principles of the mechanics of multibody system
  - Equations of motion
  - Unilateral constraints
  - Energetic balance
- Fit qualitatively and quantitatively the experiments
  - Measurable parameters with a physical meaning
- Efficient numerical treatment
  - Formulation in terms of a mathematical programm with constraints

Previous works

- Sequential approach [Han & Gilmore , 1993]
  - Heuristic choice of solutions
  - Existence and uniqueness problems (closed-loop systems)
- Generalization to multi-constrained systems
  (Newton (Moreau, 1988), Poisson (Glocker & Pfeiffer, 1995)).
  - Efficient numerical algorithm
  - Do not fit the experiments.
- Introduction of an impulse ratio [Hurmuzlu, 2001]
  - Lack of efficient numerical algorithm
  - Do not ensure the fundamental principles of mechanics
- Non local formulation of multi-body systems with impacts [Frémond, 1995].
  - Ensure the fundamental principles of mechanics
  - Lack of physical interpretation of the parameter
- Geometric framework for impacting systems [Aeberhard & Glocker]

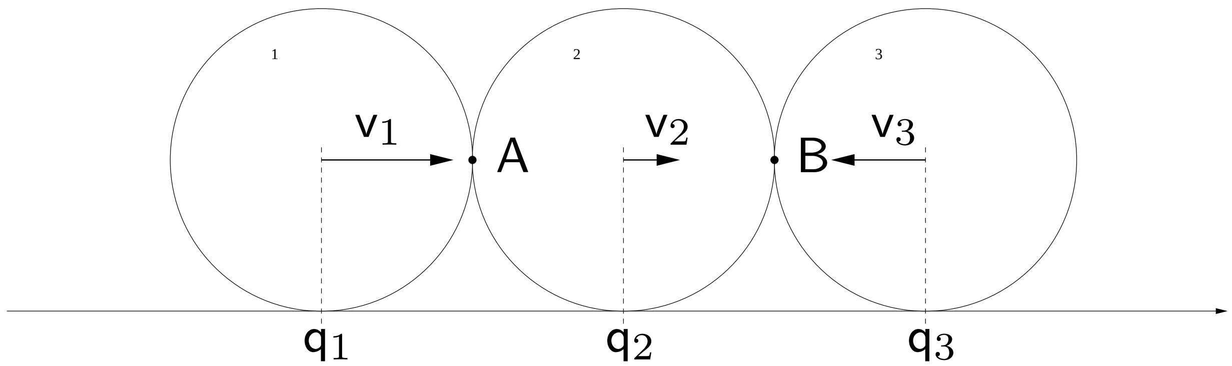
Proposed approach

- Fundamental problem:
  - Taking into account the impact propagation by wave effects.
  - Incorporating a continuum mechanics model for the deformation behaviour
- Quasi-rigid body model: general assumption of the Hertz's contact
  - Local deformation in the area of contact
  - Quasi-static behavior laws (no inertia effects in the contact region)
  - Negligible wave effects inside the bodies (focusing on bodies without special aspect ratios)

  ➜ Multibody systems are modeled by a discrete dynamical systems of rigid bodies and unilateral contact force model (non-linear stiffness and damping)
- A typical example of such systems: a chain of balls [Falcon & Laroche, 1998]
  - Balls are modeled by discrete masses
  - Contact interaction: Non linear Hertzian model

Outline

The unidimensional chain of 3 balls



q_i position
$v_i = \dot{q}_i^-$ pre-impact velocities
$v_i^+ = \dot{q}_i^+$ post-impact velocities

Equation of motion at the instant of impact ($v_2 = 0$) :

$$\begin{cases} m(v_1^+ - v_1) = -p_1 \\ m(v_2^+) = p_1 - p_2 \\ m(v_3^+ - v_3) = p_2 \end{cases} \tag{1}$$

Unilateral constraints

$$\begin{cases} v_1^+ \leq v_2^+ \leq v_3^+ \\ p_1 \geq 0, \quad p_2 \geq 0 \end{cases} \tag{2}$$

