Time-Integration methods used for nonsmooth contact dynamics with friction and impact: from the Moreau-Jean integration scheme to nonsmooth $\alpha-$generalized methods.

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Objectives & Motivations

1. Nonsmooth dynamical systems in the large :
   - What is a nonsmooth dynamical system?
     Focus on mechanical systems with contact and friction
   - Why using the nonsmooth modeling framework?
     Why not regularizing?
   - Archetypal example : the bouncing ball (a.k.a. the ping pong ball)
2. Time integration scheme for nonsmooth dynamical systems
   - Formulation of nonsmooth Lagrangian systems
   - Numerical time integration: Event-driven and time-stepping schemes
   - Principles of Moreau–Jean Time–stepping schemes
   - Extension to Nonsmooth Generalized-α schemes.
3. Among possible applications in Geotechnics
   - Rolling friction and fracture for rock-fall trajectory
   - Rock interaction with elasto-plastic obstacles
   - Debris flows with rigid objects and obstacles
   - High performance computing

What is a NonSmooth Dynamical System (NSDS) ?

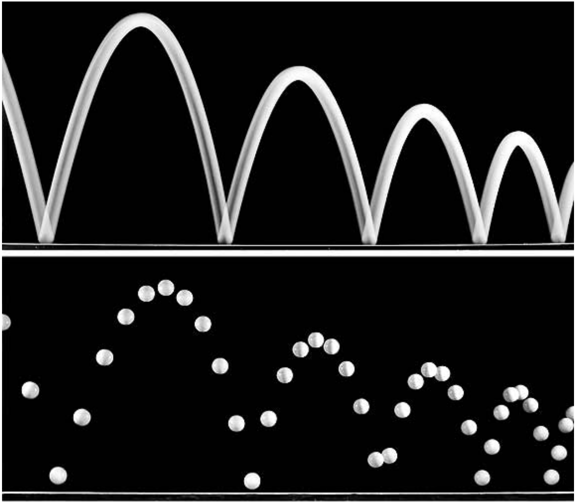
Nonsmooth

- Graphs with kinks, jumps, peaks

Dynamics

- The branch of mechanics that is concerned with the effects of forces on the motion of objects.
- Systems that evolves with time.

What is a Non Smooth Dynamical System (NSDS) ?



What is a NonSmooth Dynamical System (NSDS) ?

A NSDS is a dynamical system characterized by two correlated features:

- a nonsmooth evolution with the respect to time:
  - jumps in the state and/or in its derivatives w.r.t. time
  - generalized solutions (distributions)
- a set of non smooth laws (Generalized equations, inclusions) constraining the state x

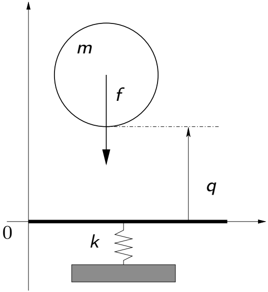
It is a modeling assumption based on two separate time–scales in the evolution of a dynamical system:

1. a small time scale where abrupt changes are located (*e.g.* impacting times)
2. a large time scale where the evolution is slower (*e.g.* free flight motion)

Remarks

It may be the result of a idealization or a passage to the limit.
Similar the continuum Mechanics modeling assumption.

A famous nonsmooth dynamical system: the bouncing ball



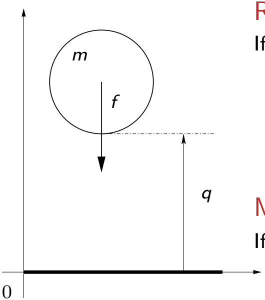
Compliant model

$$m\ddot{q}(t) = f(t) + \lambda = f(t) + \begin{cases} -kq(t) & \text{if} \quad q(t) < 0 \\ 0 & \text{if} \quad q(t) \geq 0 \end{cases} \tag{1}$$

Complementarity formulation

$$\lambda = \begin{cases} -kq & \text{if} \quad q < 0 \\ 0 & \text{if} \quad q \geq 0 \end{cases} \Longleftrightarrow 0 \leq \lambda \perp \lambda + kq \geq 0 \tag{2}$$

A famous nonsmooth dynamical system: the bouncing ball



Rigid limit

If we let $k \to +\infty$ (rigid contact with restitution) we get

$$\begin{cases} m\ddot{q}(t) = f(t) + \lambda(t) \\ 0 \leq q(t) \perp \lambda(t) \geq 0 \end{cases} \tag{3}$$

Mandatory impact law (for discrete systems)

If $\dot{q}(t^-) < 0$ and $q(t) = 0$

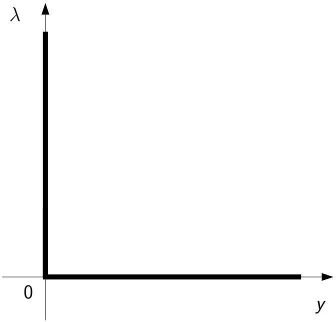
$$\dot{q}(t^+) = -e\dot{q}(t^-) \tag{4}$$

Therefore we pass from a piecewise linear system to a complementarity system

What do we gain doing so (compliance replaced by rigidity)?

Complementarity condition

Signorini's condition in contact mechanics



$$0 \leq y \quad \perp \quad \lambda \geq 0 \tag{5}$$
$$\Updownarrow$$
$$-y \quad \in \quad N_{\boldsymbol{R}_+}(\lambda) \tag{6}$$
$$\Updownarrow$$
$$-\lambda \quad \in \quad N_{\boldsymbol{R}_+}(y) \tag{7}$$
$$\Updownarrow$$
$$\lambda^T(y' - y) \quad \geq \quad 0, \text{ for all } y' \in \boldsymbol{R}_+ \tag{8}$$
$$\Updownarrow$$
$$y^T(\lambda' - \lambda) \quad \geq \quad 0, \text{ for all } \lambda' \in \boldsymbol{R}_+ \tag{9}$$

A well-know concept in Optimization

- Numerous theoretical tools (variational inequalities, complementarity problems, proximal point techniques)
- Numerous numerical tools (pivoting techniques, projected over–relaxation (Gauss–Seidel), semi-smooth Newton methods, interior point methods, ...)

Compliant vs. rigid model

Compliant model

- ⊕ Possibly more realistic models.
  - are we able to accurately know the behavior at contact (relation force/indentation) ?
  - Hertz's contact model for spheres (limited validity !) dissipation ?
- ⊖ Complex contact phenomena.
  - space and time scales are difficult to handle
  - numerous inner variables
- ⊖ Numerical implementation ostensibly more simpler, but numerous issues
  - stiff model, high frequency dynamics (most of the time non physical), stability of integrators, small time–steps, . . .
  - high sensitivity to contact parameters
  - limited smoothness : issues in order and adaptive time–step strategy

Rigid model

- ⊖ Limited description of the contact behavior
- ⊕ Modeling of threshold effects
- ⊕ Simple set of parameters with limited sensitivity
- ⊕ Stable and robust numerical implementation
  - no spurious high frequency dynamics.

Numerical simulation: Stiff problems versus complementarity

Euler discretization of the compliant system (finite k)

$$\begin{cases} \frac{\dot q_{i+1}-\dot q_i}{h} = kq_{i+1} \\ \frac{q_{i+1}-q_i}{h} = \dot q_i \end{cases} \Leftrightarrow \begin{pmatrix} \dot q_{i+1} \\ q_{i+1} \end{pmatrix} = \begin{pmatrix} kh^2+1 & kh \\ h & 1 \end{pmatrix} \begin{pmatrix} \dot q_i \\ q_i \end{pmatrix} \tag{10}$$

This problem is **stiff** because the eigenvalues γ_1 and γ_2 of $\begin{pmatrix} kh^2+1 & kh \\ h & 1 \end{pmatrix}$ satisfy $\frac{\gamma_1}{\gamma_2} \to +\infty$ when $k \to +\infty$.

