Time-Integration methods for nonsmooth contact dynamics with friction and impact.

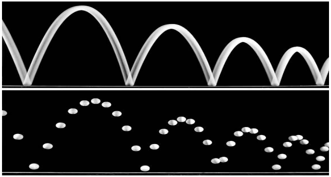
Vincent Acary

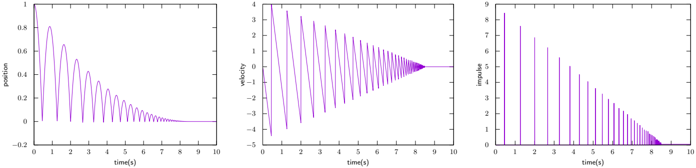
TRIPOP. Inria Grenoble - Laboratoire Jean Kuntzmann

February 17, 2020

Nonsmooth dynamical systems

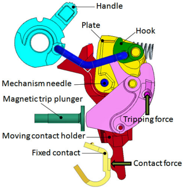
$\boxed{\text{nonsmooth} = \text{lack of continuity/differentiability}}$

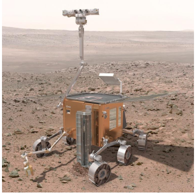




- nonsmooth solutions in time (jumps, kinks, distributions, measures)
- nonsmooth modeling and constitutive laws (set–valued mapping, inequality constraints, complementarity, impact laws)

Application fields.



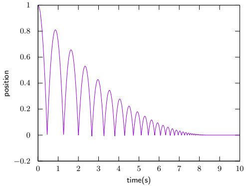


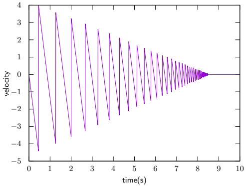
- Computational mechanics. Plasticity. Unilateral contact, Coulomb friction and impacts : multi-body systems, robotic systems, frictional contact oscillators, granular materials.
- Electronics. Switched electrical circuits (digital/analog converters and power electronics, diodes, transistors, switchs).
- Computer science. Hybrid and Cyber–physical systems
- Bio-mathematics. Gene regulatory networks
- Transportation science. Fluid transportation networks with queues.
- Economy and Finance. Oligopolistic market equilibrium

Nonsmooth approach is crucial for a correct modeling and a efficient simulation

Sources of nonsmoothness

- Two largely different time-scales of evolution:
  1. a slow smooth dynamics (free flight of the bouncing ball)
  2. a very fast dynamics (events, transitions, impacts) that can be modeled as a punctual event.





Nonsmooth dynamical systems

Difficulty

Standard tools of numerical analysis and simulation (in finite dimension) are no longer suitable due to the lack of regularity.

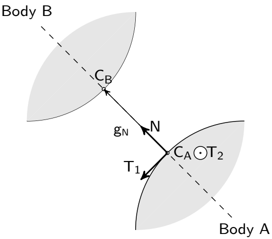
Specific tools

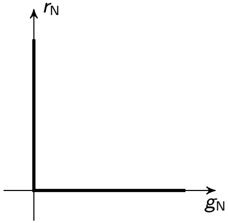
Differential measure theory. Convex, nonsmooth and variational Analysis (Clarke, Wets & Rockafellar). Complementarity theory. Maximal monotone operators.

Examples of nonsmooth dynamical systems

- Piecewise smooth systems
- Complementarity systems and differential variational inequality.
- Specific differential inclusions (Filippov, Moreau sweeping process, Normal cone inclusion).

Unilateral contact and impact





- gap function $g_N = (C_B - C_A)N$.
- reaction forces

$$r = r_N N + r_T, \quad \text{with } r_N \in \mathbb{R} \text{ and } r_T \in \mathbb{R}^2.$$

- Signorini condition at position level

$$0 \leq g_N \perp r_N \geq 0.$$

- relative velocity

$$u = u_N N + u_T, \quad \text{with } u_N \in \mathbb{R} \text{ and } u_T \in \mathbb{R}^2.$$

- Signorini condition at velocity level

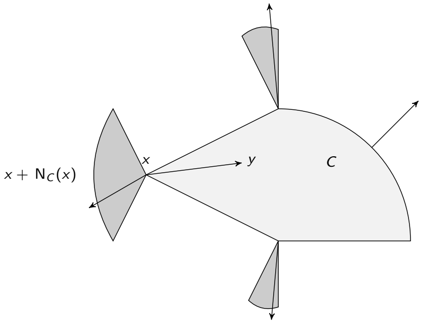
$$\begin{cases} 0 \leq u_N \perp r_N \geq 0 & \text{if } g_N \leq 0 \\ r_N = 0 & \text{otherwise.} \end{cases}$$