Choice of a parametrization

- One global energetic coefficient e :

$$(v_1^+)^2 + (v_2^+)^2 + (v_3^+)^2 \quad = \quad e\,(v_1^2 + v_3^2) \tag{3}$$

- One impulse ratio

$$\alpha = \frac{p_2}{p_1} \tag{4}$$

➜ Parametrization of the solutions :

$$\begin{aligned} p_1 &= \frac{(1+\sqrt{\Delta})m}{2(1-\alpha+\alpha^2)} \\ p_2 &= \frac{(1+\sqrt{\Delta})m\alpha}{2(1-\alpha+\alpha^2)} \end{aligned} \tag{5}$$

$$\begin{aligned} \dot{q}_1^+ &= -\frac{1+\sqrt{\Delta}}{2(1-\alpha+\alpha^2)} + \dot{q}_1^- \\ \dot{q}_2^+ &= -\frac{(1+\sqrt{\Delta})(\alpha-1)}{2(1-\alpha+\alpha^2)} \\ \dot{q}_3^+ &= -\frac{(1+\sqrt{\Delta})\alpha}{2(1-\alpha+\alpha^2)} \end{aligned} \tag{6}$$

with $\Delta = -1 + 2e - 2\alpha e + 2\alpha + 2\alpha^2 e - 2\alpha^2$.

Evaluation of e and α

- Bounds imposed by $\Delta > 0$ and unilateral constraints

$$\tfrac{1}{2} \leq \alpha \leq \frac{2e - 2 - \sqrt{6e-2}}{e-3} \quad , \qquad \tfrac{1}{3} \leq e \leq 1 \tag{7}$$

- Evaluation by a regularized system with Hertzian springs.

$$\begin{cases} m\ddot{\delta}_1 = -2f_1(\delta_1) + f_2(\delta_2) \\ m\ddot{\delta}_2 = -2f_2(\delta_2) + f_1(\delta_1) \\ 0 \leq \boldsymbol{f} \perp \boldsymbol{f} - \boldsymbol{K}(\boldsymbol{\delta}).\boldsymbol{\delta} \geq 0 \end{cases} \tag{8}$$

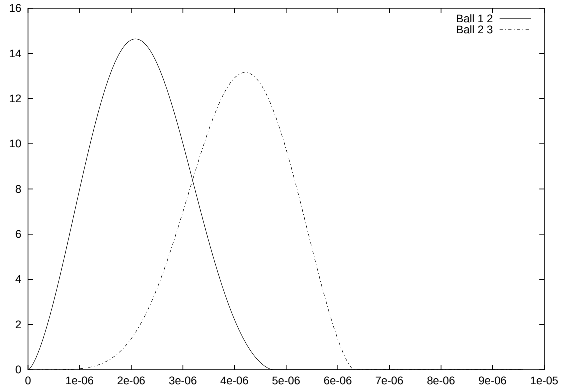
$$\delta_i = q_{i+1} - q_i \text{ Indentation,} \quad K(\boldsymbol{\delta}) = \begin{bmatrix} k_1\delta_1^{1/2} & 0 \\ 0 & k_2\delta_2^{1/2} \end{bmatrix} \quad \kappa = \frac{k_1}{k_2} \tag{9}$$

- Evaluation of the impulse ratio

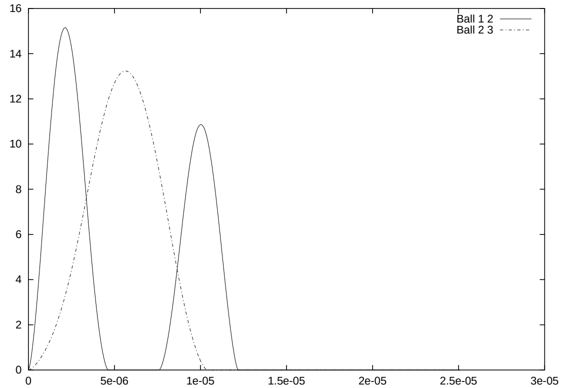
$$\alpha = \frac{\displaystyle\int_0^{t_f} f_1(t)\,dt}{\displaystyle\int_0^{t_f} f_2(t)\,dt} \tag{10}$$

Evaluation of e and α (Continued ...)

