Time-integration methods for nonsmooth contact dynamics : beyond the seminal Moreau-Jean scheme

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Introduction

Topic

Dynamics with unilateral contacts (one-sided inequalities)

for finite and infinite freedom dynamics.

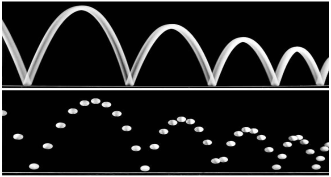
Original motivation: flexible multi-body systems and divided materials.

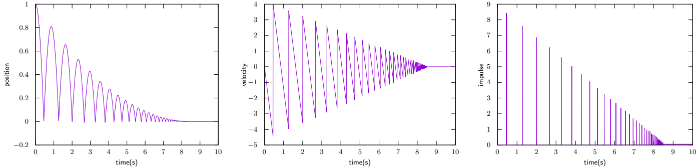
Objectives

- Dynamics and unilateral constraints imply nonsmoothness
  - velocities are discontinuous function of time
  - possibly, impulsive forces (percussions)
- Numerical time integration of nonsmooth systems must be done with care
  - Issues with standard and higher order methods
  - Moreau–Jean scheme answers to several questions
- Extensions of Moreau–Jean scheme improve order and qualitative aspects
  - Focus on nonsmooth generalized α–scheme
- An open question : nonsmoothness and percussions in elastic solids

Nonsmooth dynamical systems

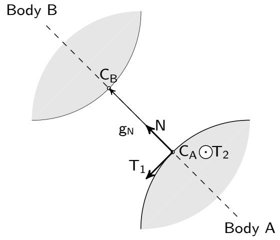
nonsmooth = lack of continuity/differentiability

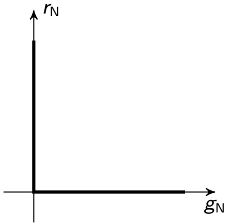




- nonsmooth solutions in time (jumps, kinks, distributions, measures)
- nonsmooth modeling and constitutive laws (set–valued mapping, inequality constraints, complementarity, impact laws)

Unilateral contact and impact





- gap function $g_N = (C_B - C_A)N$.
- reaction forces
  $r = r_N N + r_T, \quad r_N \in \mathbb{R}$ and $r_T \in \mathbb{R}^2$.
- Signorini condition at position level

$$
\begin{array}{c}
0 \leq g_N \perp r_N \geq 0 \\
\Updownarrow \\
-r_N \in N_{\mathbb{R}_+}(g_N) \\
\Updownarrow \\
r_N = \mathrm{proj}_{\mathbb{R}_+}(r_N - \rho g_n)
\end{array}
$$

- relative velocity
  $u = u_N N + u_T, \quad u_N \in \mathbb{R}$ and $u_T \in \mathbb{R}^2$.
- Signorini condition at velocity level

$$
\begin{cases}
0 \leq u_N \perp r_N \geq 0 & \text{if } g_N \leq 0 \\
r_N = 0 & \text{otherwise.}
\end{cases}
$$

- Impact law if needed $u_N^+ = -e\, u_N^-$
  e is the coefficient of restitution.

Multiple constraints

- $q \in \boldsymbol{R}^n$ coordinates that describes the state of the system in finite-dimension
- Notion of admissible set $C(t)$

$$\mathcal{C}(t) = \{q \in \boldsymbol{R}^n, g_\alpha(q,t) \geq 0, \alpha \in \{1 \ldots \nu\}\}$$

- Normal cone inclusion

$$-r \in \mathrm{N}_{\mathcal{C}(t)}(q)$$

where the normal cone to $C(t)$

$$\mathrm{N}_{\mathcal{C}(t)}(q) = \{y \mid y = -\nabla_q g(q,t)\,\lambda, \quad 0 \leq g_\alpha(q,t) \perp \lambda_\alpha \geq 0\}$$

- Coulomb's friction : Second-Order Cone Complementarity condition

$$K^\star \ni \hat{u} \perp r \in K \Longleftrightarrow -r \in N_{K^\star}(\hat{u}) \qquad (1)$$

with $K = \{r \in \boldsymbol{R}^3 \mid \|r_\mathrm{T}\| \leq \mu r_n\}$ and $\hat{u} = u + \mu \|u_\mathrm{T}\| \mathrm{N}$.

Nonsmooth dynamics in finite dimension.