$\boxed{\textit{stiff integrators}}$

Numerical simulation: Stiff problems versus complementarity

Euler discretization (Moreau's scheme) of the complementarity system (infinite k)

$$\begin{cases} \dot{q}_{i+1} - \dot{q}_i = hf_{i+1} + \lambda_{i+1} \\ 0 \leq \dot{q}_{i+1} + e\dot{q}_i \perp \lambda_{i+1} \geq 0 \\ \\ q_{i+1} = q_i + h\dot{q}_i \end{cases} \tag{11}$$

which is nothing else but solving a simple Linear complementarity systems (LCP) (or a quadratic program QP) at each step!!!

Definition (Linear Complementarity Problem (LCP))

Given $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$, the Linear Complementarity Problem, is to find a vector $z \in \mathbb{R}^n$, denoted by $\mathrm{LCP}(M, q)$ such that

$$0 \leq z \perp Mz + q \geq 0 \tag{12}$$

The inequalities have to be understood component-wise and the relation $x \perp y$ means $x^T y = 0$.

NonSmooth Multibody Systems (NSMBS)

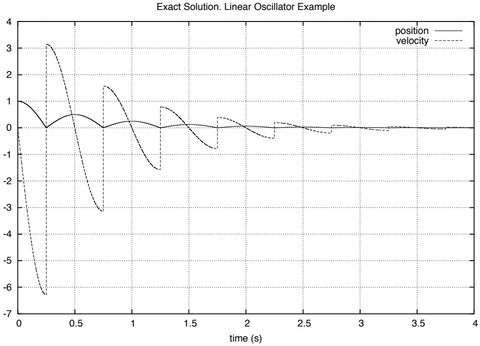


Figure: Analytical solution. Linear Oscillator

NonSmooth Multibody Systems (NSMBS)

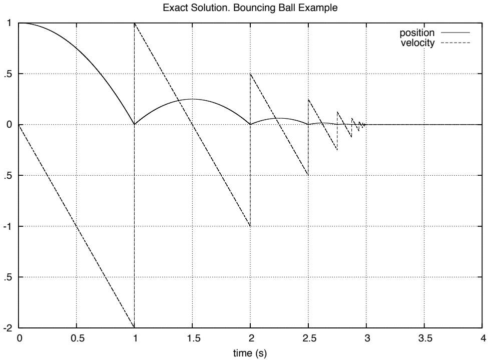
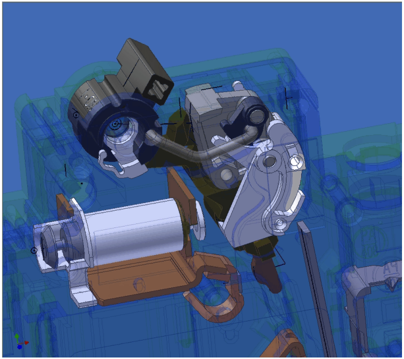


Figure: Analytical solution. Bouncing ball example

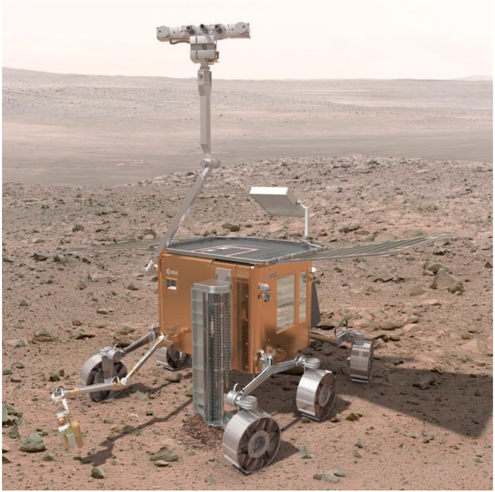
Mechanical systems with contact, impact and friction

Simulation of Circuit breakers (INRIA/Schneider Electric)



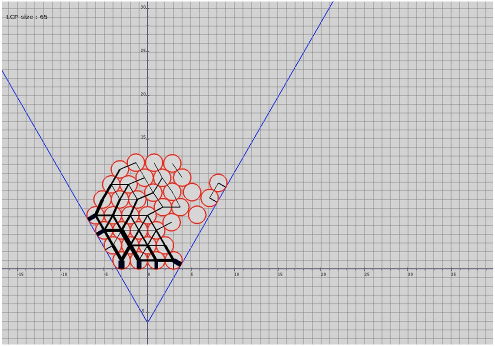
Mechanical systems with contact, impact and friction

Simulation of the ExoMars Rover (INRIA/Trasys Space/ESA)

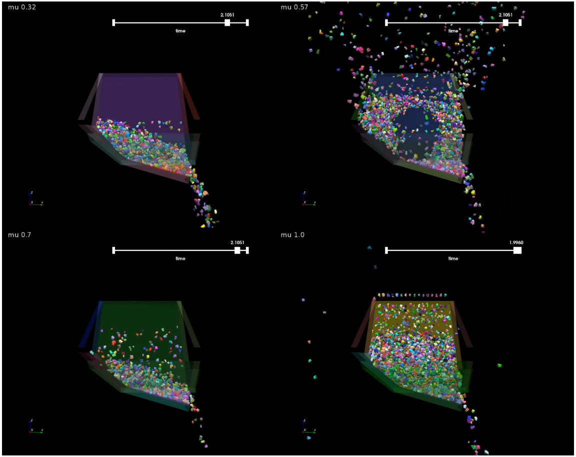


Granular and divided materials

Stack of beads with perturbation

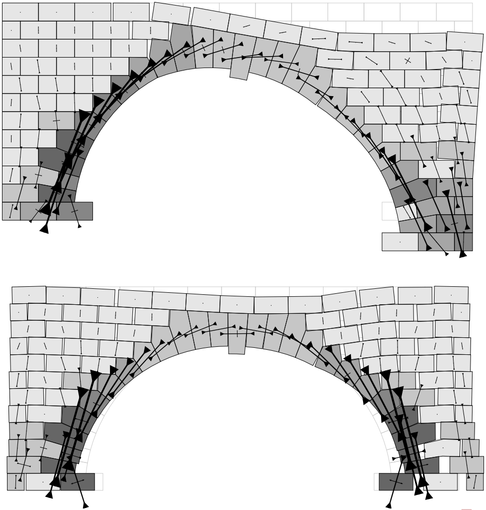


Granular and divided materials



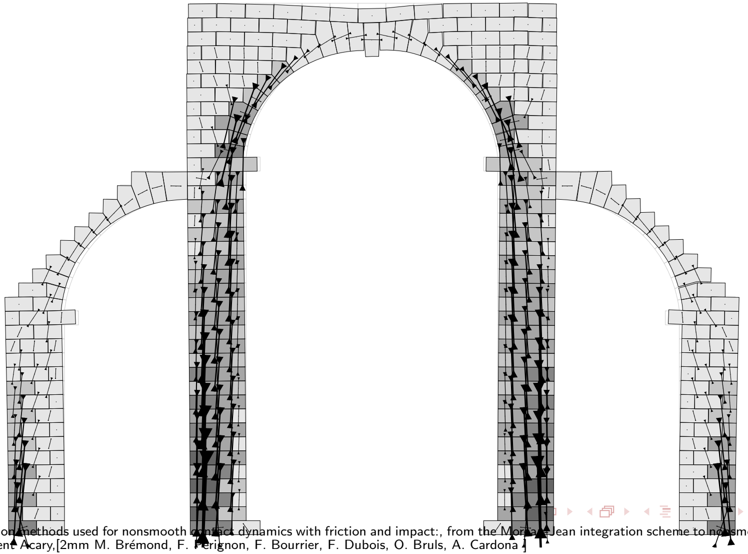
Mechanical systems with contact, impact and friction