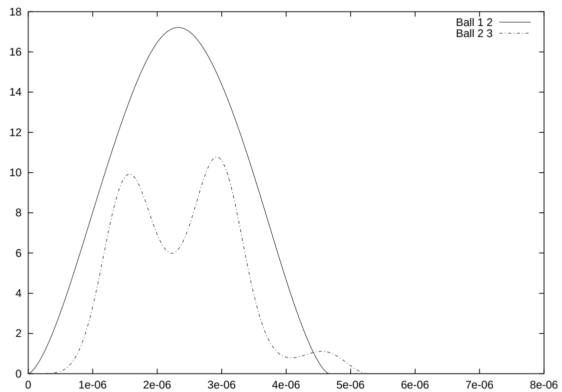
Numerical integration of 3 balls chain. Hertzian spring contact.



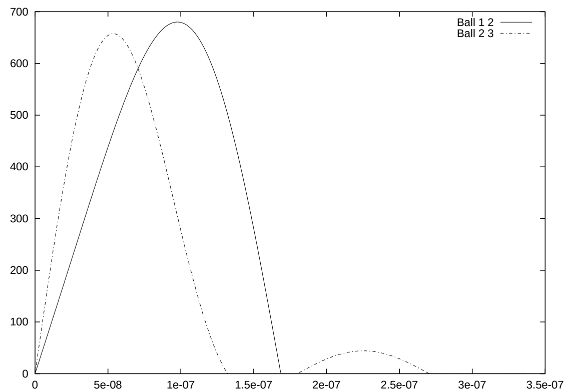
$\kappa = 1, v_1 = 1, v_3 = 0$



$\kappa = 0.1, v_1 = 1, v_3 = -1$



$\kappa = 100, v_1 = 1, v_3 = 0$



$\kappa = 2.24, v_1 = 1, v_3 = -1$

Forces between balls versus time.

➜ No sequential or simultaneous process

The unidimensional chain of n balls

- Rigid body model of a unidimensional chain of n balls :
  - Unknowns: n velocities $\dot{q}_i$, $n-1$ impulses p_i
  - Equations:
    - n equations of motion
    - 1 Energetic balance e
    - $n-2$ impulse ratios, $\alpha_{ij} = p_i/p_j$

  ➜ Unique determination of post-impact velocities and impulses
- Regularised quasi-rigid model:
  - Visco-elastic contact model

$$f = K\delta^{\nu} + C\delta^{\nu-1}\dot{\delta} \tag{11}$$

  - Evaluation of the impulse ratios

$$\alpha_{\gamma\beta} = \frac{\displaystyle\int_0^{t_f} f_\gamma(t)\,dt}{\displaystyle\int_0^{t_f} f_\beta(t)\,dt} \tag{12}$$

The unidimensional chain of n balls (Continued ...)