Space–discretized equations (by FEM, for instance)
or discrete mechanical systems (rigid bodies, linkages, . . .)

Definition (Perfect unilateral constraints on the smooth dynamics)

$$\begin{cases} \dot{q} = v \\ M(q)\dfrac{dv}{dt} + F(t,q,v) = r \\ -r \in N_{\mathcal{C}(t)}(q(t)) \end{cases} \tag{2}$$

where r it the generalized force or generalized reaction due to the constraints.

Remark

- Second order differential inclusion.
- The unilateral constraints are said to be perfect due to the normality condition.

Nonsmooth dynamics in finite dimension.

Fundamental assumptions.

- The velocity $v = \dot{q}$ is of Bounded Variations (B.V)
  ➜ The equation are written in terms of a right continuous B.V. (R.C.B.V.) function, v^+ such that
  $$v^+ = \dot{q}^+ \tag{3}$$
- q is related to this velocity by
  $$q(t) = q(t_0) + \int_{t_0}^{t} v^+(t)\, dt \tag{4}$$
- The acceleration, ($\ddot{q}$ in the usual sense) is hence a differential measure dv associated with v such that
  $$dv(]a,b]) = \int_{]a,b]} dv = v^+(b) - v^+(a) \tag{5}$$

Nonsmooth dynamics in finite dimension.

Definition (Nonsmooth Lagrangian Dynamics)

$$\begin{cases} M(q)dv + F(t, q, v^+)dt = di \\ v^+ = \dot{q}^+ \end{cases} \tag{6}$$

where di is the reaction measure and dt is the Lebesgue measure.

Remarks

- The nonsmooth Dynamics contains the impact equations and the smooth evolution in a single equation.
- The formulation allows one to take into account very complex behaviors, especially, finite accumulation (Zeno-state).
- This formulation is sound from a mathematical Analysis point of view.

References: [19, 20, 15, 16]

Nonsmooth dynamics in finite dimension.

Measures Decomposition

Densities w.r.t. Lebesgue measure dt

$$\frac{dv}{dt} = \gamma = \ddot{q} \quad \text{acceleration defined in the usual sense a.e.}$$

$$\frac{di}{dt} = f \quad \text{Lebesgue measurable force}$$

Densities w.r.t. purely atomic measure $d\nu = \sum_i \delta_{t_i}$

$d\nu$ is a purely atomic measure concentrated at the instants t_i of discontinuities of v

$$\frac{dv}{d\nu} = v^+ - v^- \quad \text{velocity jump}$$

$$\frac{di}{d\nu} = p \quad \text{percussion (impulsive force)}$$

Impact equations and non impulsive dynamics

Using the densities in the nonsmooth Lagrangian Dynamics, one obtains

Definition (Impact equations at any time t_i)

$$M(q)(v^+ - v^-)d\nu = pd\nu, \tag{7}$$

or, equivalently,

$$M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \tag{8}$$

Definition (Smooth Dynamics almost everywhere)

$$M(q)\gamma dt + F(t, q, v)dt = fdt \tag{9}$$

or

$$M(q)\gamma^+ + F(t, q, v^+) \quad = \quad f^+ \quad [dt - a.e.] \tag{10}$$

The Moreau's sweeping process of second order

Definition (Moreau's sweeping process of second order [15, 16])

The inclusion (2) is "replaced" by the following inclusion

$$\begin{cases} M(q)dv + F(t,q,v^+)dt = di \\ v^+ = \dot{q}^+ \\ -di \in N_{T_C(q)}(v^+ + ev^-) \end{cases} \tag{11}$$

Inclusion at the velocity level of the measure

A key stone of this formulation is the inclusion in terms of velocity. For $C = \boldsymbol{R}_+$,

$$-di \in N_{T_{\boldsymbol{R}_+}(q)}(v^+ + ev^-) \Longleftrightarrow \text{if } q \leq 0, \text{ then } 0 \leq v^+ + ev^- \perp di \geq 0$$

➜ Foundation for the Moreau–Jean time–stepping approach.

Mathematical results

Finite dimension

- Counter example to uniqueness with $\mathcal{C}^\infty$ data (Schatzman, Percivale)
- Existence and uniqueness in the frictionless case with analytic data (Ballard[3])
- Frictional case.
  - No result in the general case
  - Existence and uniqueness with lumped mass system

Infinite dimension. Elastodynamics (or elasto-plastic) dynamics.