Divided Materials and Masonry



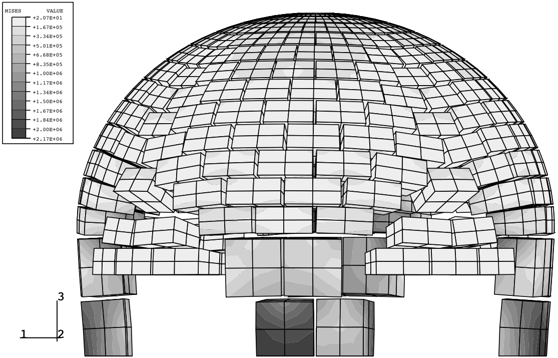
Mechanical systems with contact, impact and friction

Divided Materials and Masonry



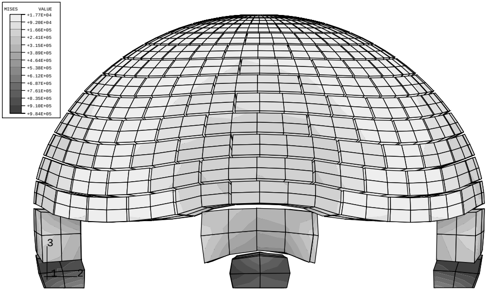
Mechanical systems with contact, impact and friction

Divided Materials and Masonry



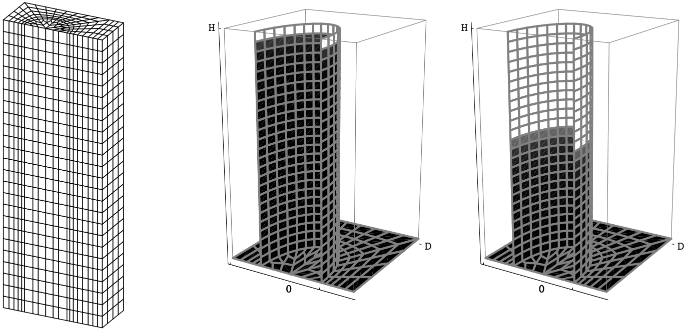
Mechanical systems with contact, impact and friction

Divided Materials and Masonry



Mechanical systems with contact, impact and friction

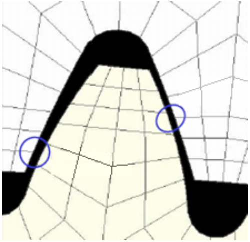
FEM models with contact, friction cohesion, etc...

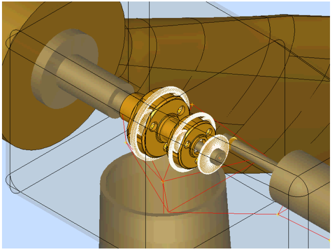


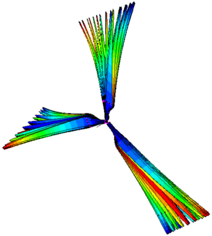
Joint work with Y. Monerie, IRSN.

Mechanical systems with contact, impact and friction

Simulation of wind turbines (DYNAWIND project) collaboration with O. Brüls (Université de Liège)

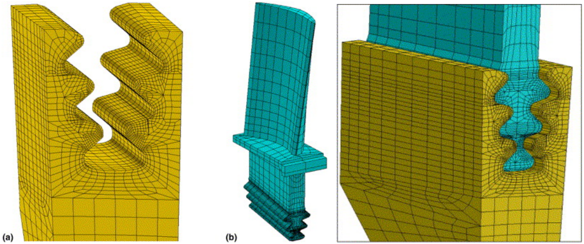






Mechanical systems with contact, impact and friction

Simulation of blades



Principle of nonsmooth event tracking methods (Event-driven schemes)

Time-decomposition of the dynamics in

- *modes*, time-intervals in which the dynamics is smooth,
- discrete events, times where the dynamics is nonsmooth.

Comments

On the numerical point of view, we need

- detect events with for instance root-finding procedure.
  - Dichotomy and interval arithmetic
  - Newton procedure for C^2 function and polynomials
- solve the non smooth dynamics at events with a reinitialization rule of the state,
- integrate the smooth dynamics between two events with any ODE solvers.

Principle of nonsmooth event capturing methods (Time–stepping schemes)

1. A unique formulation of the dynamics is considered. For instance, a dynamics in terms of measures.

$$\begin{cases} -mdv + fdt = di \\ \dot{q} = v^+ \\ 0 \leq di \perp v^+ \geq 0 \text{ if } q \leq 0 \end{cases} \tag{13}$$

2. The time-integration is based on a consistent approximation of the equations in terms of measures. For instance,

$$\int_{]t_k, t_{k+1}]} dv = \int_{]t_k, t_{k+1}]} dv = (v^+(t_{k+1}) - v^+(t_k)) \approx (v_{k+1} - v_k) \tag{14}$$

3. Consistent approximation of measure inclusion.

$$0 \leq di \perp v^+ \geq 0 \text{ if } q \leq 0 \quad \rightarrow \quad \begin{cases} p_{k+1} \approx \displaystyle\int_{]t_k, t_{k+1}]} di \\ 0 \leq p_{k+1} \perp v_{k+1} \geq 0 \quad \text{if } \tilde{q}_k \leq 0 \end{cases} \tag{15}$$

Nonsmooth Lagrangian Dynamics

Fundamental assumptions.

- The velocity $v = \dot{q}$ is of Bounded Variations (B.V)
  ➜ The equation are written in terms of a right continuous B.V. (R.C.B.V.) function, v^+ such that
$$v^+ = \dot{q}^+ \tag{16}$$
- q is related to this velocity by
$$q(t) = q(t_0) + \int_{t_0}^{t} v^+(t)\,dt \tag{17}$$
- The acceleration, ($\ddot{q}$ in the usual sense) is hence a differential measure dv associated with v such that
$$dv(]a,b]) = \int_{]a,b]} dv = v^+(b) - v^+(a) \tag{18}$$

NonSmooth Multibody Systems

Scleronomous holonomic perfect unilateral constraints

$$\begin{cases} M(q(t))\dot{v} = F(t, q(t), v(t)) + G(q(t))\,\lambda(t), \ \text{a.e} \\ \dot{q}(t) = v(t), \\ g(t) = g(q(t)), \quad \dot{g}(t) = G^T(q(t))v(t), \\ 0 \leq g(t) \perp \lambda(t) \geq 0, \\ \dot{g}^+(t) = -e\dot{g}^-(t), \end{cases} \tag{19}$$

where $G(q) = \nabla g(q)$ and e is the coefficient of restitution.

Nonsmooth Lagrangian Dynamics

Definition (Nonsmooth Lagrangian Dynamics)

$$\begin{cases} M(q)dv + F(t,q,v^+)dt = di \\ v^+ = \dot{q}^+ \end{cases} \tag{20}$$

where di is the reaction measure and dt is the Lebesgue measure.

Remarks

- The nonsmooth Dynamics contains the impact equations and the smooth evolution in a single equation.
- The formulation allows one to take into account very complex behaviors, especially, finite accumulation (Zeno-state).
- This formulation is sound from a mathematical Analysis point of view.

References

[10, 11, 6, 7]

Nonsmooth Lagrangian Dynamics

Measures Decomposition (for dummies)

$$\begin{cases} dv = & \gamma\, dt + & (v^+ - v^-)\, d\nu + & dv_s \\ di = & f\, dt + & p\, d\nu + & di_s \end{cases} \tag{21}$$

where

- $\gamma = \ddot{q}$ is the acceleration defined in the usual sense.
- f is the Lebesgue measurable force,
- $v^+ - v^-$ is the difference between the right continuous and the left continuous functions associated with the B.V. function $v = \dot{q}$,
- $d\nu$ is a purely atomic measure concentrated at the time t_i of discontinuities of v, i.e. where $(v^+ - v^-) \neq 0$,i.e. $d\nu = \sum_i \delta_{t_i}$
- p is the purely atomic impact percussions such that $pd\nu = \sum_i p_i \delta_{t_i}$
- dv_S and di_S are singular measures with the respect to $dt + d\eta$.

