Computational methods for the simulation of nonsmooth cable dynamics in ropeways transportation systems

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Context

Dynamics of a translating cable subjected to unilateral constraints, friction and punctual loads

Engineering applications

- Aerial ropeways as an alternative for public transportation with increasing velocities
- maintenance and support of existing infrastructures

Scientific issues and open questions

- Need for efficient numerical tools for highly stiff and nonlinear systems
- A design tool based on constrained dynamics with contact and friction
- Understand sudden large amplitudes observed in practice
- Comparison of models (thin beams or cable?).

Photo credit: POMA, Eiffage

Outline

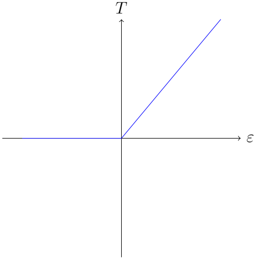
Assumptions

The cable is assumed to be

- linear elastic
- a curvilinear domain
- bending and torsion moments vanish
- a tension-only material ($T \geqslant 0$)

As a consequence,

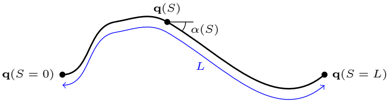
- only one space variable S, curvilinear abscissa is needed
- each tangent has a left and right limit, but kinks are possible



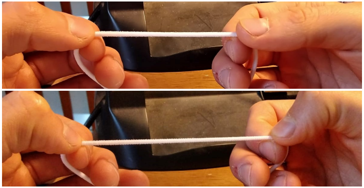
Unilateral elasticity

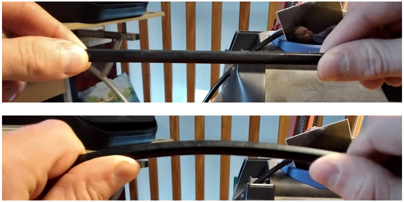
Kinematics

Curvilinear domain $\longrightarrow$ domain assimilated to a curve.



Two main mechanisms:





Dilatation: $\varepsilon(S) = \left\|\mathbf{q}'(S)\right\| - 1$ (1)

Flexure: $\kappa(S) = \omega'(S) = \alpha'(S) - \alpha_0'(S)$ $\longrightarrow$ Useless here (2)

Governing equations

Lagrangian with unilateral constraints

$$\mathcal{L}\left(\dot{\mathbf{q}}, \mathbf{q}', \mathbf{q}, \lambda\right) = \mathcal{L}^*\left(\dot{\mathbf{q}}, \mathbf{q}', \mathbf{q}\right) - \lambda^T \mathbf{a}(\mathbf{q}, \mathbf{q}', \dot{\mathbf{q}})$$

$$\text{with } \mathcal{L}^*\left(\dot{\mathbf{q}}, \mathbf{q}', \mathbf{q}\right) = \mathcal{T}\left(\dot{\mathbf{q}}, \mathbf{q}\right) - \mathcal{V}\left(\mathbf{q}', \mathbf{q}\right) \qquad (3)$$

$$\mathbf{a}(\mathbf{q}, \mathbf{q}', \dot{\mathbf{q}}) \geq 0$$

Euler-Lagrange equations with unilateral constraints

$$\begin{cases} \mathbf{0} = \dfrac{\partial \mathcal{L}^*}{\partial \mathbf{q}} - \dfrac{\mathrm{d}}{\mathrm{d}S}\left(\dfrac{\partial \mathcal{L}^*}{\partial \mathbf{q}'}\right) - \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \mathcal{L}^*}{\partial \dot{\mathbf{q}}}\right) - \dfrac{\partial \mathbf{a}}{\partial \mathbf{q}}^\top \lambda + \dfrac{\mathrm{d}}{\mathrm{d}S}\left[\dfrac{\partial \mathbf{a}}{\partial \mathbf{q}'}^\top \lambda\right] + \dfrac{\mathrm{d}}{\mathrm{d}t}\left[\dfrac{\partial \mathbf{a}}{\partial \dot{\mathbf{q}}}^\top \lambda\right] \\ \mathbf{0} \leqslant \mathbf{a}(\mathbf{q}, \mathbf{q}', \dot{\mathbf{q}}) \perp \lambda \geqslant 0 \end{cases} \qquad (4)$$

Governing equations

Lagrangian

- Kinetic energy
- External work

$$\mathcal{L}^* \left(\dot{\mathbf{q}}, \mathbf{q}', \mathbf{q}, \lambda\right) = \frac{\rho}{2} \dot{\mathbf{q}} \cdot \dot{\mathbf{q}} \tag{5}$$

Governing equations

Lagrangian

- Kinetic energy
- External work

$$\mathcal{L}^* \left(\dot{\mathbf{q}}, \mathbf{q}', \mathbf{q}, \lambda\right) = \frac{\rho}{2}\dot{\mathbf{q}} \cdot \dot{\mathbf{q}} + \mathbf{f}_e \cdot \mathbf{q} \tag{5}$$