The situation if far more complex. See the discussion at the end of the talk.

How to perform numerical time integration ?

1 - Smoothing dynamics

- Regularization techniques with penalty paramater
- Advanced and sophisticated methods for FEM discretizations.
  - Singular mass method and mass redistribution method.
    Renard, Laborde and co-workers [14, 18]
  - Nitsche's method.
    Wriggers, Zavarise, Chouly, Hild, Renard among others [23, 11, 12]

⊕ convergence proof

⊕ enable the use of standard time discretization methods.

⊖ numerical stiff ordinary differential equations.

⊖ spurious oscillations of contact forces, and then stresses.

⊖ issue with inelastic impacts.

How to perform numerical time integration ?

2 - Event detecting schemes (Event-driven)

- accurate time detection of events
- standard smooth numerical time integration methods.

⊕ higher order integration of free flight motions

⊖ sensibility to numerical thresholds

⊖ reformulation of constraints at higher kinematic levels.

⊖ unable to deal with finite accumulation, or very large number of events.

How to perform numerical time integration ?

3 - Event-capturing schemes (a.k.a. time stepping schemes)

- ⊕ robust, stable and proof of convergence
- ⊕ low kinematic level for the constraints
- ⊕ able to deal with finite accumulation
- ⊖ very low order of accuracy even in free flight motions
- ⊖ drift of the constraints at position, velocity or acceleration levels.

Main schemes

- Moreau–Jean
- Schatzman–Paoli

Principle of nonsmooth event capturing methods (Time–stepping schemes)

1. A unique formulation of the dynamics is considered. For instance, a dynamics in terms of measures.

$$\begin{cases} -mdv + fdt = di \\ \dot{q} = v^+ \\ 0 \leq di \perp v^+ \geq 0 \text{ if } q \leq 0 \end{cases} \tag{12}$$

2. The time-integration is based on a consistent approximation of the equations in terms of measures. For instance,

$$\int_{]t_k,t_{k+1}]} dv = \int_{]t_k,t_{k+1}]} dv = (v^+(t_{k+1}) - v^+(t_k)) \approx (v_{k+1} - v_k) \tag{13}$$

3. Consistent approximation of measure inclusion.

$$0 \leq di \perp v^+ \geq 0 \text{ if } q \leq 0 \quad \rightarrow \quad \begin{cases} p_{k+1} \approx \displaystyle\int_{]t_k,t_{k+1}]} di \\ 0 \leq p_{k+1} \perp v_{k+1} \geq 0 \quad \text{if } \tilde{q}_k \leq 0 \end{cases} \tag{14}$$

Moreau–Jean's Time stepping scheme [16, 13]

Principle

$$\begin{cases} M(q_{k+\theta})(v_{k+1}-v_k) - hF_{k+\theta} = p_{k+1} = G(q_{k+\theta})P_{k+1}, & (15a)\\ q_{k+1} = q_k + hv_{k+\theta}, & (15b)\\ u_{k+1} = G^T(q_{k+\theta})\, v_{k+1} & (15c)\\ \begin{array}{lll} 0 \leq u^\alpha_{k+1} + eU^\alpha_k \perp P^\alpha_{k+1} \geq 0 & \text{if} & \bar{g}^\alpha_{k,\gamma} \leq 0 \\ P^\alpha_{k+1} = 0 & \text{otherwise} & \end{array} . & (15d) \end{cases}$$

with

- $G(q) = \nabla_q g(q)$
- $\theta \in [0,1]$
- $x_{k+\theta} = (1-\theta)x_{k+1} + \theta x_k$
- $F_{k+\theta} = F(t_{k+\theta}, q_{k+\theta}, v_{k+\theta})$
- $\bar{g}_{k,\gamma} = g_k + \gamma h u_k, , \gamma \geq 0$ is a prediction of the constraints.

Schatzman–Paoli's Time stepping scheme [17]

Principle

$$\begin{cases} M(q_{k+1})(q_{k+1} - 2q_k + q_{k-1}) - h^2 F_{k+\theta} = p_{k+1}, & \text{(16a)} \\ v_{k+1} = \dfrac{q_{k+1} - q_{k-1}}{2h}, & \text{(16b)} \\ -p_{k+1} \in N_K\left(\dfrac{q_{k+1} + e q_{k-1}}{1+e}\right), & \text{(16c)} \end{cases}$$

where N_K defined the normal cone to K.
For $K = \{q \in \mathbb{R}^n, y = g(q) \geq 0\}$

$$0 \leq g\left(\frac{q_{k+1} + e q_{k-1}}{1+e}\right) \perp \nabla g\left(\frac{q_{k+1} + e q_{k-1}}{1+e}\right) P_{k+1} \geq 0 \tag{17}$$

But ...