Normal cone to a convex set

Definition (Normal cone to a convex set)

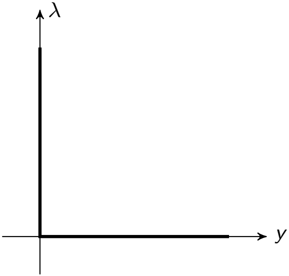
C a nonempty convex set in $\boldsymbol{R}^n$ and $x \in C$

$$\mathrm{N}_C(x) = \{s \in \boldsymbol{R}^n \mid s^T(y-x) \leq 0 \text{ for all } y \in C\} \tag{1}$$



Complementarity condition

Signorini's condition in contact mechanics



$$
\begin{array}{rcll}
0 \leq y & \perp & \lambda \geq 0 & (2) \\
& \Updownarrow & & \\
-y & \in & N_{\boldsymbol{R}_+}(\lambda) & (3) \\
& \Updownarrow & & \\
-\lambda & \in & N_{\boldsymbol{R}_+}(y) & (4) \\
& \Updownarrow & & \\
\lambda^T(y'-y) & \geq & 0, \text{ for all } y' \in \boldsymbol{R}_+ & (5) \\
& \Updownarrow & & \\
y^T(\lambda'-\lambda) & \geq & 0, \text{ for all } \lambda' \in \boldsymbol{R}_+ & (6)
\end{array}
$$

A well-known concept in Optimization

- Numerous theoretical tools (variational inequalities, complementarity problems, proximal point techniques)
- Numerous numerical tools (pivoting techniques, projected over–relaxation (Gauss–Seidel), semi-smooth Newton methods, interior point methods, ...)

Coulomb's friction

Modeling assumption

Let μ be the coefficient of friction. Let us define the Coulomb friction cone K which is chosen as the isotropic second order cone

$$K = \{r \in \boldsymbol{R}^3 \mid \|r_{\mathrm{T}}\| \leq \mu r_n\}. \tag{7}$$

The Coulomb friction states

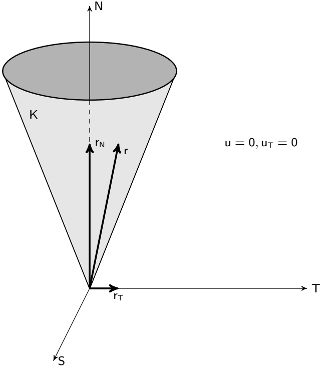
- for the sticking case that
$$u_{\mathrm{T}} = 0, \quad r \in K \tag{8}$$
- and for the sliding case that
$$u_{\mathrm{T}} \neq 0, \quad r \in \partial K, \text{ and } \quad r_{\mathrm{T}}\|u_{\mathrm{T}}\| = -u_{\mathrm{T}}\|r_{\mathrm{T}}\| \tag{9}$$

Maximum dissipation principle in the tangent plane [14].

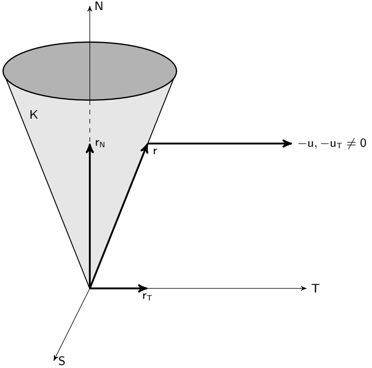
$$\max_{r_{\mathrm{T}} \in D(\mu r_{\mathrm{N}})} -r_{\mathrm{T}}^T u_{\mathrm{T}} \qquad \Longleftrightarrow -u_{\mathrm{T}} \in N_{D(\mu r_{\mathrm{N}})}(r_{\mathrm{T}}) \tag{10}$$

where $D(\mu r_{\mathrm{N}}) = \{r_T \in \boldsymbol{R}^2, \|r_{\mathrm{T}}\| \leq \mu |r_{\mathrm{N}}| \}$ is the Coulomb friction disk.