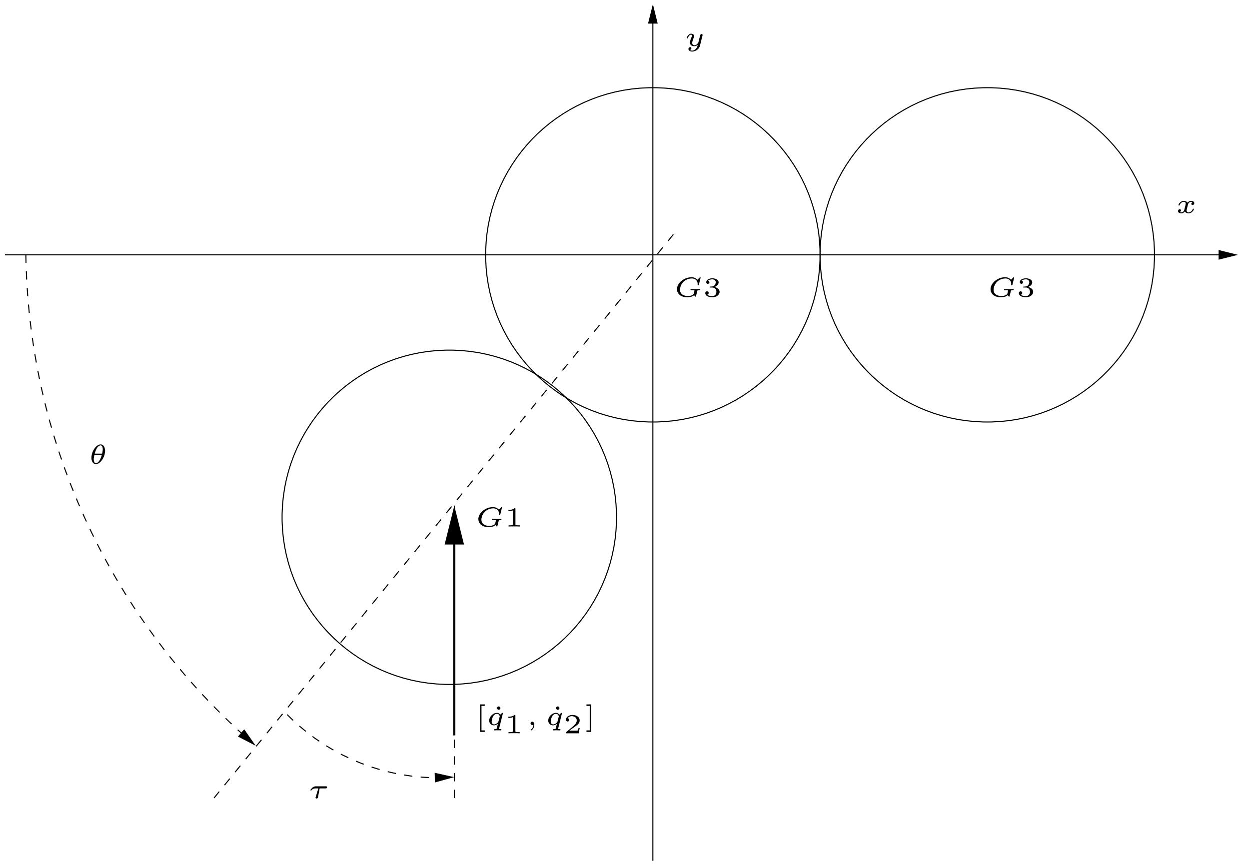
Major results:

- The ratio of impulses is finite and the subspace $E = \{\delta \geq 0, \dot{\delta} \geq 0\}$ is globally attractive.
- The impulse ratios are independent of the absolute value of stiffness and masses but only function of the ratio of stiffness and mass.
- In the linear case $\nu = 1$, the impulse ratios are completely determined by the natural modes of the regularized dynamical system and the pre-impact velocities.

Conclusion:

- The definition of the impulse ratio is relevant in the rigid limit.
- The impulse ratio provide us with some informations on the dynamical process
- The independence of the absolute value of stiffness avoids stiff numerical problem.