Both schemes are

- quite inaccurate (at most first order)
- "conserve" or "dissipate" a lot of energy

This is a consequence of the first order approximation of the smooth forces term F

Recent improvements

- Time discontinuous Galerkin methods, with T. Schindler et al. [21, 22]
- Stabilized index-2 formulation [2, 1]
- Stabilized index-1 formulation, with O. Brüls and A. Cardona [5]
- Time finite element method and variational integrators. G. Cappobianco, S. Eugster et al.[7, 6]
- Nonsmooth generalized–α schemes, with O. Brüls, A. Cardona et al. [10, 4, 8]

Nonsmooth generalized-α schemes

- Direct application of Newmark scheme or generalized α-scheme
- Splitting the dynamics between smooth and nonsmooth part
- GGL stabilization
- Numerical illustrations

Direct application of Newmark scheme or generalized α-scheme

Inconsistent results in discrete time with a direct application

- Contact condition at velocity level is wrong
- Pseudo-artificial impact law.
- Contact stresses are polluted with spurious oscillations.

Do no direclty apply generalized α-scheme with unilateral contact !!

The nonsmooth generalized α scheme

Splitting the dynamics between smooth and nonsmooth part

$$\mathrm{d}\mathbf{w} = \mathrm{d}\mathbf{v} - \dot{\tilde{\mathbf{v}}}\,\mathrm{d}t \tag{18}$$

Smooth (non-impulsive) part

Solutions of the following DAE

$$\begin{aligned}
\dot{\tilde{\mathbf{q}}} &= \tilde{\mathbf{v}} && (19a)\\
\mathbf{M}(\mathbf{q})\,\dot{\tilde{\mathbf{v}}} - \mathbf{g}_{\mathbf{q}}^{T}(\mathbf{q})\,\tilde{\boldsymbol{\lambda}} &= \mathbf{f}(\mathbf{q}, \mathbf{v}, t) && (19b)\\
\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}(\mathbf{q})\,\tilde{\mathbf{v}} &= \mathbf{0} && (19c)\\
\tilde{\boldsymbol{\lambda}}^{\mathcal{U}} &= \mathbf{0} && (19d)
\end{aligned}$$

with the initial value $\tilde{\mathbf{v}}(t_n) = \mathbf{v}(t_n)$, $\tilde{\mathbf{q}}(t_n) = \mathbf{q}(t_n)$.

The nonsmooth generalized α scheme

Splitting the dynamics between smooth and nonsmooth part

$$\dot{\mathbf{q}} = \mathbf{v} \tag{20a}$$

$$\mathrm{d}\mathbf{v} = \mathrm{d}\mathbf{w} + \dot{\tilde{\mathbf{v}}}\,\mathrm{d}t \tag{20b}$$

$$\mathbf{M}(\mathbf{q})\,\dot{\tilde{\mathbf{v}}} - \mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}},T}\,\tilde{\boldsymbol{\lambda}}^{\overline{\mathcal{U}}} = \mathbf{f}(\mathbf{q},\mathbf{v},t) \tag{20c}$$

$$\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}\,\tilde{\mathbf{v}} = \mathbf{0} \tag{20d}$$

$$\tilde{\boldsymbol{\lambda}}^{\mathcal{U}} = \mathbf{0} \tag{20e}$$

$$\mathbf{M}(\mathbf{q})\,\mathrm{d}\mathbf{w} - \mathbf{g}_{\mathbf{q}}^{T}\,(\mathrm{d}\mathbf{i} - \tilde{\boldsymbol{\lambda}}\,\mathrm{d}t) = \mathbf{0} \tag{20f}$$

$$\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}\mathbf{v} = \mathbf{0} \tag{20g}$$

$$\text{if } g^j(\mathbf{q}) \leq 0 \text{ then } 0 \leq g_{\mathbf{q}}^j\,\mathbf{v} + e\,g_{\mathbf{q}}^j\,\mathbf{v}^- \perp \mathrm{d}i^j \geq 0, \quad \forall j \in \mathcal{U} \tag{20h}$$