Impact equations and Smooth Lagrangian dynamics

Substituting the decomposition of measures into the nonsmooth Lagrangian Dynamics, one obtains

Definition (Impact equations)

$$M(q)(v^+ - v^-)d\nu = pd\nu, \tag{22}$$

or

$$M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \tag{23}$$

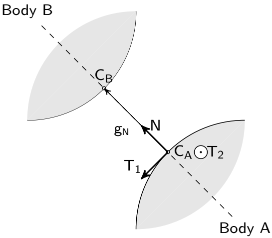
Definition (Smooth Dynamics between impacts)

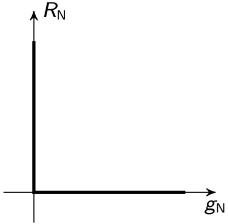
$$M(q)\gamma dt + F(t,q,v)dt = fdt \tag{24}$$

or

$$M(q)\gamma^+ + F(t,q,v^+) \quad = \quad f^+ \quad [dt - a.e.] \tag{25}$$

Unilateral contact and impact





- Unilateral contact (Signorini condition)

$$0 \leq g_N(q) \perp R_N \geq 0 \tag{26}$$

  Complementarity condition
- Local relative velocity at contact

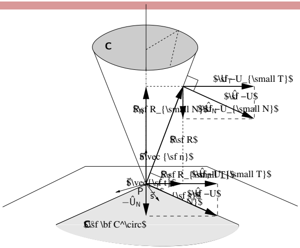
$$U = \begin{bmatrix} U_N \\ U_T \end{bmatrix} = G^T(q)v \tag{27}$$

- Impact Law (Newton Impact law)

$$U_N^+ = -e\, U_N^- \tag{28}$$

  e is the coefficient of restitution.

Coulomb's friction



Coulomb's friction

Coulomb's friction says the following:
If $g_N(q) = 0$ then:

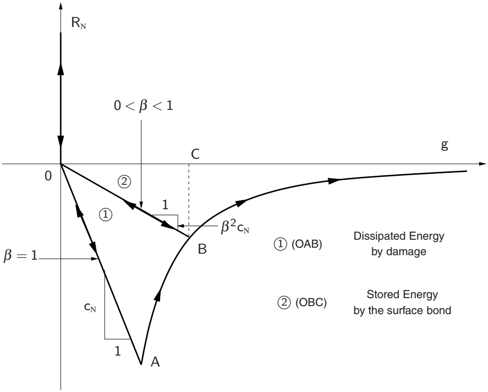
$$\left\{\begin{array}{ll} \text{If } U_T = 0 & \text{then } R \in \ \mathbf{K} \\ \text{If } U_T \neq 0 & \text{then } ||R_T(t)|| = \mu |R_N| \text{ and there exists a scalar } a \geq 0 \\ & \text{such that } R_T = -aU_T \end{array}\right. \tag{29}$$

where $K = \{R, ||R_T|| \leq \mu |R_N| \ \}$ is the Coulomb friction cone

Maximum dissipation principle in the tangent plane [8].

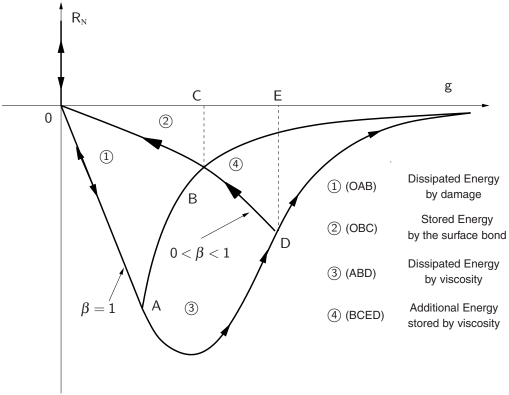
$$\max_{R_T \in D(\mu R_N)} -U_T^T R_T \tag{30}$$

Nonsmooth cohesive zone model



(a) Rate independent law

Nonsmooth cohesive zone model



(b) Rate dependent law (viscosity)

Why a nonsmooth modeling rather a smooth one ?

Tribological reasons

- complexity of the behavior of the interface/interphase : elasticity, viscosity, damage, plasticity, wear, ...
- parameters are difficult to identify and to measure
- multi–scale problems: high stiffness coefficients, uncertainties on parameters.

smoothing techniques and regularized models

Regularization enables to use of standard PDE and/or ODE solvers, BUT

- the regularization parameters are in general not physical
- the results are highly sensible the regularization parameters
- the numerical tools are inefficient: stiff ODES, numerical instabilities.
- the intrinsic set–valuedness of the model is not well-represented (sticking state).
- the quasi–static process needs also unrealistic viscosity regularization.

State–of–the–art

Numerical time–integration methods for Nonsmooth Multibody systems (NSMBS):

Nonsmooth event capturing methods (Time–stepping methods)

- ⊕ robust, stable and proof of convergence
- ⊕ low kinematic level for the constraints
- ⊕ able to deal with finite accumulation
- ⊖ very low order of accuracy even in free flight motions

Nonsmooth event tracking methods (Event–driven methods)

- ⊕ high level integration of free flight motions
- ⊖ no proof of convergence
- ⊖ sensibility to numerical thresholds
- ⊖ reformulation of constraints at higher kinematic levels.
- ⊖ unable to deal with finite accumulation

Two main implementations

- Moreau–Jean time–stepping scheme
- Schatzman–Paoli time–stepping scheme

Moreau–Jean's Time stepping scheme [7, 5]

Principle

$$\begin{cases} M(q_{k+\theta})(v_{k+1} - v_k) - hF_{k+\theta} = p_{k+1} = G(q_{k+\theta})P_{k+1}, & (31a) \\ q_{k+1} = q_k + hv_{k+\theta}, & (31b) \\ U_{k+1} = G^T(q_{k+\theta})\, v_{k+1} & (31c) \\ \begin{array}{ll} 0 \leq U^\alpha_{k+1} + eU^\alpha_k \perp P^\alpha_{k+1} \geq 0 & \text{if} \quad \bar{g}^\alpha_{k,\gamma} \leq 0 \\ P^\alpha_{k+1} = 0 & \text{otherwise} \end{array} . & (31d) \end{cases}$$

with

- $\theta \in [0, 1]$
- $x_{k+\theta} = (1-\theta)x_{k+1} + \theta x_k$
- $F_{k+\theta} = F(t_{k+\theta}, q_{k+\theta}, v_{k+\theta})$
- $\bar{g}_{k,\gamma} = g_k + \gamma h U_k, , \gamma \geq 0$ is a prediction of the constraints.

Schatzman–Paoli's Time stepping scheme [9]

Principle

$$\begin{cases} M(q_{k+1})(q_{k+1} - 2q_k + q_{k-1}) - h^2 F_{k+\theta} = p_{k+1}, & (32a) \\ v_{k+1} = \dfrac{q_{k+1} - q_{k-1}}{2h}, & (32b) \\ -p_{k+1} \in N_K\left(\dfrac{q_{k+1} + e q_{k-1}}{1+e}\right), & (32c) \end{cases}$$

where N_K defined the normal cone to K.
For $K = \{q \in \mathbb{R}^n, y = g(q) \geq 0\}$

$$0 \leq g\left(\frac{q_{k+1} + e q_{k-1}}{1+e}\right) \perp \nabla g\left(\frac{q_{k+1} + e q_{k-1}}{1+e}\right) P_{k+1} \geq 0 \qquad (33)$$

Comparison

Shared mathematical properties

- Convergence results for one constraints
- Convergence results for multiple constraints problems with acute kinetic angles
- No theoretical proof of order

Mechanical properties

- Position vs. velocity constraints
- Respect of the impact law in one step (Moreau) vs. Two-steps(Schatzman)
- Linearized constraints rather than nonlinear.

But ...

But

Both schemes are quite inaccurate and "dissipate" a lot of energy of vibrations. This is a consequence of the first order approximation of the smooth forces term F

Recent improvements

- Nonsmooth generalized α schemes [4, 3]
- Time discontinuous Galerkin methods [12, 13]
- Stabilized index-2 formulation [2, 1]
- Stabilized index-1 formulation []
- Geometric integrators.