The pool: A bidimensional chain of balls



3-balls in the plane

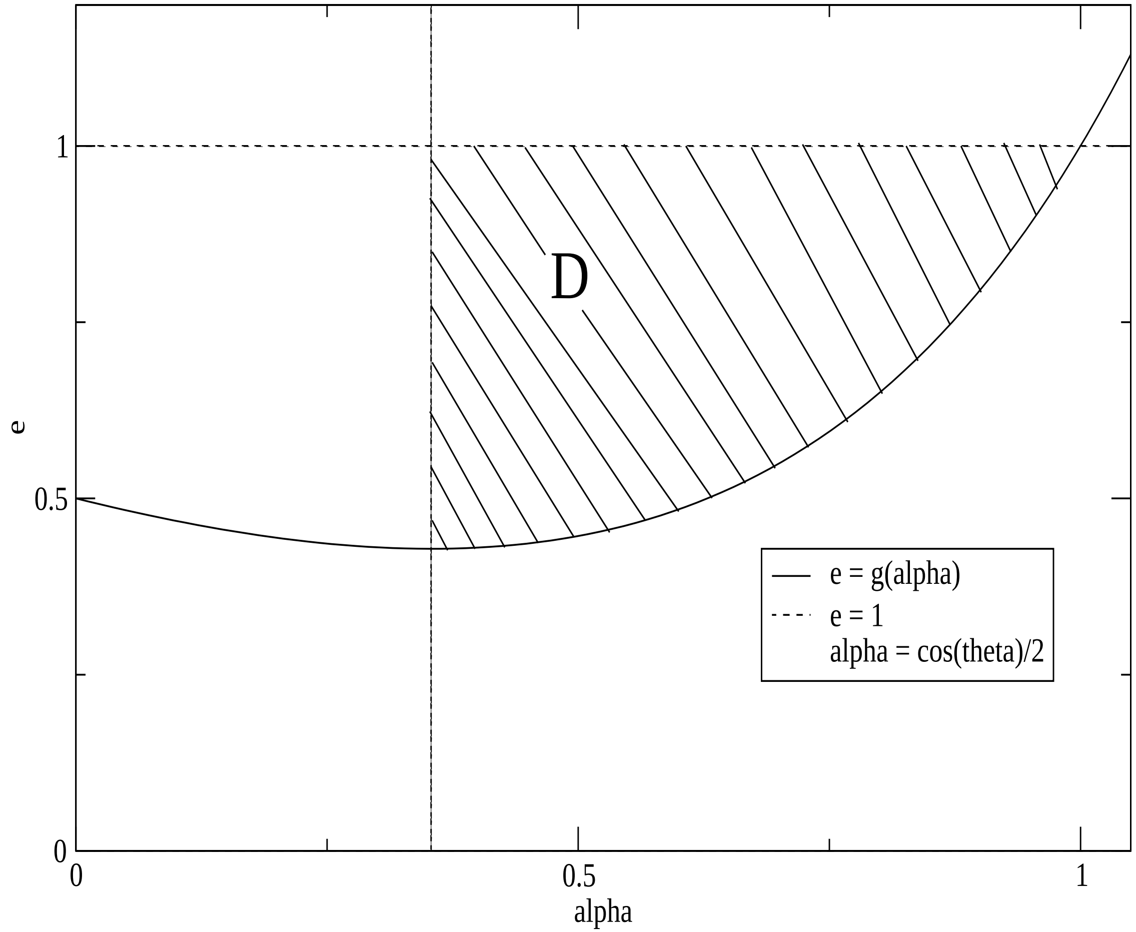
The pool: A bidimensional chain of balls

Solution for $\tau = 0$ and $v_1 = 1$

$$
\left\{
\begin{array}{l}
p_1 = \dfrac{1+\sqrt{\Delta}}{-2\alpha_{12}\cos\theta + 2\,{\alpha_{12}}^2 + 2} \\
p_2 = \dfrac{(1+\sqrt{\Delta})\alpha_{12}}{-2\alpha_{12}\cos\theta + 2\,{\alpha_{12}}^2 + 2} \\
\dot{x}_1^+ = \cos\theta(1 - \dfrac{1+\sqrt{\Delta}}{-2\alpha_{12}\cos\theta + 2\,{\alpha_{12}}^2 + 2}) \\
\dot{y}_1^+ = \sin\theta(1 - \dfrac{1+\sqrt{\Delta}}{-2\alpha_{12}\cos\theta + 2\,{\alpha_{12}}^2 + 2}) \\
\dot{x}_2^+ = (\cos\theta - \alpha_{12})(\dfrac{1+\sqrt{\Delta}}{-2\alpha_{12}\cos\theta + 2\,{\alpha_{12}}^2 + 2}) \\
\dot{y}_2^+ = \sin\theta(\dfrac{1+\sqrt{\Delta}}{-2\alpha_{12}\cos\theta + 2\,{\alpha_{12}}^2 + 2}) \\
\dot{x}_3^+ = \alpha_{12}(\dfrac{1+\sqrt{\Delta}}{-2\alpha_{12}\cos\theta + 2\,{\alpha_{12}}^2 + 2}) \\
\dot{y}_3^+ = 0 \\
\text{with} \\
\Delta = 2(e-1)(\alpha_{12}^2 - \alpha_{12}\cos\theta + 1) + 1
\end{array}
\right. \tag{13}
$$

Valid domain for e and α

- Bounds imposed by $\Delta > 0$ and unilateral constraints



Valid domain of e and α for $\theta = \frac{\pi}{4}$.