The nonsmooth generalized α scheme

GGL approach to stabilize the constraints at the position level

The equations of motion become

$$\begin{aligned}
\mathbf{M}(\mathbf{q})\,\dot{\mathbf{q}} - \mathbf{g}_{\mathbf{q}}^T\,\boldsymbol{\mu} &= \mathbf{M}(\mathbf{q})\,\mathbf{v} && (21a)\\
\xcancel{\dot{\mathbf{q}} = \mathbf{v}} \rightarrow \qquad \mathbf{g}^{\overline{\mathcal{U}}}(\mathbf{q}) &= \mathbf{0} && (21b)\\
\mathbf{0} \leq \mathbf{g}^{\mathcal{U}}(\mathbf{q}) &\perp \boldsymbol{\mu}^{\mathcal{U}} \geq \mathbf{0} && (21c)\\
\mathrm{d}\mathbf{v} &= \mathrm{d}\mathbf{w} + \dot{\tilde{\mathbf{v}}}\,\mathrm{d}t && (21d)\\
\mathbf{M}(\mathbf{q})\,\dot{\tilde{\mathbf{v}}} - \mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}},T}\,\tilde{\boldsymbol{\lambda}}^{\overline{\mathcal{U}}} &= \mathbf{f}(\mathbf{q},\mathbf{v},t) && (21e)\\
\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}\,\tilde{\mathbf{v}} &= \mathbf{0} && (21f)\\
\tilde{\boldsymbol{\lambda}}^{\mathcal{U}} &= \mathbf{0} && (21g)\\
\mathbf{M}(\mathbf{q})\,\mathrm{d}\mathbf{w} - \mathbf{g}_{\mathbf{q}}^T\,(\mathrm{d}\mathbf{i} - \tilde{\boldsymbol{\lambda}}\,\mathrm{d}t) &= \mathbf{0} && (21h)\\
\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}\mathbf{v} &= \mathbf{0} && (21i)\\
\text{if } g^j(\mathbf{q}) \leq 0 \text{ then } 0 \leq g_{\mathbf{q}}^j\,\mathbf{v} + e\,g_{\mathbf{q}}^j\,\mathbf{v}^- &\perp \mathrm{d}i^j \geq 0, \quad \forall j \in \mathcal{U} && (21j)
\end{aligned}$$

The nonsmooth generalized α scheme

Velocity jumps and position correction

The multipliers $\mathbf{\Lambda}(t_n;t)$ and $\boldsymbol{\nu}(t_n;t)$ are defined as

$$\mathbf{\Lambda}(t_n;t) = \int_{(t_n,t]} (\mathrm{d}\mathbf{i} - \tilde{\boldsymbol{\lambda}}(\tau)\,\mathrm{d}\tau) \tag{22a}$$

$$\boldsymbol{\nu}(t_n;t) = \int_{t_n}^{t} (\boldsymbol{\mu}(\tau) + \mathbf{\Lambda}(t_n;\tau))\,\mathrm{d}\tau \tag{22b}$$

with $\mathbf{\Lambda}(t_n;t_n) = \boldsymbol{\nu}(t_n;t_n) = \mathbf{0}$.
The velocity jump and position correction variables

$$\mathbf{W}(t_n;t) = \int_{(t_n,t]} \mathrm{d}\mathbf{w} = \mathbf{v}(t) - \tilde{\mathbf{v}}(t) \tag{23a}$$

$$\mathbf{U}(t_n;t) = \int_{t_n}^{t} (\dot{\mathbf{q}} - \tilde{\mathbf{v}})\mathrm{d}t = \mathbf{q}(t) - \tilde{\mathbf{q}}(t) \tag{23b}$$

➜ Low-order approximation of impulsive terms.
➜ Higher–order approximation of non impulsive terms.