Coulomb's friction



Sticking case



Sliding case

Disjunctive formulation of the frictional contact behavior

$$
\begin{cases}
r = 0 & \text{if } g_{\mathrm{N}} > 0 \quad \text{(no contact)} \\
r = 0, u_{\mathrm{N}} \geq 0 & \text{if } g_{\mathrm{N}} \leq 0 \quad \text{(take–off)} \\
r \in K, u = 0 & \text{if } g_{\mathrm{N}} \leq 0 \quad \text{(sticking)} \\
r \in \partial K, u_{\mathrm{N}} = 0, r_{\mathrm{T}} \|u_{\mathrm{T}}\| = -u_{\mathrm{T}} \|r_{\mathrm{T}}\| & \text{if } g_{\mathrm{N}} \leq 0 \quad \text{(sliding)}
\end{cases} \tag{11}
$$

Signorini's condition and Coulomb's friction

Second Order Cone Complementarity (SOCCP) formulation [9]

- Modified relative velocity $\hat{u} \in \boldsymbol{R}^3$ defined by

$$\hat{u} = u + \mu \|u_{\mathrm{T}}\| \mathrm{N}. \tag{12}$$

- Second-Order Cone Complementarity condition

$$K^\star \ni \hat{u} \perp r \in K \tag{13}$$

if $g_{\mathrm{N}} \leq 0$ and $r = 0$ otherwise. The set $K^\star$ is the dual convex cone to K defined by

$$K^\star = \{u \in \boldsymbol{R}^3 \mid r^\top u \geq 0, \quad \text{for all } r \in K\}. \tag{14}$$

- Normal cone inclusion

$$-\hat{u} \in N_K(r) \tag{15}$$

- Nonassociated character of the friction (loss of monotony)

$$-(u + \mu \|u_{\mathrm{T}}\| \mathrm{N}) \in \mathrm{N}_{\mathrm{K}}(\mathrm{r}) \tag{16}$$

Signorini's condition and Coulomb's friction

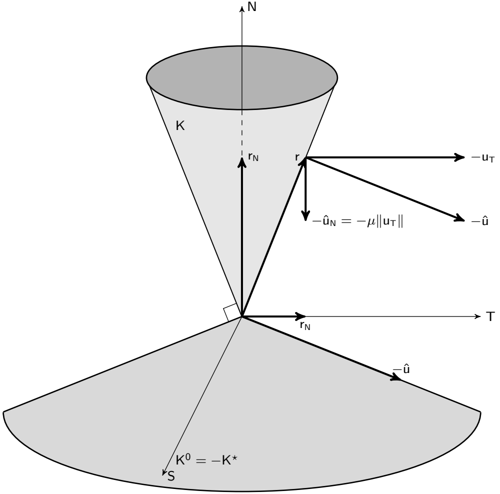
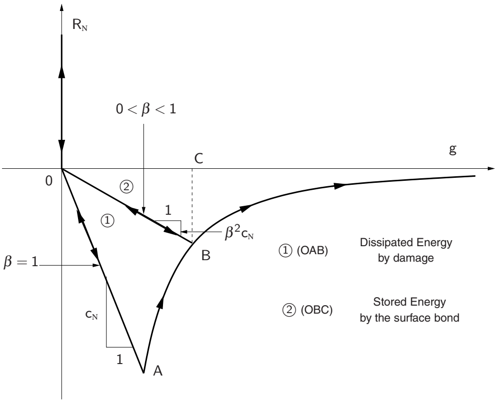


Figure: Coulomb's friction and the modified velocity $\hat{u}$. The sliding case.

Nonsmooth cohesive zone model



(a) Rate independent law

Multiple constraints

- $q \in \boldsymbol{R}^n$ coordinates that describes the state of the system in finite-dimension
- Notion of admissible set $C(t)$

$$\mathcal{C}(t) = \{q \in \boldsymbol{R}^n, g_\alpha(q,t) \geq 0, \alpha \in \{1 \ldots \nu\}\}$$

- Normal cone to $C(t)$

$$\mathrm{N}_{\mathcal{C}(t)}(q) = \{y \mid y = -\nabla_q g(q,t)\,\lambda, 0 \leq g_\alpha(q,t) \geq 0 \perp \lambda_\alpha \geq 0\}$$

- Normal cone inclusion

$$-r \in \mathrm{N}_{\mathcal{C}(t)}(q)$$

Unilateral constraints as an inclusion

Definition (Perfect unilateral constraints on the smooth dynamics)

$$\begin{cases} \dot{q} = v \\ M(q)\dfrac{dv}{dt} + F(t,q,v) = r \\ -r \in N_{\mathcal{C}(t)}(q(t)) \end{cases} \tag{17}$$

where r it the generalized force or generalized reaction due to the constraints.