Influence of the configuration and the initial conditions

- Influence of the angle τ.
  - The angle τ determines the occurrence of an impact (single or multiple).

$$-v_1^- \in \mathcal{T}_1 = \{p | -\nabla h_1(q).p \geq 0\}. \tag{13}$$

  - The model leads to consistent results whatever the angle τ.

- Influence of the angle θ for τ such that $-v_1^- \in \mathcal{T}_1$:
  1. $\theta \in [0, \frac{\pi}{2})$, one multiple impact
     - We obtain a unique solution. The set of solutions is parametrized by e and α.
  2. $\theta \in [\frac{\pi}{2}, \frac{2\pi}{3})$, two single impacts
     - We must choose a correct value, i.e. $\alpha_{12} = 0$ to have consistent results.

     or
     - We must choose two standards law for single impact.
  3. $\theta = \frac{2\pi}{3}$, The Bernoulli's problem
     - A impulse ratio $\alpha_{13} = p_3/p_1$ is added.
     - The choice $\alpha_{12} = \alpha_{13}$ leads to consistent results in the symmetric case (same masses and stiffnesses.).

Preliminary conclusion

- Validity of the model.
  - In the case of one multiple impact, the set of solutions which respect the equations of motion, the unilateral constraints and the energetic balance is correctly parametrized by the parameter e and α_{12},
  - In the case of two single impacts, standard single laws must be applied or correct values of the impulse ratios must be chosen.
- Open problems
  - How to separate multiple or single impact cases ?
  - How to write a single model for all impacting cases ?

Outline

General model

- equations of motion at the instant of impact.

$$M(\dot{q}^{+} - \dot{q}^{-}) = P, \quad q \in \mathbb{R}^{n} \tag{14}$$

- unilateral constraints

$$h_i(q) \geq 0, \quad i = 1, \dots, m \tag{15}$$

- kinematic relations :

$$v = H\dot{q} \tag{16}$$

$$P = H^{T} p \tag{17}$$

$$H = \nabla h(q(t_k)) \tag{18}$$

- unilateral constraints on the relative velocity and the percussions

$$\text{if} \quad h_i(q) = 0, \quad v_i^{+} \geq 0, p_i \geq 0 \tag{19}$$

$$\text{if} \quad h_i(q) > 0, \quad p_i = 0 \tag{20}$$

Angles between constraints in the kinetic metric

- Gradient to a constraint in the kinetic metric :

$$n_i = \frac{M^{-1}(q)\nabla_q h_i(q)}{\sqrt{\nabla_q h_i(q)^T M(q)^{-1} \nabla_q h_i(q)}} \quad (21)$$

- Angle between two constraints in the kinetic metric:

$$\cos\theta_{ij} = \langle n_i, n_j \rangle_M = \frac{\nabla_q^T h_i(q) M^{-1}(q) \nabla_q h_j(q)}{\sqrt{\nabla_q h_i(q)^T M(q)^{-1} \nabla_q h_i(q)} \sqrt{\nabla_q h_j(q)^T M(q)^{-1} \nabla_q h_j(q)}} \quad (22)$$

It is noteworthy that $\nabla_q^T h_i(q) M^{-1}(q) \nabla_q h_j(q)$ is a coefficient of the Delassus matrix $H^T M^{-1} H$

- Coupling between two adjacent constraints:

$$\cos\theta_{ij} = \langle n_i, n_j \rangle_M < 0 \implies \text{coupling} \quad \alpha_{ij} = p_i/p_j \quad (23)$$

$$\cos\theta_{ij} = \langle n_i, n_j \rangle_M \geq 0 \implies \text{no coupling due to unilaterality} \quad (24)$$