The nonsmooth generalized α scheme

Splitting the dynamics between smooth and nonsmooth part

$$\mathrm{d}\mathbf{w} = \mathrm{d}\mathbf{v} - \dot{\tilde{\mathbf{v}}}\,\mathrm{d}t \tag{34}$$

Smooth (non-impulsive) part

Solutions of the following DAE

$$\begin{aligned}
\dot{\tilde{\mathbf{q}}} &= \tilde{\mathbf{v}} && (35a)\\
\mathbf{M}(\mathbf{q})\,\dot{\tilde{\mathbf{v}}} - \mathbf{g}_{\mathbf{q}}^{T}(\mathbf{q})\,\tilde{\boldsymbol{\lambda}} &= \mathbf{f}(\mathbf{q}, \mathbf{v}, t) && (35b)\\
\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}(\mathbf{q})\,\tilde{\mathbf{v}} &= \mathbf{0} && (35c)\\
\tilde{\boldsymbol{\lambda}}^{\mathcal{U}} &= \mathbf{0} && (35d)
\end{aligned}$$

with the initial value $\tilde{\mathbf{v}}(t_n) = \mathbf{v}(t_n)$, $\tilde{\mathbf{q}}(t_n) = \mathbf{q}(t_n)$.

The nonsmooth generalized α scheme

Splitting the dynamics between smooth and nonsmooth part

$$
\begin{aligned}
\dot{\mathbf{q}} &= \mathbf{v} && \text{(36a)}\\
\mathrm{d}\mathbf{v} &= \mathrm{d}\mathbf{w} + \dot{\tilde{\mathbf{v}}}\,\mathrm{d}t && \text{(36b)}\\
\mathbf{M}(\mathbf{q})\,\dot{\tilde{\mathbf{v}}} - \mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}},T}\,\tilde{\boldsymbol{\lambda}}^{\overline{\mathcal{U}}} &= \mathbf{f}(\mathbf{q},\mathbf{v},t) && \text{(36c)}\\
\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}\,\tilde{\mathbf{v}} &= \mathbf{0} && \text{(36d)}\\
\tilde{\boldsymbol{\lambda}}^{\mathcal{U}} &= \mathbf{0} && \text{(36e)}\\
\mathbf{M}(\mathbf{q})\,\mathrm{d}\mathbf{w} - \mathbf{g}_{\mathbf{q}}^{T}\,(\mathrm{d}\mathbf{i} - \tilde{\boldsymbol{\lambda}}\,\mathrm{d}t) &= \mathbf{0} && \text{(36f)}\\
\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}\mathbf{v} &= \mathbf{0} && \text{(36g)}\\
\text{if } g^{j}(\mathbf{q}) \leq 0 \text{ then } 0 \leq g_{\mathbf{q}}^{j}\,\mathbf{v} + e\,g_{\mathbf{q}}^{j}\,\mathbf{v}^{-} &\perp \mathrm{d}i^{j} \geq 0, \quad \forall j \in \mathcal{U} && \text{(36h)}
\end{aligned}
$$

The nonsmooth generalized α scheme

GGL approach to stabilize the constraints at the position level

The equations of motion become

$$\cancel{\dot{\mathbf{q}} = \mathbf{v}} \rightarrow$$

$$\begin{aligned}
\mathbf{M}(\mathbf{q})\,\dot{\mathbf{q}} - \mathbf{g}_{\mathbf{q}}^{T}\,\boldsymbol{\mu} &= \mathbf{M}(\mathbf{q})\,\mathbf{v} && \text{(37a)}\\
\mathbf{g}^{\overline{\mathcal{U}}}(\mathbf{q}) &= \mathbf{0} && \text{(37b)}\\
\mathbf{0} \leq \mathbf{g}^{\mathcal{U}}(\mathbf{q}) &\perp \boldsymbol{\mu}^{\mathcal{U}} \geq \mathbf{0} && \text{(37c)}\\
\mathrm{d}\mathbf{v} &= \mathrm{d}\mathbf{w} + \dot{\tilde{\mathbf{v}}}\,\mathrm{d}t && \text{(37d)}\\
\mathbf{M}(\mathbf{q})\,\dot{\tilde{\mathbf{v}}} - \mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}},T}\,\tilde{\boldsymbol{\lambda}}^{\overline{\mathcal{U}}} &= \mathbf{f}(\mathbf{q},\mathbf{v},t) && \text{(37e)}\\
\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}\,\tilde{\mathbf{v}} &= \mathbf{0} && \text{(37f)}\\
\tilde{\boldsymbol{\lambda}}^{\mathcal{U}} &= \mathbf{0} && \text{(37g)}\\
\mathbf{M}(\mathbf{q})\,\mathrm{d}\mathbf{w} - \mathbf{g}_{\mathbf{q}}^{T}\,(\mathrm{d}\mathbf{i} - \tilde{\boldsymbol{\lambda}}\,\mathrm{d}t) &= \mathbf{0} && \text{(37h)}\\
\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}\mathbf{v} &= \mathbf{0} && \text{(37i)}\\
\text{if } g^{j}(\mathbf{q}) \leq 0 \text{ then } 0 \leq g_{\mathbf{q}}^{j}\,\mathbf{v} + e\,g_{\mathbf{q}}^{j}\,\mathbf{v}^{-} &\perp \mathrm{d}i^{j} \geq 0, \quad \forall j \in \mathcal{U} && \text{(37j)}
\end{aligned}$$

The nonsmooth generalized α scheme

Velocity jumps and position correction

The multipliers $\mathbf{\Lambda}(t_n;t)$ and $\boldsymbol{\nu}(t_n;t)$ are defined as

$$\mathbf{\Lambda}(t_n;t) = \int_{(t_n,t]} (\mathrm{d}\mathbf{i} - \tilde{\boldsymbol{\lambda}}(\tau)\,\mathrm{d}\tau) \tag{38a}$$

$$\boldsymbol{\nu}(t_n;t) = \int_{t_n}^{t} (\boldsymbol{\mu}(\tau) + \mathbf{\Lambda}(t_n;\tau))\,\mathrm{d}\tau \tag{38b}$$

with $\mathbf{\Lambda}(t_n;t_n) = \boldsymbol{\nu}(t_n;t_n) = \mathbf{0}$.
The velocity jump and position correction variables

$$\mathbf{W}(t_n;t) = \int_{(t_n,t]} \mathrm{d}\mathbf{w} = \mathbf{v}(t) - \tilde{\mathbf{v}}(t) \tag{39a}$$

$$\mathbf{U}(t_n;t) = \int_{t_n}^{t} (\dot{\mathbf{q}} - \tilde{\mathbf{v}})\mathrm{d}t = \mathbf{q}(t) - \tilde{\mathbf{q}}(t) \tag{39b}$$

➜ Low-order approximation of impulsive terms.
➜ Higher–order approximation of non impulsive terms.