Governing equations

Lagrangian

- Kinetic energy
- External work
- Elastic energy

$$\mathcal{L}^{*}\left(\dot{\mathbf{q}}, \mathbf{q}', \mathbf{q}, \lambda\right) = \frac{\rho}{2}\dot{\mathbf{q}} \cdot \dot{\mathbf{q}} + \mathbf{f}_e \cdot \mathbf{q} - \frac{EA}{2}\left(\|\mathbf{q}'\| - 1\right)^2 \tag{5}$$

Dynamical equations for the (standard) elastic cable

$$\left\{ \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\dot{\mathbf{q}}\right) = \frac{\mathrm{d}}{\mathrm{d}S}\left(\left[EA\left(\|\mathbf{q}'\| - 1\right)\right]\frac{\mathbf{q}'}{\|\mathbf{q}'\|}\right) + \mathbf{f}_e \right. \tag{6}$$

with $T = \left[EA\left(\|\mathbf{q}'\| - 1\right)\right]$ the tension.

Governing equations

Lagrangian

- Kinetic energy
- External work
- Inextensibility

$$\mathcal{L}^* \left(\dot{\mathbf{q}}, \mathbf{q}', \mathbf{q}, \lambda\right) = \frac{\rho}{2}\dot{\mathbf{q}} \cdot \dot{\mathbf{q}} + \mathbf{f}_e \cdot \mathbf{q} - \lambda \left(1 - \left\|\mathbf{q}'\right\|\right) \tag{5}$$

Dynamical equations for the inextensible cable

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\rho\dot{\mathbf{q}}\right) = \dfrac{\mathrm{d}}{\mathrm{d}S}\left(\lambda \dfrac{\mathbf{q}'}{\left\|\mathbf{q}'\right\|}\right) + \mathbf{f}_e \\ 0 \leqslant 1 - \left\|\mathbf{q}'\right\| \perp \lambda \geqslant 0 \end{cases} \tag{6}$$

with $T = \lambda \geq 0$ the tension.

Governing equations

Lagrangian

- Kinetic energy
- External work
- Unilateral elasticity

$$\mathcal{L}^*\left(\dot{\mathbf{q}}, \mathbf{q}', \mathbf{q}, \tilde{\mathbf{q}}', \lambda\right) = \frac{\rho}{2}\dot{\mathbf{q}}\cdot\dot{\mathbf{q}} + \mathbf{f}_e\cdot\mathbf{q} - \frac{EA}{2}\left(\|\mathbf{q}'\| - \|\tilde{\mathbf{q}}'\|\right)^2 - \lambda\left(1 - \|\tilde{\mathbf{q}}'\|\right) \quad (5)$$

Dynamical equations for the unilateral elastic cable

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\rho\dot{\mathbf{q}}\right) = \dfrac{\mathrm{d}}{\mathrm{d}S}\left(\left[EA\left(\|\mathbf{q}'\| - \|\tilde{\mathbf{q}}'\|\right)\right]\dfrac{\mathbf{q}'}{\|\mathbf{q}'\|}\right) + \mathbf{f}_e \\ 0 = \left[EA\left(\|\mathbf{q}'\| - \|\tilde{\mathbf{q}}'\|\right) - \lambda\right]\dfrac{\tilde{\mathbf{q}}'}{\|\tilde{\mathbf{q}}'\|} \\ 0 \leqslant 1 - \|\tilde{\mathbf{q}}'\| \perp \lambda \geqslant 0 \end{cases} \quad (6)$$

Governing equations

Lagrangian

- Kinetic energy
- External work
- Unilateral elasticity

$$\mathcal{L}^*\left(\dot{\mathbf{q}},\mathbf{q}',\mathbf{q},\tilde{\mathbf{q}}',\lambda\right)=\frac{\rho}{2}\dot{\mathbf{q}}\cdot\dot{\mathbf{q}}+\mathbf{f}_e\cdot\mathbf{q}-\frac{EA}{2}\left(\|\mathbf{q}'\|-\|\tilde{\mathbf{q}}'\|\right)^2-\lambda\left(1-\|\tilde{\mathbf{q}}'\|\right) \quad (5)$$

Dynamical equations for the unilateral elastic cable

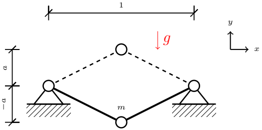
Simplification by eliminating $\tilde{\mathbf{q}}'$

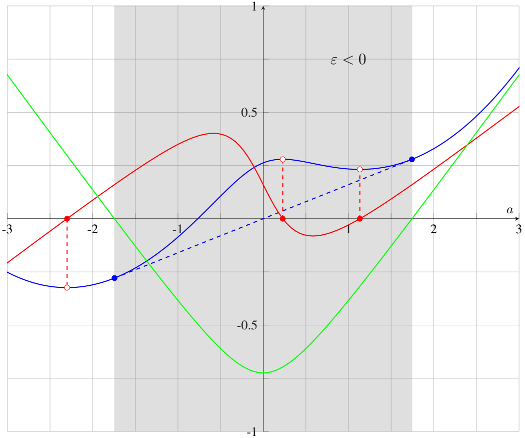
$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}(\rho\dot{\mathbf{q}})=\dfrac{\mathrm{d}}{\mathrm{d}S}\left(\lambda\dfrac{\mathbf{q}'}{\|\mathbf{q}'\|}\right)+\mathbf{f}_e\\ 0\leqslant\lambda-EA(\|\mathbf{q}'\|-1)\perp\lambda\geqslant 0\end{cases} \quad (6)$$

with $T=\lambda\geq 0$ the tension.