Remark

- Second order differential inclusion.
- The unilateral constraints are said to be perfect due to the normality condition.
- Notion of normal cones can be extended to more general sets. see [7, 8, 11]

Nonsmooth Lagrangian Dynamics

Fundamental assumptions.

- The velocity $v = \dot{q}$ is of Bounded Variations (B.V)
  ➜ The equation are written in terms of a right continuous B.V. (R.C.B.V.) function, v^+ such that
  $$v^+ = \dot{q}^+ \tag{18}$$
- q is related to this velocity by
  $$q(t) = q(t_0) + \int_{t_0}^{t} v^+(t)\,dt \tag{19}$$
- The acceleration, ($\ddot{q}$ in the usual sense) is hence a differential measure dv associated with v such that
  $$dv(]a,b]) = \int_{]a,b]} dv = v^+(b) - v^+(a) \tag{20}$$

Nonsmooth Lagrangian Dynamics

Definition (Nonsmooth Lagrangian Dynamics)

$$\begin{cases} M(q)dv + F(t, q, v^+)dt = di \\ \\ v^+ = \dot{q}^+ \end{cases} \tag{21}$$

where di is the reaction measure and dt is the Lebesgue measure.

Remarks

- The nonsmooth Dynamics contains the impact equations and the smooth evolution in a single equation.
- The formulation allows one to take into account very complex behaviors, especially, finite accumulation (Zeno-state).
- This formulation is sound from a mathematical Analysis point of view.

References

[16, 17, 12, 13]

Nonsmooth Lagrangian Dynamics

Measures Decomposition (for dummies)

$$\left\{\begin{array}{rccc} dv = & \gamma\, dt + & (v^+ - v^-)\, d\nu + & dv_s \\ di = & f\, dt + & p\, d\nu + & di_s \end{array}\right. \tag{22}$$

where

- $\gamma = \ddot{q}$ is the acceleration defined in the usual sense.
- f is the Lebesgue measurable force,
- $v^+ - v^-$ is the difference between the right continuous and the left continuous functions associated with the B.V. function $v = \dot{q}$,
- $d\nu$ is a purely atomic measure concentrated at the time t_i of discontinuities of v, i.e. where $(v^+ - v^-) \neq 0$,i.e. $d\nu = \sum_i \delta_{t_i}$
- p is the purely atomic impact percussions such that $pd\nu = \sum_i p_i \delta_{t_i}$
- dv_S and di_S are singular measures with the respect to $dt + d\eta$.

Impact equations and Smooth Lagrangian dynamics

Substituting the decomposition of measures into the nonsmooth Lagrangian Dynamics, one obtains

Definition (Impact equations)

$$M(q)(v^+ - v^-)d\nu = pd\nu, \tag{23}$$

or

$$M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \tag{24}$$

Definition (Smooth Dynamics between impacts)

$$M(q)\gamma dt + F(t,q,v)dt = fdt \tag{25}$$

or

$$M(q)\gamma^+ + F(t,q,v^+) \quad = \quad f^+ \quad [dt - a.e.] \tag{26}$$

The Moreau's sweeping process of second order

Definition ([12, 13])

A key stone of this formulation is the inclusion in terms of velocity. Indeed, the inclusion (17) is "replaced" by the following inclusion

$$\begin{cases} M(q)dv + F(t,q,v^+)dt = di \\ v^+ = \dot{q}^+ \\ -di \in N_{T_C(q)}(v^+) \end{cases} \tag{27}$$

Comments

This formulation provides a common framework for the nonsmooth dynamics containing inelastic impacts without decomposition.
➜ Foundation for the Moreau–Jean time–stepping approach.

The Moreau's sweeping process of second order

Definition (Tangent cone to a convex set)

C a nonempty convex set in $\boldsymbol{R}^n$ and $x \in C$

$$\mathrm{T}_C(x) = \{t \in \boldsymbol{R}^n \mid t^T s \leq 0 \text{ for all } s \in \mathrm{N}_C(x)\} \tag{28}$$

Interpretation

- Inclusion in terms of the velocity. Viability Lemma
  If $q(t_0) \in C(t_0)$, then

  $$v^+ \in T_C(q), t \geq t_0 \Rightarrow q(t) \in C(t), t \geq t_0$$

  ➜ The unilateral constraints on q are satisfied. The equivalence needs at least an impact inelastic rule.