The nonsmooth generalized α scheme

$$\mathbf{M}(\mathbf{q}_{n+1})\mathbf{U}_{n+1} - \mathbf{g}^{T}_{\mathbf{q},n+1}\,\boldsymbol{\nu}_{n+1} = \mathbf{0} \tag{24a}$$

$$\mathbf{g}^{\overline{\mathcal{U}}}(\mathbf{q}_{n+1}) = \mathbf{0} \tag{24b}$$

$$\mathbf{0} \leq \mathbf{g}^{\mathcal{U}}(\mathbf{q}_{n+1}) \perp \boldsymbol{\nu}^{\mathcal{U}}_{n+1} \geq \mathbf{0} \tag{24c}$$

$$\mathbf{M}(\mathbf{q}_{n+1})\dot{\tilde{\mathbf{v}}}_{n+1} - \mathbf{f}(\mathbf{q}_{n+1}, \mathbf{v}_{n+1}, t_{n+1}) - \mathbf{g}^{\overline{\mathcal{U}},T}_{\mathbf{q},n+1}\,\tilde{\boldsymbol{\lambda}}^{\overline{\mathcal{U}}}_{n+1} = \mathbf{0} \tag{24d}$$

$$\mathbf{g}^{\overline{\mathcal{U}}}_{\mathbf{q},n+1}\,\tilde{\mathbf{v}}_{n+1} = \mathbf{0} \tag{24e}$$

$$\mathbf{M}(\mathbf{q}_{n+1})\mathbf{W}_{n+1} - \mathbf{g}^{T}_{\mathbf{q},n+1}\boldsymbol{\Lambda}_{n+1} = \mathbf{0} \tag{24f}$$

$$\mathbf{g}^{\overline{\mathcal{U}}}_{\mathbf{q},n+1}\mathbf{v}_{n+1} = \mathbf{0} \tag{24g}$$

$$\text{if } g^{j}(\mathbf{q}^{*}_{n+1}) \leq 0 \text{ then } 0 \leq g^{j}_{\mathbf{q},n+1}\,\mathbf{v}_{n+1} + e\,g^{j}_{\mathbf{q},n}\,\mathbf{v}_{n} \perp \Lambda^{j}_{n+1} \geq 0, \forall j \in \mathcal{U}$$

The nonsmooth generalized α scheme

Nonsmooth generalized α-scheme

$$\begin{aligned}
\tilde{\mathbf{q}}_{n+1} &= \mathbf{q}_n + h\mathbf{v}_n + h^2(0.5-\beta)\mathbf{a}_n + h^2\beta\mathbf{a}_{n+1} && \text{(25a)}\\
\mathbf{q}_{n+1} &= \tilde{\mathbf{q}}_{n+1} + \mathbf{U}_{n+1} && \text{(25b)}\\
\tilde{\mathbf{v}}_{n+1} &= \mathbf{v}_n + h(1-\gamma)\mathbf{a}_n + h\gamma\mathbf{a}_{n+1} && \text{(25c)}\\
\mathbf{v}_{n+1} &= \tilde{\mathbf{v}}_{n+1} + \mathbf{W}_{n+1} && \text{(25d)}\\
(1-\alpha_m)\mathbf{a}_{n+1} + \alpha_m\mathbf{a}_n &= (1-\alpha_f)\dot{\tilde{\mathbf{v}}}_{n+1} + \alpha_f\dot{\tilde{\mathbf{v}}}_n && \text{(25e)}
\end{aligned}$$

Special cases

- $\alpha_m = \alpha_f = 0$ ➜ Nonsmooth Newmark
- $\alpha_m = 0, \alpha_f \in [0, 1/3]$ ➜ Nonsmooth Hilber-Hughes–Taylor (HHT)

Spectral radius at infinity $\rho_\infty \in [0, 1]$

$$\alpha_m = \frac{2\rho_\infty - 1}{\rho_\infty + 1}, \quad \alpha_f = \frac{\rho_\infty}{\rho_\infty + 1}, \quad \beta = \frac{1}{4}(\gamma + \frac{1}{2})^2. \qquad (26)$$

The nonsmooth generalized α scheme

Observed properties on examples

- the scheme is consistent and globally of order one.
- the scheme seems to share the stability property as the original HHT
- the scheme dissipates energy only in high-frequency oscillations (w.r.t the time–step.)

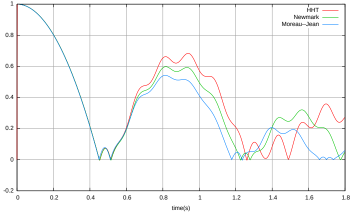
Discrete energy balance (proof in [2])

- For the Moreau–Jean, a simple variant allows us to obtain a scheme which always dissipates energy.
- For the Newmark and the HHT scheme with retrieve the dissipation properties as the smooth case. The term associated with impact is added in the balance.