Main interests

Tension-only cable with unilateral elasticity. Uniqueness can be retrieved





Elastic cable under self-weight example

Weak formulation

Unconstrained dynamics:

$$\rho \frac{d\mathbf{v}}{dt} = [T\mathbf{e}]' + \mathbf{f}_e \quad (7)$$

where:

$$\mathbf{v}\ ,\ \mathbf{q}\ \in \mathcal{H}^1 = \left\{ \mathbf{q} \in \mathbb{R}^3 \text{ t.q. } \mathbf{q} \in \mathcal{L}^2\left([0,L]\right), \mathbf{q}' \in \mathcal{L}^2\left([0,L]\right), \mathbf{q} = \int^t \mathbf{v}dt \right\} \quad (8)$$

equipped with the norm:

$$\|\mathbf{q}\|_1 = \left[\int_0^L \mathbf{q}\cdot\mathbf{q} + \mathbf{q}'\cdot\mathbf{q}'dS\right]^{\frac{1}{2}} \quad (9)$$

Then for $\varphi \in \mathcal{H}^1$:

$$\int_0^L \rho \frac{d\mathbf{v}}{dt}\cdot\varphi dS + \int_0^L T\mathbf{e}\cdot\varphi' dS = [T\mathbf{e}\cdot\varphi]_0^L + \int_0^L \mathbf{f}_e\cdot\varphi dS \quad (10)$$

i.e.:

$$\int_0^L \rho \frac{d\mathbf{v}}{dt}\cdot\varphi dS + \int_0^L T\mathbf{e}\cdot\varphi' dS = \int_0^L \mathbf{f}_e\cdot\varphi dS \quad (11)$$

Finite element approximation

FE approximation with $p1$ elements [1].

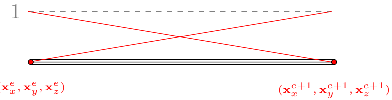
$$\mathbf{q}(S) \approx \sum_{e=1}^{N} \mathbf{N}(S)\ \mathbf{q}^e \qquad (12)$$

$$\mathbf{v}(S) \approx \sum_{e=1}^{N} \mathbf{N}(S)\ \mathbf{v}^e \qquad (13)$$

$$\varphi(S) \approx \sum_{e=1}^{N} \mathbf{N}(S)\varphi^e \qquad (14)$$

where $\mathbf{N}$ stands for:

$$\mathbf{N}(S) = \begin{bmatrix} 1-\xi^e & 0 & 0 & \xi^e & 0 & 0 \\ 0 & 1-\xi^e & 0 & 0 & \xi^e & 0 \\ 0 & 0 & 1-\xi^e & 0 & 0 & \xi^e \end{bmatrix} \quad , \quad \xi^e = \frac{S - S^e}{L_e} \qquad (15)$$



Linear interpolation on element e

[1] O.C. Zienkiewicz and R.L. Taylor. *The finite element method. Vol. 1: The basis*, 2002

Finite element approximation

The global equilibrium reads:

$$\sum_{e=1}^{N} \varphi^e \cdot \left[\mathbf{M}^e \frac{\mathrm{d}\mathbf{v}^e}{\mathrm{d}t} + \mathbf{K}^e(\mathbf{q}^e)\mathbf{q}^e - \mathbf{f}_e^e \right] = \mathbf{0} \tag{16}$$

with

$$\mathbf{M}^e = \rho \int_0^{L^e} \mathbf{N}(S)^\top \mathbf{N}(S) \mathrm{d}S \ , \ \mathbf{f}^e = \int_0^{L^e} \mathbf{N}(S)^\top \mathbf{f}_e \ \mathrm{d}S \tag{17}$$

$$\mathbf{K}^e = EA \int_0^{L^e} \left(\|\mathbf{N}'(S)\mathbf{q}^e\| - 1 \right) \frac{\mathbf{N}'(S)^\top \mathbf{N}'(S)}{\|\mathbf{N}'(S)\mathbf{q}^e\|} \mathrm{d}S \tag{18}$$

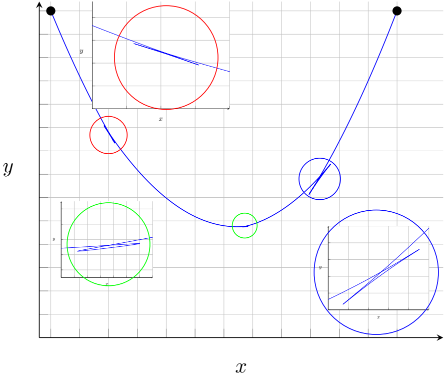
Assembly + structural damping:

$$\mathbf{0} = \mathbf{M}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \mathbf{C}(\mathbf{q}, \mathbf{v})\mathbf{v} + \mathbf{K}(\mathbf{q})\mathbf{q} - \mathbf{f} \tag{19}$$

No constraints for now.