The nonsmooth generalized α scheme

$$\mathbf{M}(\mathbf{q}_{n+1})\mathbf{U}_{n+1} - \mathbf{g}^{T}_{\mathbf{q},n+1}\,\boldsymbol{\nu}_{n+1} = \mathbf{0} \tag{40a}$$

$$\mathbf{g}^{\overline{\mathcal{U}}}(\mathbf{q}_{n+1}) = \mathbf{0} \tag{40b}$$

$$\mathbf{0} \leq \mathbf{g}^{\mathcal{U}}(\mathbf{q}_{n+1}) \perp \boldsymbol{\nu}^{\mathcal{U}}_{n+1} \geq \mathbf{0} \tag{40c}$$

$$\mathbf{M}(\mathbf{q}_{n+1})\dot{\tilde{\mathbf{v}}}_{n+1} - \mathbf{f}(\mathbf{q}_{n+1},\mathbf{v}_{n+1},t_{n+1}) - \mathbf{g}^{\overline{\mathcal{U}},T}_{\mathbf{q},n+1}\,\tilde{\boldsymbol{\lambda}}^{\overline{\mathcal{U}}}_{n+1} = \mathbf{0} \tag{40d}$$

$$\mathbf{g}^{\overline{\mathcal{U}}}_{\mathbf{q},n+1}\,\tilde{\mathbf{v}}_{n+1} = \mathbf{0} \tag{40e}$$

$$\mathbf{M}(\mathbf{q}_{n+1})\mathbf{W}_{n+1} - \mathbf{g}^{T}_{\mathbf{q},n+1}\boldsymbol{\Lambda}_{n+1} = \mathbf{0} \tag{40f}$$

$$\mathbf{g}^{\overline{\mathcal{U}}}_{\mathbf{q},n+1}\mathbf{v}_{n+1} = \mathbf{0} \tag{40g}$$

$$\text{if } g^{j}(\mathbf{q}^{*}_{n+1}) \leq 0 \text{ then } 0 \leq g^{j}_{\mathbf{q},n+1}\,\mathbf{v}_{n+1} + e\,g^{j}_{\mathbf{q},n}\,\mathbf{v}_{n} \perp \Lambda^{j}_{n+1} \geq 0, \forall j \in \mathcal{U}$$

The nonsmooth generalized α scheme

Nonsmooth generalized α-scheme

$$
\begin{aligned}
\tilde{\mathbf{q}}_{n+1} &= \mathbf{q}_n + h\mathbf{v}_n + h^2(0.5-\beta)\mathbf{a}_n + h^2\beta\mathbf{a}_{n+1} && \text{(41a)}\\
\mathbf{q}_{n+1} &= \tilde{\mathbf{q}}_{n+1} + \mathbf{U}_{n+1} && \text{(41b)}\\
\tilde{\mathbf{v}}_{n+1} &= \mathbf{v}_n + h(1-\gamma)\mathbf{a}_n + h\gamma\mathbf{a}_{n+1} && \text{(41c)}\\
\mathbf{v}_{n+1} &= \tilde{\mathbf{v}}_{n+1} + \mathbf{W}_{n+1} && \text{(41d)}\\
(1-\alpha_m)\mathbf{a}_{n+1} + \alpha_m\mathbf{a}_n &= (1-\alpha_f)\dot{\tilde{\mathbf{v}}}_{n+1} + \alpha_f\dot{\tilde{\mathbf{v}}}_n && \text{(41e)}
\end{aligned}
$$

Special cases

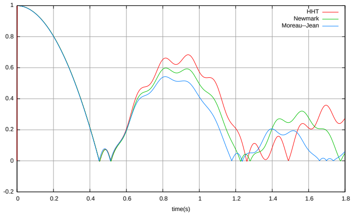
- $\alpha_m = \alpha_f = 0$ ➜ Nonsmooth Newmark
- $\alpha_m = 0, \alpha_f \in [0, 1/3]$ ➜ Nonsmooth Hilber-Hughes–Taylor (HHT)

Spectral radius at infinity $\rho_\infty \in [0, 1]$

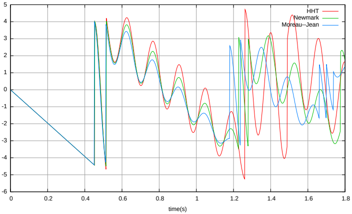
$$
\alpha_m = \frac{2\rho_\infty - 1}{\rho_\infty + 1}, \quad \alpha_f = \frac{\rho_\infty}{\rho_\infty + 1}, \quad \beta = \frac{1}{4}(\gamma + \frac{1}{2})^2. \tag{42}
$$

Numerical Illustrations

Two ball oscillator with impact.

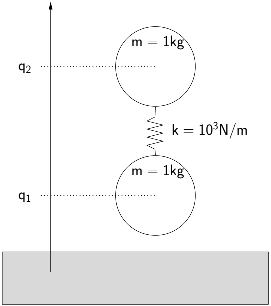


Position of the first ball



Velocity of the first ball

Time–step : $h = 2e - 3$.
Moreau ($\theta = 1.0$).
Newmark ($\gamma = 1.0, \beta = 0.5$, $\alpha_m = \alpha_f = 0$).
HHT ($\gamma = 1.0, \beta = 0.5$, $\alpha_f = 0.1, \alpha_m = 0$)



Numerical Illustrations

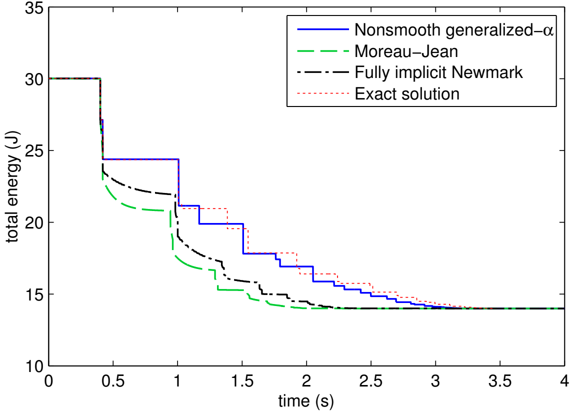
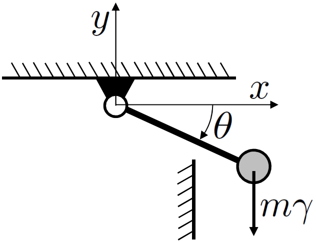


Figure 7. Numerical results for the total energy of the bouncing oscillator.

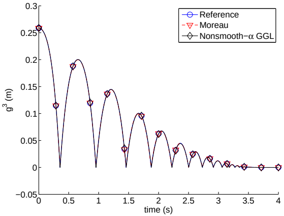
Numerical Illustrations

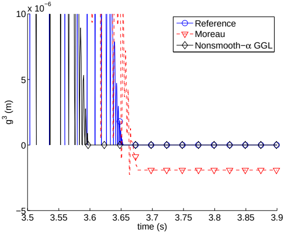
Bouncing Pendulum

$$\mathbf{q} = [x, y, \theta]^T$$
$$g_1(\mathbf{q}) = x - l\cos\theta = 0$$
$$g_2(\mathbf{q}) = y - l\sin\theta = 0$$
$$g_3(\mathbf{q}) = x - \sqrt{2}/2 \geq 0$$



Unilateral constraint



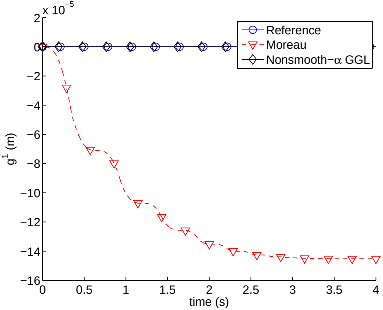


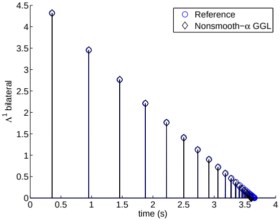
Time–step : $h = 2e-3$.
Moreau ($\theta = 1/1.8$).
α-schemes ($\rho_\infty = 0.8$)

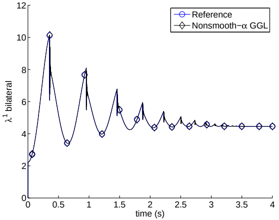
$e = 0.8$

Numerical Illustrations

Bouncing Pendulum

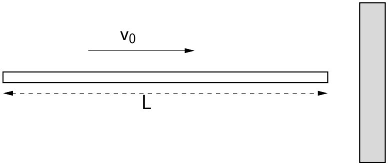






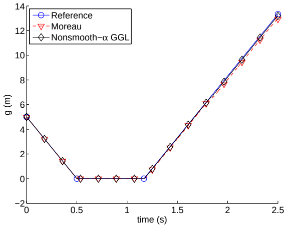
Numerical Illustrations

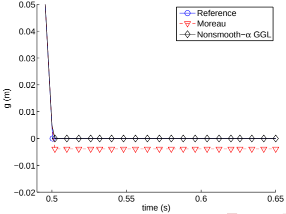
Impacting elastic bar



$g_3(\mathbf{q}) = x_1 \geq 0$
$e = 0.0$
200 finite elements
Time–step : $h = 2e - 3$.
Moreau $(\theta = 1/1.8)$.
α-schemes $(\rho_\infty = 0.8)$