Illustration on an open chain of balls

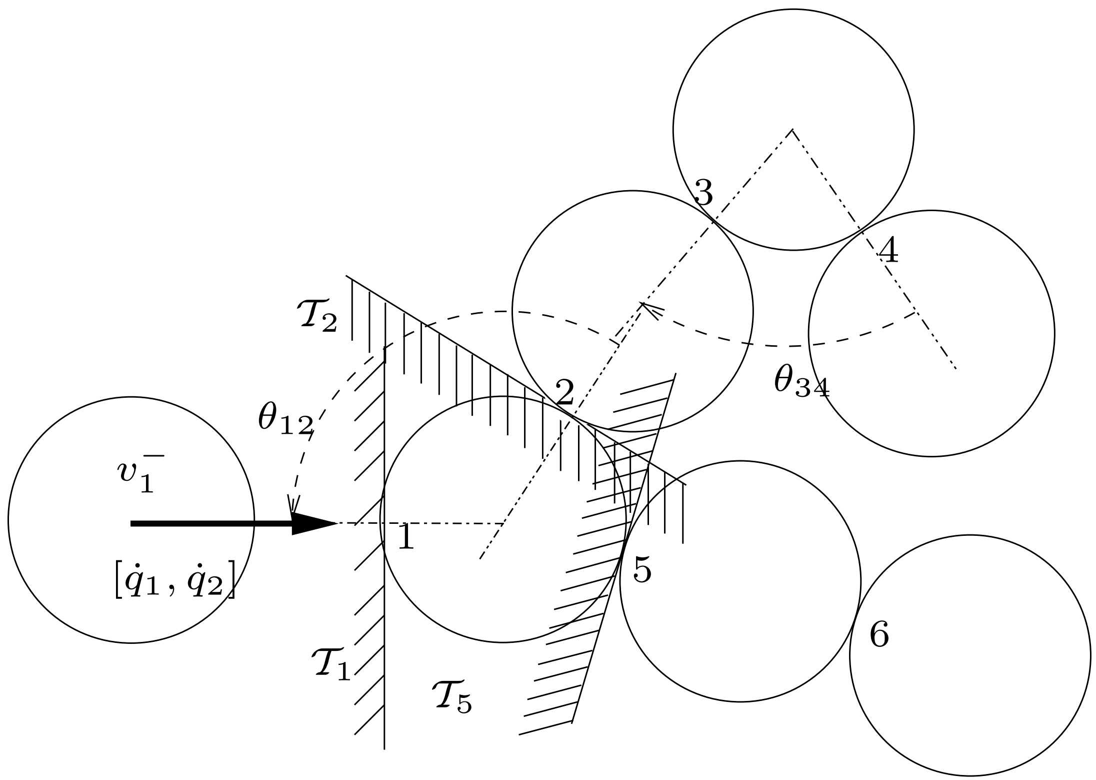


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Illustration on an open chain of balls

- First criteria of impact (single or multiple), the velocity v_1 must belong to the opposite of the tangent cone $\mathcal{T}_1(q)$

$$-v_1^- \in \mathcal{T}_1 = \{p | -\nabla h_1(q).p \geq 0\}. \quad (26)$$

- To have a multiple impact of degree k, (involving the first k-balls) the $k-1$ first kinetic angle $\theta_{i,i+1}$ must be greater than $\frac{\pi}{2}$:

$$\theta_{i,i+1} > \frac{\pi}{2}, \text{ i.e } \langle n_i, n_{i+1}\rangle_M < 0, i = 1 \dots k-1 \quad (25)$$

Outline

Conclusion and perspectives

- Model with a global energetic coefficient and impulse ratios is a good candidate for propagation of an impact
- Properties of the impulse ratio
  - Well defined in the rigid limit of a regularized model
  - Easy to evaluate from the numerical point of view.
  - Parametrization of the set of solutions after impact.
- A general impact law for multiple and single impacts is a more challenging task:
  - How to take into account the kinetic angle between constraints into the formulation ?
  - How to write a single law valid in all cases ?
  - How to deal with such law on the numerical point of view ?

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