Numerical convergence issues with standard Newton-Raphson method

- convergence issues far from the solution.
- numerous local mimina : spurious solutions



Numerical equilibrium obtained for a cable with compressed segments

$$EA = 1.10^{10}N, L = 51m, span = 50m, n = 25$$

How to cope with those situations

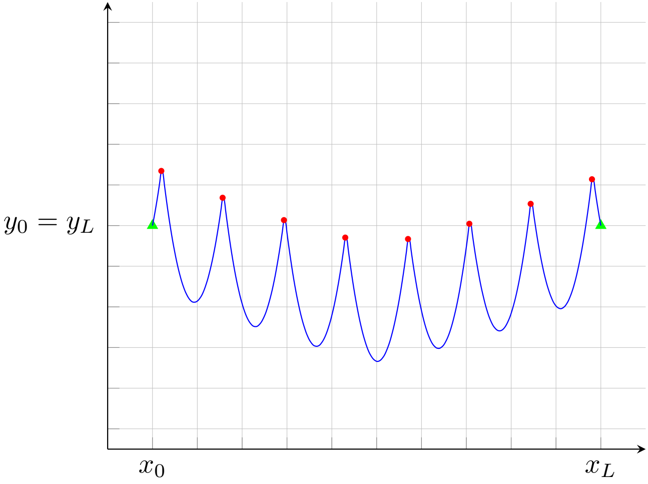
Non-smooth Newton method with a modified Jacobian:

$$\varepsilon^e(S) = \left\|\mathbf{N}'(S)\mathbf{q}^e\right\| - 1 \tag{20}$$

$$\mathbf{K}^e = \begin{cases} EA\displaystyle\int_0^{L^e} \frac{\mathbf{N}'(S)^\top \mathbf{N}'(S)}{1+|\varepsilon^e(S)|^{-1}} \mathrm{d}S & ; \quad \varepsilon^e(S) \geqslant 0 \\ \mathbf{0} & ; \quad \varepsilon^e(S) < 0 \end{cases} \tag{21}$$

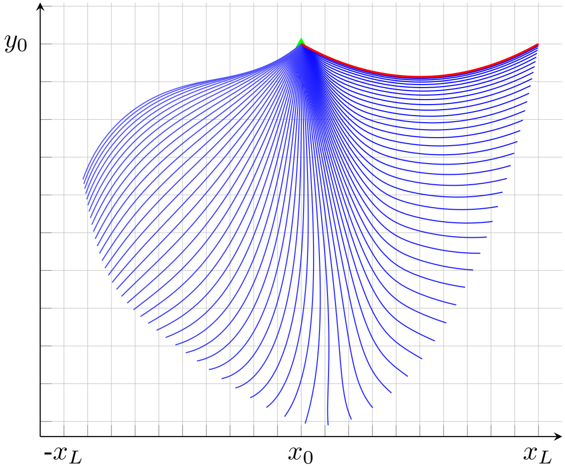
$$\Delta\mathbf{K}^e = \begin{cases} \mathbf{K}^e + EA\displaystyle\int_0^{L^e} \frac{\mathbf{N}'(S)^\top \mathbf{N}'(S)\mathbf{q}^e\mathbf{q}^{e\top}\mathbf{N}'(S)^\top \mathbf{N}'(S)}{(|\varepsilon^e(S)|+1)^3} \mathrm{d}S & ; \quad \varepsilon^e(S) \geqslant 0 \\ EA\displaystyle\int_0^{L^e} \frac{\mathbf{N}'(S)^\top \mathbf{N}'(S)}{1+|\varepsilon^e(S)|^{-1}} \mathrm{d}S & ; \quad \varepsilon^e(S) < 0 \end{cases} \tag{22}$$

Examples



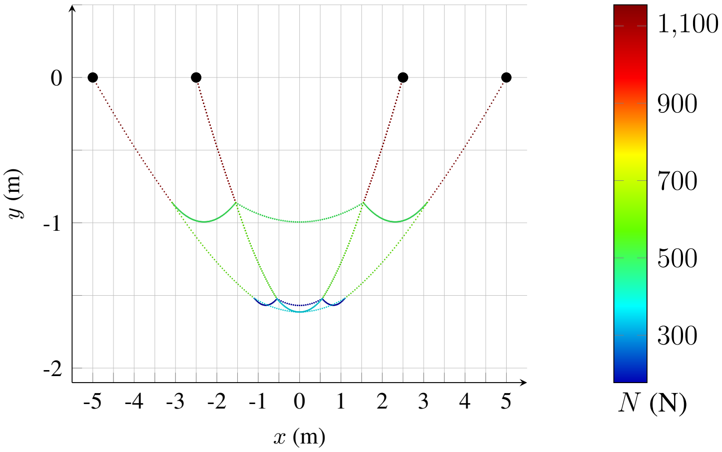
Cable subjected to vertical upwards loads and self-weight