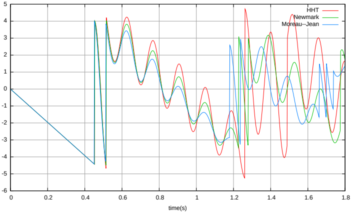
➜ nonsmooth generalized-α-schemes are stable with a controllable dissipation of mechanical energy and satisfies kinematic constraints at various levels

Numerical Illustrations

Two ball oscillator with impact.

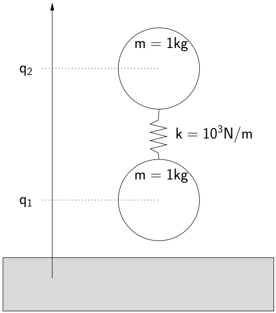


Position of the first ball



Velocity of the first ball

Time–step : $h = 2e-3$.
Moreau ($\theta = 1.0$).
Newmark ($\gamma = 1.0, \beta = 0.5, \alpha_m = \alpha_f = 0$).
HHT ($\gamma = 1.0, \beta = 0.5, \alpha_f = 0.1, \alpha_m = 0$)



Numerical Illustrations

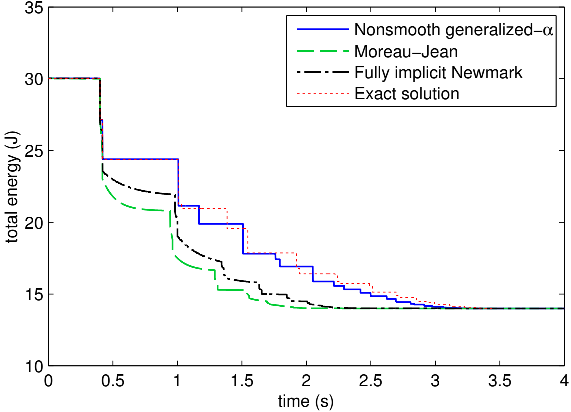
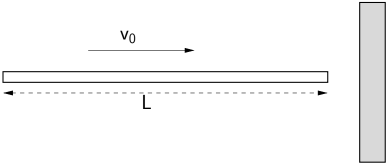


Figure 7. Numerical results for the total energy of the bouncing oscillator.

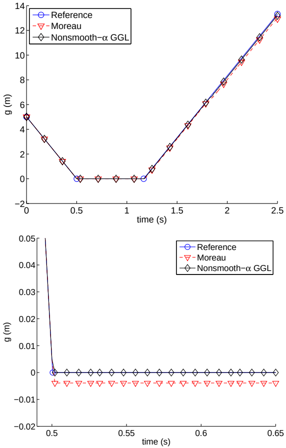
Numerical Illustrations

Impacting elastic bar



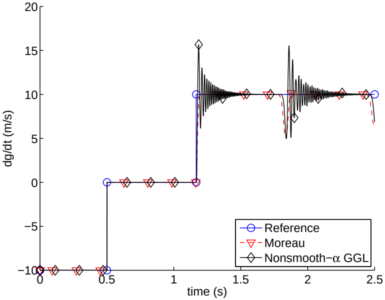
$g_3(\mathbf{q}) = x_1 \geq 0$
$e = 0.0$
200 finite elements
Time–step : $h = 2e - 3$.
Moreau ($\theta = 1/1.8$).
α-schemes ($\rho_\infty = 0.8$)

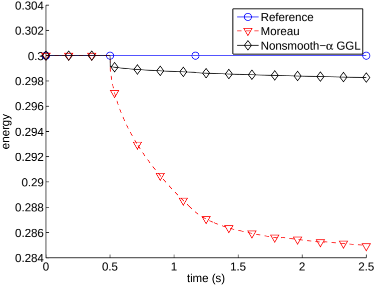
Unilateral constraint



Numerical Illustrations

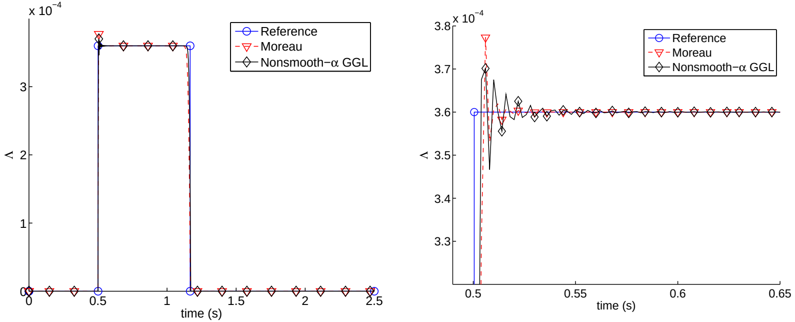
Impacting elastic bar





Numerical Illustrations

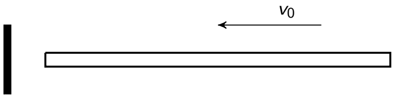
Impacting elastic bar



Open discussion: Nonsmoothness and percussions in continuum media

A modeling problem

Timoshenko's solutions



Standard assumptions in continuum mechanics

- Unilateral contact in elastic solids generates traveling surfaces (manifold of co-dimension 1) of velocity discontinuities