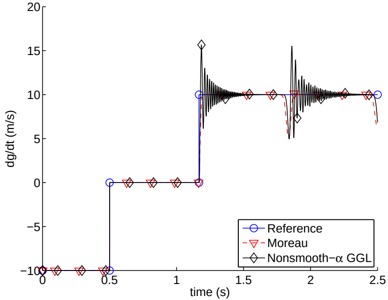
Unilateral constraint

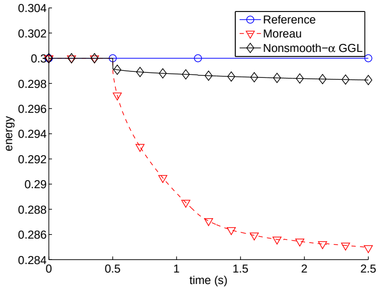




Numerical Illustrations

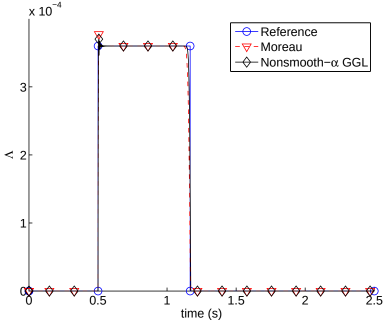
Impacting elastic bar

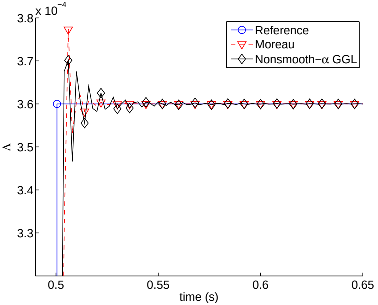




Numerical Illustrations

Impacting elastic bar





Conclusions and perspectives

Possible applications Geotechnics

What we have scheduled in Tripop

Conclusions and perspectives

- Possible applications Geotechnics
- What we have scheduled in Tripop

Fields of expertise

Mechanical systems with contact, friction, impacts or cohesive interfaces

Modelling and numerical simulations of:

- Granular matter (flows, quasi-static equilibria, dense packing)
- Fracture dynamics.
- Jointed rock mechanics.
- Fluid/Granular flows (sedimentation).
- Multibody system dynamics.

Numerical methods are a kind of Discrete Element method(DEM), but

- Hard contact laws. (Nonsmooth Dynamics)
- Real Coulomb friction
- Enhanced cohesive zone model (CZM) with elasticity, damage

Possible Applications in geotechnics.

Natural hazards

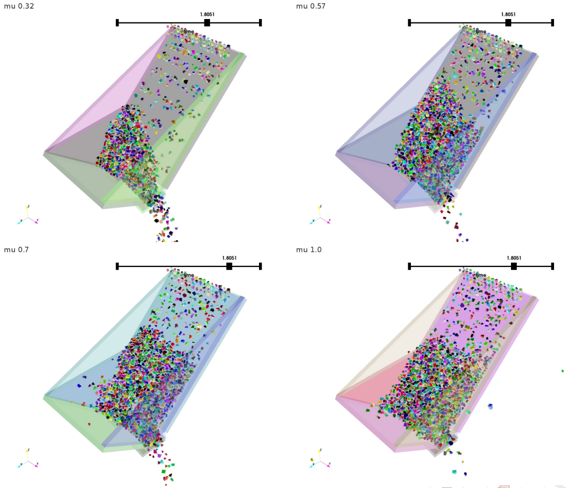
- Rocky and snow avalanches
- Stability of jointed rock mass
- Earthquake engineering (friction and instability)

Mines engineering process of ore

- Rock mechanics, fracture mechanics for block caving techniques
- Ore (granular) transport and transfer chutes (conveyor)
- Stirred mills, SAG mills, crushers and high pressure grinding rolls
- Efficient separation, screening performance,
- Fluid flows with grains (sedimentation and transports)
- Soil Mechanics and tailing dams

Possible Applications in geotechnics.

Flow of granular material (Siconos, INRIA Chile)



Possible Applications in geotechnics.

Rockfall (F. Bourrier, IRSTEA)

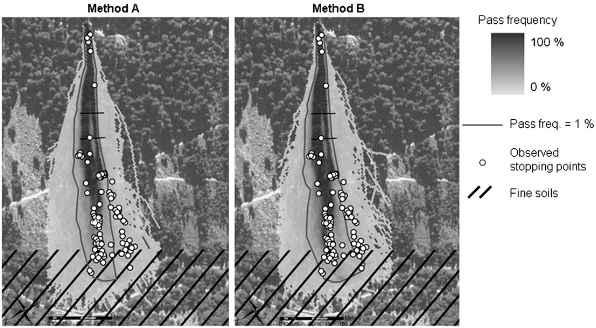
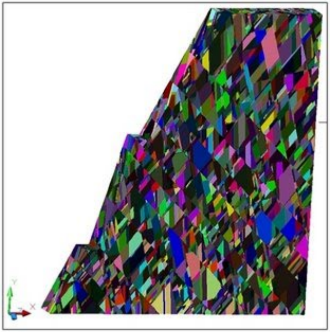
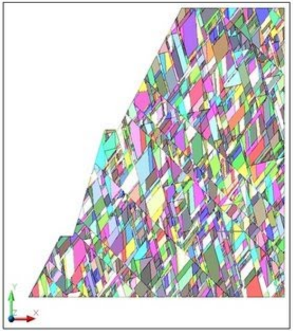


Fig. 10. Map of the simulated pass frequencies for methods A (“size classes”) and B (“mean size”) and the observed stopping points (white dots).

Possible Applications in geotechnics.

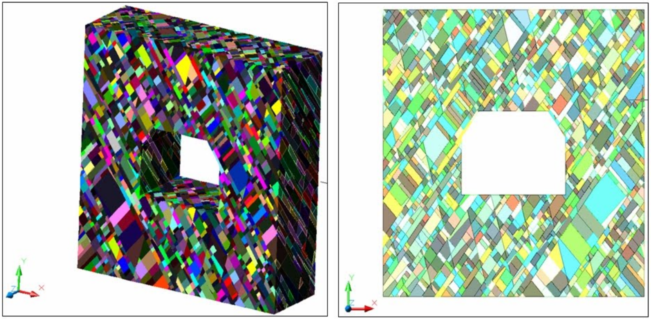
Stability of Rock masses (LMGC90, Mines d'Ales Ali Rafiee, M. Vinches, F. Dubois)





Possible Applications in geotechnics.

Stability of Rock masses (LMGC90, Mines d'Ales Ali Rafiee, M. Vinches, F. Dubois)



Possible Applications in geotechnics.

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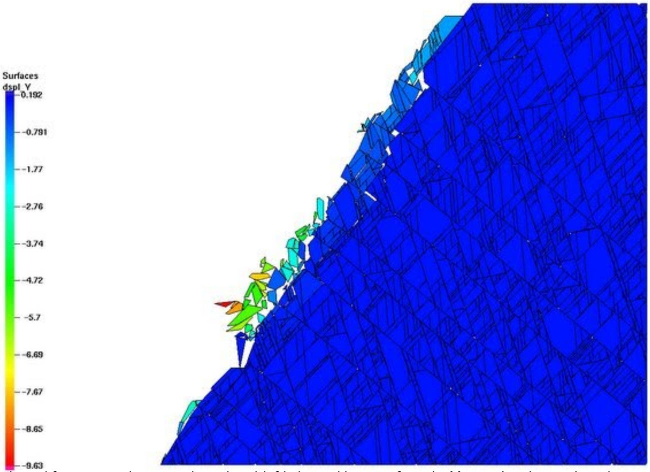

Possible Applications in geotechnics.