Examples



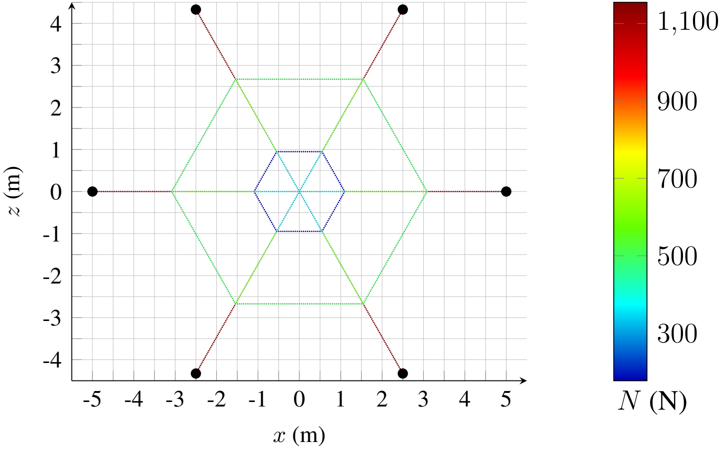
Pendulum cable trajectory

Examples



Tension fields in a cable network - $x - y$

Examples

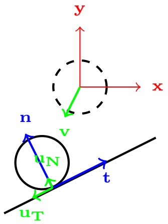


Tension fields in a cable network - $x - z$

Contact kinematics - 1

The cable moves at velocity $\mathbf{v}$ and the obstacle at $\mathbf{v}_{obs}$. The relative velocity from one cable section, M, to one obstacle point, M', reads:

$$\mathbf{u}(M, M') = \mathbf{v}(M) - \mathbf{v}_{obs}(M') \qquad (23)$$



The relative velocity reads in the local basis:

$$\mathbf{u}(M, M') \rightarrow \begin{cases} \mathbf{H}_N & : \quad \mathbf{u}_N(M, M') = \mathbf{H}_N(M, M')\mathbf{u}(M, M') \\ \mathbf{H}_T & : \quad \mathbf{u}_T(M, M') = \mathbf{H}_T(M, M')\mathbf{u}(M, M') \end{cases} \qquad (24)$$

where $\mathbf{t} = [\mathbf{t}_1, \mathbf{t}_2]^\top$ and $\mathbf{n} \perp \mathbf{t}_1 \perp \mathbf{t}_2 \perp \mathbf{n}$. The contact reaction in local basis reads:

$$\mathbf{p} = \mathbf{H}_N{}^\top \mathbf{r}_N + \mathbf{H}_T{}^\top \mathbf{r}_T = \mathbf{H}^\top r \qquad (25)$$

Contact kinematics - 2

The link between local and global formulation is done via **H** and

$$\mathbf{M}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \mathbf{f} + \mathbf{p} \quad \Leftrightarrow \quad \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \tilde{\mathbf{f}} + \widehat{\mathbf{W}}\mathbf{r} \tag{26}$$

yields the Delassus operator $\widehat{\mathbf{W}}$:

$$\widehat{\mathbf{W}} = \mathbf{H}\mathbf{M}^{-1}\mathbf{H}^{\top}\mathbf{f} \tag{27}$$

and

$$\tilde{\mathbf{f}} = \mathbf{H}\mathbf{M}^{-1} \tag{28}$$

Coulomb friction with contact at the velocity level

Coulomb's second order cone

$$\mathbf{K} = \left\{\mathbf{r} \in \mathbb{R}^3 \ , \ \|\mathbf{r}_T\| \leqslant \mu \mathbf{r}_N\right\} \tag{29}$$

Coulomb friction with contact at the veloicty level

three distinct cases:

- No contact i.e. $\mathbf{r} = 0$ and $\mathbf{u}_N \geqslant 0$
- The cable sticks $\mathbf{r} \in \mathbf{K}$ and $\mathbf{u} = 0$
- The cable slips at contact $\mathbf{r} \in \partial\mathbf{K}/\{\mathbf{0}\}$ and $\mathbf{r}_T = -\alpha \mathbf{u}_T$

De Saxcé et Feng's change of variable

$$\tilde{\mathbf{u}} = \mathbf{u} + \mu \|\mathbf{u}_T\| \mathbf{n} \tag{30}$$

Coulomb friction is recast in a second order cone complementarity

$$\mathbf{K}^* \ni \tilde{\mathbf{u}} \perp \mathbf{r} \in \mathbf{K} \tag{31}$$

where $\mathbf{K}^* = \left\{\mathbf{u} \in \mathbb{R}^3 \ , \ \forall \mathbf{r} \in \mathbf{K} \ , \ \mathbf{u} \cdot \mathbf{r} \geqslant 0\right\}$ is the dual cone.