$$\mathcal{S} = \{x, t \in \Omega \times [0, T] \mid \boldsymbol{v}^+(x,t) \neq \boldsymbol{v}^-(x,t)\}$$

- Discontinuity surfaces have no mass (negligible sets with respect to the mass measure (no discrete mass))
  Theory of shock waves in continuum media. Germain et al.
- The conservation equation and principles of virtual power implies that is no impulsive stresses or forces. Virtual power of inertial forces at time of jump t

$$\int_\Omega \frac{\boldsymbol{v}^{\star,+}(x,t) + \boldsymbol{v}^{\star,-}(x,t)}{2} \cdot (\boldsymbol{v}^+(x,t) - \boldsymbol{v}^-(x,t))\mathrm{d}m = 0$$

Mathematical results

Few generic mathematical results

- 1D, 2D, or 3D elastic half-spaces, or tubes:
  - Existence and uniqueness obtained by Lebeau and Schatzman
  - Conservation of energy is obtained (and not imposed).
  - No impact law in the models.

General implicit assumption

No impulsive forces and no need of impact law for closing the system.

Consequences

- Impact law in space-discretized systems should not change the results at convergence in time and space
- Coefficient of restitution acts as a numerical parameter that is convenient in discrete time and space.

Mathematical results

but, because there is a but

Quote from the Lebeau and Schatzman's article :

> There seems to be a great difference between unilateral constraints on an open subset of Ω and on a submanifold of lower dimension of Ω.

$\Omega = \{x \in \boldsymbol{R}^d, x_d > 0\}$ is the ambient half-space of study

Mathematical results

Submanifold of lower dimension : strings, beams, plates and shells

The solutions are far more complicated and raise modeling issues.

- Strings (1D wave equation) (Cabannes, Schatzman et al.)
  - punctual obstacles: no impulsive forces
  - convex and concave obstacle : possible impulse force and the need for an impact law or a conservation of energy rule
- Beams
  - Paoli and Shillor: Existence of solutions for a clamped beam with two stop at free end.
  - Reaction forces are impulsive.
  - no uniqueness : an ingredient is missing ?
- distributed discrete masses

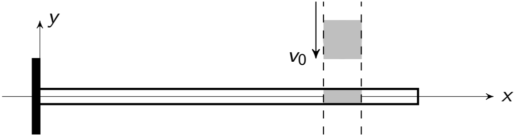
Consequences

- An impact law, of conservation law, may be needed in the model.
- Coefficient of restitution is no longer a numerical parameter , and should influence the results at convergence.

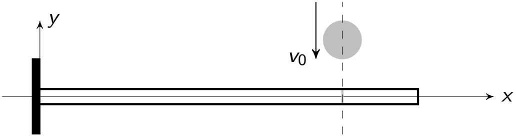
Some illustrations

Impact on a cantilever beam

- surface contact



- punctual contact



Chatterjee's work [9]

Response of infinite beam to a punctual impulse by Fourier transform
Elastic infinite beams show velocity discontinuity for the whole beam at initial time .
(infinite speed of bending wave)

$$-u_{tt} + \alpha u_{xxxx} = p$$

Conclusions

- Space discretized systems are finite dimensional systems :
  - Moreau-Jean scheme deals consistently with nonsmoothness
  - impact law or velocity level constraints is a good stabilization technique (index-reduction)
  - coefficient of restitution is a useful numerical parameter
- Nonsmooth generalized α-scheme extends all the good properties of original schemes (both Moreau–Jean and generalized α)
- If the continuous model needs for an impact law, the discrete scheme can take into account this feature, otherwise it enables stabilization of the constraints.
- For multi-body systems (flexible+rigid+joints+contact+friction), we have a monolithic scheme that is consistent.

Thank you for your attention.

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