The Moreau's sweeping process of second order

Velocity level formulation. Index reduction

$$\begin{array}{c} 0 \leq y \perp \lambda \geq 0 \\ \Updownarrow \\ -\lambda \in N_{\boldsymbol{R}^+}(y) \\ \Uparrow \\ -\lambda \in N_{T_{\boldsymbol{R}^+}(y)}(\dot{y}) \\ \Updownarrow \\ \text{if } y \leq 0 \text{ then } 0 \leq \dot{y} \perp \lambda \geq 0 \end{array} \tag{29}$$

The Moreau's sweeping process of second order

The Newton impact rule

$$v^+(t) = -ev^-(t) \tag{30}$$

where e is a coefficient of restitution.

The Newton-Moreau impact rule

$$-\,di \in N_{T_C(q(t))}(v^+(t) + ev^-(t)) \tag{31}$$

where e is a coefficient of restitution.

The Newton-Moreau impact rule in terms of complementarity for C finitely represented

$$\begin{cases} di = \nabla_q g(q,t) dI \\ u(t) = \nabla_q^\top g(q,t) v(t) + \dfrac{\partial g(q,t)}{\partial t} \\ \text{if } g(q(t)) = 0, \mathit{then} \leq dI_\alpha \perp u^+_\alpha(t) + eu^-_\alpha(t) \geq 0 \end{cases} \tag{32}$$

where e is a coefficient of restitution.

The Moreau's sweeping process of second order

Comments

- *The inclusion concerns measures.* Therefore, it is necessary to define what is the inclusion of a measure into a cone.
- *The inclusion in terms of velocity* v^+ rather than of the coordinates q.

Interpretation

- Inclusion of measure, $-di \in K$
  - Case $di = r' dt = f dt$.
$$-f \in K \tag{33}$$
  - Case $di = p_i \delta_i$.
$$-p_i \in K \tag{34}$$

Mathematical results

Finite dimension

- Counter example to uniqueness with $\mathcal{C}^\infty$ data (Schatzman, Percivale)
- Existence and uniqueness in the frictionless case with analytic data (Ballard[3])
- Frictional case.
  - No result in the general case
  - Existence and uniqueness with lumped mass system

Elastodynamics. Infinite dimension

- One dimensional system (wave equation) (Schatzman et al.)
  Question of the impact law and law for the conservation of energy
- Elastic half-space without friction. Existence and uniqueness obtained by Lebeau and schatzman.

Principle of nonsmooth event capturing methods (Time–stepping schemes)

1. A unique formulation of the dynamics is considered. For instance, a dynamics in terms of measures.

$$\begin{cases} -mdv + fdt = di \\ \dot{q} = v^+ \\ 0 \leq di \perp v^+ \geq 0 \text{ if } q \leq 0 \end{cases} \tag{35}$$

2. The time-integration is based on a consistent approximation of the equations in terms of measures. For instance,

$$\int_{]t_k,t_{k+1}]} dv = \int_{]t_k,t_{k+1}]} dv = (v^+(t_{k+1}) - v^+(t_k)) \approx (v_{k+1} - v_k) \tag{36}$$

3. Consistent approximation of measure inclusion.

$$0 \leq di \perp v^+ \geq 0 \text{ if } q \leq 0 \quad \rightarrow \quad \begin{cases} p_{k+1} \approx \displaystyle\int_{]t_k,t_{k+1}]} di \\ 0 \leq p_{k+1} \perp v_{k+1} \geq 0 \quad \text{ if } \tilde{q}_k \leq 0 \end{cases} \tag{37}$$

State–of–the–art

Numerical time–integration methods for Nonsmooth Multibody systems (NSMBS):

Nonsmooth event capturing methods (Time–stepping methods)

- ⊕ robust, stable and proof of convergence
- ⊕ low kinematic level for the constraints
- ⊕ able to deal with finite accumulation
- ⊖ very low order of accuracy even in free flight motions

Nonsmooth event tracking methods (Event–driven methods)

- ⊕ high level integration of free flight motions
- ⊖ no proof of convergence
- ⊖ sensibility to numerical thresholds
- ⊖ reformulation of constraints at higher kinematic levels.
- ⊖ unable to deal with finite accumulation