Numerical scheme development - 1

We go back to the FEM:

$$\begin{cases} \mathbf{0} = \mathbf{M}\dfrac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \mathbf{C}(\mathbf{q},\mathbf{v})\mathbf{v} + \mathbf{K}(\mathbf{q})\mathbf{q} - \mathbf{f} \\ \text{such that } \ \mathbf{g}(\mathbf{q},t) \geqslant \mathbf{0} \quad , \quad \text{where } \mathbf{g} \ \text{ is given} \end{cases} \tag{32}$$

Traditionally:

$$\begin{cases} \mathbf{M}\dfrac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \mathbf{C}(\mathbf{q},\mathbf{v})\mathbf{v} + \mathbf{K}(\mathbf{q})\mathbf{q} = \mathbf{f} + \mathbf{p} \\ \mathbf{p} = \nabla_{\mathbf{g}}(\mathbf{q},t)\lambda \\ \mathbf{0} \leqslant \lambda \perp \mathbf{g}(\mathbf{q}) \geqslant \mathbf{0} \end{cases} \tag{33}$$

Inequality $\longrightarrow$ non-smooth velocity (bounded variations)

$$\mathbf{M}\mathrm{d}\mathbf{v} + [\mathbf{C}(\mathbf{q},\mathbf{v})\mathbf{v} + \mathbf{K}(\mathbf{q})\mathbf{q}]\,\mathrm{d}t = \mathbf{f}\mathrm{d}t + \mathrm{d}\mathbf{p} \tag{34}$$

where measures $\mathrm{d}\mathbf{v}$ and $\mathrm{d}\mathbf{p}$ are discomposed as:

$$\mathrm{d}\mathbf{v} = \gamma \mathrm{d}t + \left(\mathbf{v}^{+} - \mathbf{v}^{-}\right)\mathrm{d}\nu + \mathrm{d}\mathbf{v}_{s} \tag{35}$$

$$\mathrm{d}\mathbf{p} = \mathbf{p}\mathrm{d}\nu + \mathrm{d}\mathbf{p}_{s} \tag{36}$$

where γ is the acceleration in the usual sense and $\mathrm{d}t$ the Lebesgue's measure.

Numerical scheme development - 2

Time integration on $[t_k, t_{k+1}]$ of the linearized model uses the θ-method for the smooth terms:

$$\begin{cases} \widehat{\mathbf{M}}_k \left(\mathbf{v}_{k+1} - \mathbf{v}_k\right) - \widehat{\mathbf{f}}_k = \mathbf{p}_{k+1} \\ \mathbf{q}_{k+1} = \mathbf{q}_k + h\theta\mathbf{v}_k + h(1-\theta)\mathbf{v}_{k+1} \end{cases} \tag{37}$$

where

$$\widehat{\mathbf{M}}_k = \mathbf{M} + h\theta\mathbf{C} + h^2\theta^2\Delta\mathbf{K}_k \tag{38}$$

$$\widehat{\mathbf{f}}_k = h\theta\mathbf{f}_{k+1} + h(1-\theta)\mathbf{f}_k - h\mathbf{C}\mathbf{v}_k - h\mathbf{K}_k\mathbf{q}_k - h^2\theta\Delta\mathbf{K}_k\mathbf{v}_k \tag{39}$$

$$\mathbf{p}_{k+1} = \int_{t_k}^{t_{k+1}} \mathrm{d}\mathbf{p} \quad , \quad h = t_{k+1} - t_k \tag{40}$$

Introducing the free velocity $\mathbf{v}_f$ as

$$\mathbf{v}_f = \mathbf{v}_k + \widehat{\mathbf{M}}_k^{-1}\widehat{\mathbf{f}}_k, \tag{41}$$

we obtain

$$\widehat{\mathbf{M}_k} \left(\mathbf{v}_{k+1} - \mathbf{v}_f\right) = \mathbf{p}_{k+1}. \tag{42}$$

More details are available in the work of Moreau et Jean[2].

[2] M. Jean and J.J. Moreau. *Dynamics in the presence of unilateral contacts and dry friction: a numerical approach*, Unilateral problems in structural analysis. II, pages 151–196. CISM 304, Spinger Verlag, 1987.