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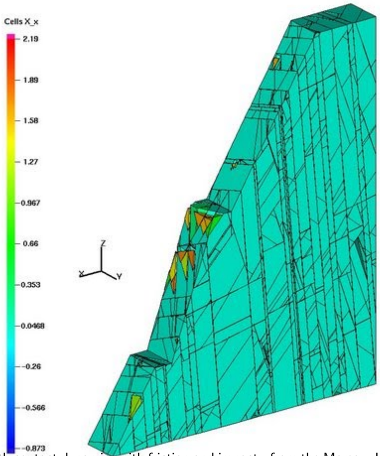

Possible Applications in geotechnics.

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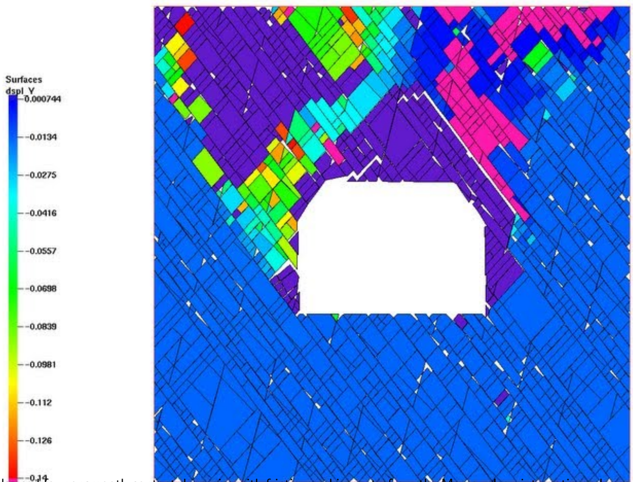
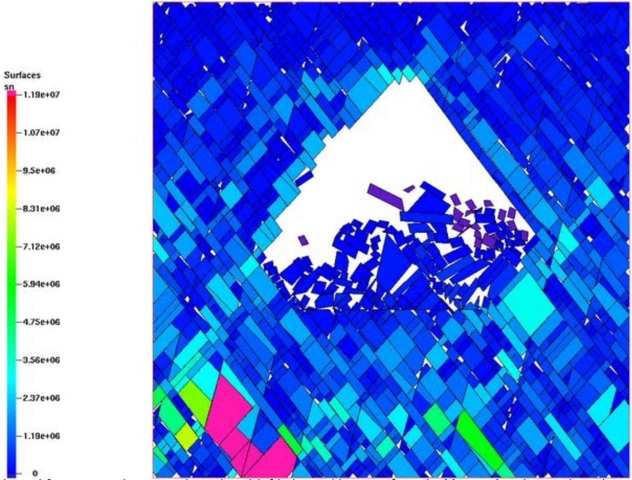


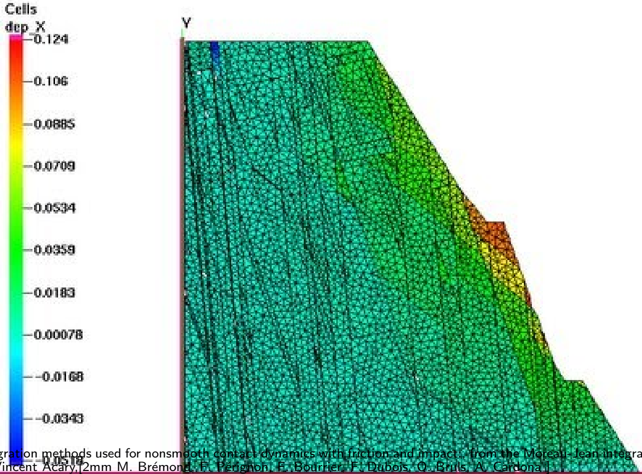
Possible Applications in geotechnics.

Stability of Rock masses (LMGC90, Mines d'Ales Ali Rafiee, M. Vinches, F. Dubois)



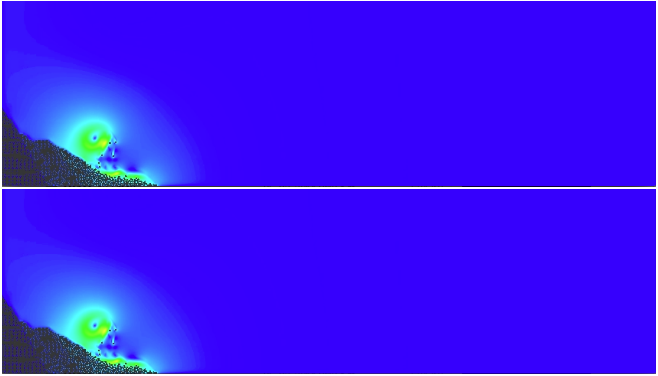
Possible Applications in geotechnics.

Stability of Rock masses (LMGC90, Mines d'Ales Ali Rafiee, M. Vinches, F. Dubois)



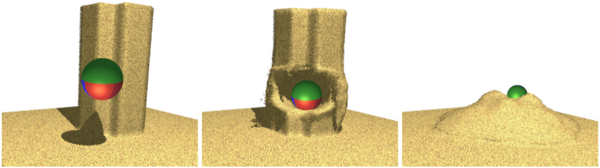
Possible Applications in geotechnics.

Fluid Grains interactions (LMGC90 IFPEN Topin, Dubois)

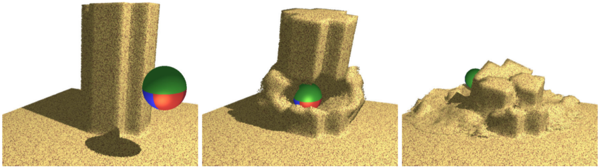


Possible Applications in geotechnics.

Fluid mechanics of Granular material (G. Daviet, F. Descoubes, P. Saramito INRIA/LJK)



(a) *Trilinear stress shape functions and cohesion decay*



(b) *Particle-based stress shape functions without cohesion decay*

Figure 7.22: *A sphere impacting a sand tower initially standing up thanks to a high cohesion coefficient.*

Possible Applications in geotechnics.

Fluid mechanics of Granular material (G. Daviet, F. Descoubes, P. Saramito INRIA/LJK

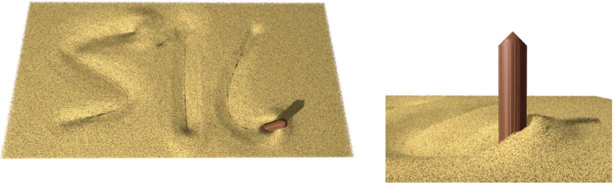


Figure 7.20: *Letters drawn by dragging a stick in the sand. Right: a typical mound grows at the front of the stick.*

Possible Applications in geotechnics.

Fluid mechanics of Granular material (G. Daviet, F. Descoubes, P. Saramito INRIA/LJK

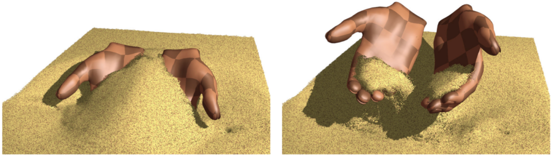


Figure 7.21: *Picking-up sand and letting it flow away.*

What we have scheduled in tripop

1. Rolling friction and fracture for rock-fall trajectory
   - Numerical algorithms for second order cones.
   - Cohesive zone modeling of interfaces with damage, contact and friction (Frémond-like).
2. Rock interaction with elasto-plastic obstacles
   - Plasticity and damage as nonsmooth behavior law (complementarity and differential inclusions)
   - Numerical methods based on modern optimization techniques
3. Debris flows with rigid bodies and obstacles
   - Debris Flows with large objects and accumulation and contact.
   - Non Newtonian fluids with non-associated plasticity (Bingham, Drucker-Prager, Mohr-Coulomb)
   - Material Point Method with behavior laws based in second order cones for elastic domains.
4. High performance computing

Thank you for your attention.

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