Numerical scheme development - 3

In the local basis:

$$\mathbf{u}_{k+1} = \mathbf{u}_f + \widehat{\mathbf{W}}\mathbf{r}_{k+1} \tag{43}$$

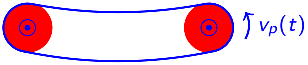
The reaction, $\mathbf{r}$, is found by solving the following second order cone complementarity problem:

$$\begin{cases} \mathbf{u}_{k+1} = \mathbf{u}_f + \widehat{\mathbf{W}}\mathbf{r}_{k+1} \\ \tilde{\mathbf{u}}_{k+1} = \mathbf{u}_{k+1} + \mu \left\| \mathbf{u}_T \right\| \mathbf{n} \\ \mathbf{K}^* \ni \tilde{\mathbf{u}}_{k+1} \perp \mathbf{r}_{k+1} \in \mathbf{K} \end{cases} \tag{44}$$

The latter is solved using the Siconos platform (INRIA):

- Block projected Gauss-Seidel
- Alternating Direction Method of Multipliers (ADMM)
- Interior Point Methods for SOCP (IPM)

Belt dynamics - 1

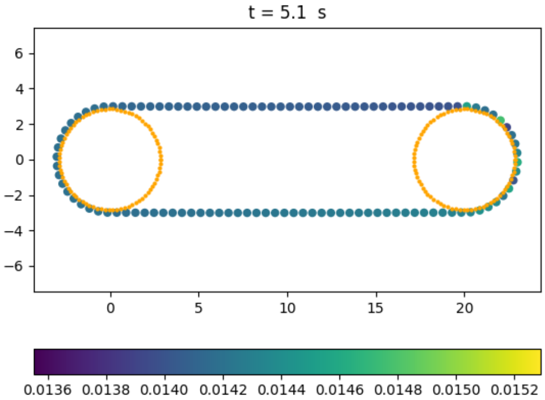


Model

- Low friction coefficient for the driven pulley
- Friction coefficient close to 1 for the drive pulley
- Mesh assembly with first and last node identical
- The velocity of the drive pulley is given
- Cylinder for the pulley
- Rayleigh damping

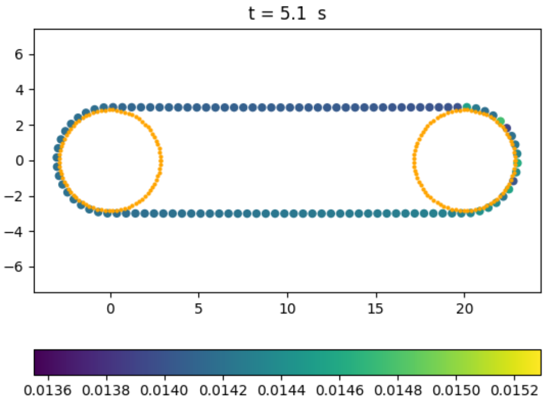
EA	L	ρ	Horizontal span	v_p	Radius
30100 N	1.2 m	0.096 kg/m	0.45 m	60 rad/s	0.05 m

Belt dynamics - 2



Strain in the belt at 5 s

Belt dynamics - 2



Frictional dynamics at the pulley cable interface

Constrained modes

The mode is seen as a vibration around an equilibrium:

$$\begin{cases} \mathbf{0} = \mathbf{M}\dfrac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \mathbf{C}\mathbf{v} + \mathbf{K}(\mathbf{q})\mathbf{q} - \mathbf{f} - (\nabla_{\mathbf{q}}\mathbf{a})^{\top}\boldsymbol{\lambda} - (\nabla_{\mathbf{q}}\mathbf{g})^{\top}\bar{\boldsymbol{\lambda}} \\ \mathbf{0} = \dfrac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} - \mathbf{v} \\ \mathbf{0} = \mathbf{a}(\mathbf{q}) \\ \mathbf{0} \leqslant \mathbf{g}(\mathbf{q}) \perp \bar{\boldsymbol{\lambda}} \geqslant \mathbf{0} \end{cases} . \tag{45}$$

The active constraints sets are denoted with $\cdot_{\mathcal{A}}$. For a given equilibrium $\mathbf{q}$, we use the following relation:

$$\begin{cases} \mathbf{0} = \mathbf{K}(\mathbf{q})\mathbf{q} - \mathbf{f} - (\nabla_{\mathbf{q}}\mathbf{a})^{\top}\boldsymbol{\lambda} - (\nabla_{\mathbf{q}}\mathbf{g})^{\top}\bar{\boldsymbol{\lambda}} \\ \mathbf{0} = \mathbf{a}(\mathbf{q}) \\ \mathbf{0} = \mathbf{g}_{\mathcal{A}}(\mathbf{q}) \quad \text{and} \quad \mathbf{0} = \bar{\boldsymbol{\lambda}}_{\bar{\mathcal{A}}} \end{cases} \tag{46}$$

where we assume that:

$$\mathbf{g}(\mathbf{q}) = \begin{bmatrix} \mathbf{g}_{\mathcal{A}}(\mathbf{q}) \\ \mathbf{g}_{\bar{\mathcal{A}}}(\mathbf{q}) \end{bmatrix} \quad ; \quad \bar{\boldsymbol{\lambda}} = \begin{bmatrix} \bar{\boldsymbol{\lambda}}_{\mathcal{A}} \\ \bar{\boldsymbol{\lambda}}_{\bar{\mathcal{A}}} \end{bmatrix} \tag{47}$$

An incremental dynamics around the latter is written as:

$$\begin{cases} \mathbf{0} = \mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}(\mathbf{q}+\mathbf{u})(\mathbf{q}+\mathbf{u}) - \mathbf{f} - (\nabla_{\mathbf{q}}\mathbf{a})^{\top}(\boldsymbol{\lambda}_{\mathbf{q}} + \boldsymbol{\lambda}_{\mathbf{u}}) - (\nabla_{\mathbf{q}}\mathbf{g})^{\top}(\bar{\boldsymbol{\lambda}}_{\mathbf{q}} + \bar{\boldsymbol{\lambda}}_{\mathbf{u}}) \\ \mathbf{0} = \mathbf{a}(\mathbf{q}+\mathbf{u}) \\ \mathbf{0} = \mathbf{g}_{\mathcal{A}}(\mathbf{q}+\mathbf{u}) \quad \text{and} \quad \mathbf{0} = \bar{\boldsymbol{\lambda}}_{\bar{\mathcal{A}}} \end{cases} \tag{48}$$

With projection method [3] we enforce the dynamics to satisfy the constraints as:

$$\mathbf{0} = \mathbf{P}^{\top}\mathbf{Q}^{\top}\mathbf{M}\mathbf{Q}\mathbf{P}\ddot{\tilde{\mathbf{u}}} + \mathbf{P}^{\top}\mathbf{Q}^{\top}\Delta\mathbf{K}(\mathbf{q})\mathbf{Q}\mathbf{P}\tilde{\mathbf{u}} - \mathbf{P}^{\top}\mathbf{Q}^{\top}(\nabla_{\mathbf{q}}\mathbf{a})^{\top}\boldsymbol{\lambda}_{\mathbf{u}} - \mathbf{P}^{\top}\mathbf{Q}^{\top}(\nabla_{\mathbf{q}}\mathbf{g})^{\top}\bar{\boldsymbol{\lambda}}_{\mathbf{u}} \, . \tag{49}$$

which simplifies as:

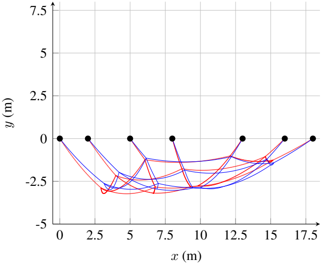
$$\left(\widetilde{\mathbf{M}}^{-1}\widetilde{\Delta\mathbf{K}}(\mathbf{q}) - \omega^2\mathbf{I}\right)\tilde{\mathbf{u}} = \mathbf{0} \, . \tag{50}$$

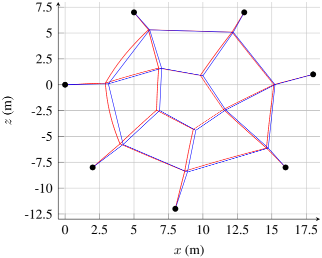
where:

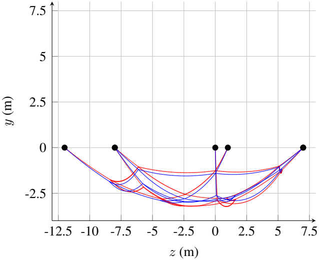
$$\begin{cases} \widetilde{\mathbf{M}} = (\mathbf{Q}\mathbf{P})^{\top}\mathbf{M}\,(\mathbf{Q}\mathbf{P}) \\ \widetilde{\Delta\mathbf{K}}(\mathbf{q}) = (\mathbf{Q}\mathbf{P})^{\top}\Delta\mathbf{K}(\mathbf{q})\,(\mathbf{Q}\mathbf{P}) \end{cases} . \tag{51}$$

[3] B. Fraejis de Veubeke, M. Gérardin, and A. Huck. *Structural dynamics*. LTAS, Liège, 1974.

Global modes examples



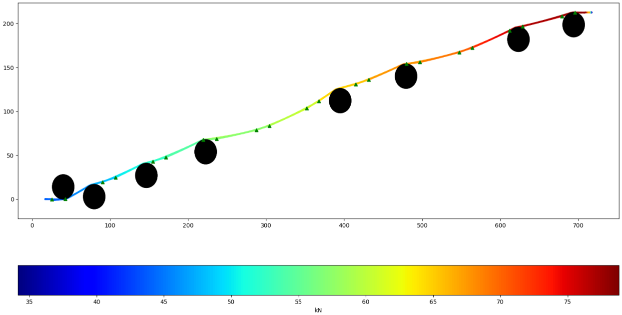




First mode of the "Spyder" web obtained with MEF

Global modes examples

Equilibrium of an aerial ropeways



One mode where vibrations happen on several spans simultaneously.

Conclusion and perspectives

Conclusion

- Lagrangian formalism for the cable:
- Unilateral elastic (tension only) FE for the cable
- Frictional contact dynamics for the constrained cable
- Global modes for constrained cable systems

Perspectives

- Convergence, existence and uniqueness of solution
- Use of higher order schemes such as nonsmooth generealized-α scheme
- Full development of inextensible with linear and nonlinear modes computations
- Question of impulsive forces (percussions) in elastic systems.

- Integration of roller batteries as rigid MBS for supports
  Photo credit: LEITNER

Thanks for your attention

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