Two main implementations

- Moreau–Jean time–stepping scheme
- Schatzman–Paoli time–stepping scheme

Moreau–Jean's Time stepping scheme [13, 10]

Principle

$$\begin{cases} M(q_{k+\theta})(v_{k+1}-v_k) - hF_{k+\theta} = p_{k+1} = G(q_{k+\theta})P_{k+1}, & (38a)\\ q_{k+1} = q_k + hv_{k+\theta}, & (38b)\\ u_{k+1} = G^T(q_{k+\theta})\, v_{k+1} & (38c)\\ \begin{array}{ll} 0 \leq u^\alpha_{k+1} + eU^\alpha_k \perp P^\alpha_{k+1} \geq 0 & \text{if} \quad \bar{g}^\alpha_{k,\gamma} \leq 0 \\ P^\alpha_{k+1} = 0 & \text{otherwise} \end{array} . & (38d) \end{cases}$$

with

- $G(q) = \nabla_q g(q)$
- $\theta \in [0,1]$
- $x_{k+\theta} = (1-\theta)x_{k+1} + \theta x_k$
- $F_{k+\theta} = F(t_{k+\theta}, q_{k+\theta}, v_{k+\theta})$
- $\bar{g}_{k,\gamma} = g_k + \gamma h U_k, , \gamma \geq 0$ is a prediction of the constraints.

Schatzman–Paoli's Time stepping scheme [15]

Principle

$$
\begin{cases}
M(q_{k+1})(q_{k+1}-2q_k+q_{k-1}) - h^2 F_{k+\theta} = p_{k+1}, & (39a)\\[2ex]
v_{k+1} = \dfrac{q_{k+1}-q_{k-1}}{2h}, & (39b)\\[2ex]
-p_{k+1} \in N_K\left(\dfrac{q_{k+1}+eq_{k-1}}{1+e}\right), & (39c)
\end{cases}
$$

where N_K defined the normal cone to K.
For $K = \{q \in \mathbb{R}^n, y = g(q) \geq 0\}$

$$
0 \leq g\left(\frac{q_{k+1}+eq_{k-1}}{1+e}\right) \perp \nabla g\left(\frac{q_{k+1}+eq_{k-1}}{1+e}\right) P_{k+1} \geq 0 \qquad (40)
$$

Comparison

Shared mathematical properties

- Convergence results for one constraints
- Convergence results for multiple constraints problems with acute kinetic angles
- No theoretical proof of order

Mechanical properties

- Position vs. velocity constraints
- Respect of the impact law in one step (Moreau) vs. Two-steps(Schatzman)
- Linearized constraints rather than nonlinear.

But ...

But

Both schemes are quite inaccurate and "dissipate" a lot of energy of vibrations. This is a consequence of the first order approximation of the smooth forces term F

Recent improvements

- Nonsmooth generalized α schemes [6, 4]
- Time discontinuous Galerkin methods [18, 19]
- Stabilized index-2 formulation [2, 1]
- Stabilized index-1 formulation [5]

Nonsmooth generalized-α schemes

The nonsmooth generalized α scheme

Splitting the dynamics between smooth and nonsmooth part

$$\mathrm{d}\mathbf{w} = \mathrm{d}\mathbf{v} - \dot{\tilde{\mathbf{v}}}\,\mathrm{d}t \tag{41}$$

Smooth (non-impulsive) part

Solutions of the following DAE

$$\begin{aligned}
\dot{\tilde{\mathbf{q}}} &= \tilde{\mathbf{v}} && (42a)\\
\mathbf{M}(\mathbf{q})\,\dot{\tilde{\mathbf{v}}} - \mathbf{g}_{\mathbf{q}}^{T}(\mathbf{q})\,\tilde{\boldsymbol{\lambda}} &= \mathbf{f}(\mathbf{q},\mathbf{v},t) && (42b)\\
\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}(\mathbf{q})\,\tilde{\mathbf{v}} &= \mathbf{0} && (42c)\\
\tilde{\boldsymbol{\lambda}}^{\mathcal{U}} &= \mathbf{0} && (42d)
\end{aligned}$$

with the initial value $\tilde{\mathbf{v}}(t_n) = \mathbf{v}(t_n)$, $\tilde{\mathbf{q}}(t_n) = \mathbf{q}(t_n)$.

The nonsmooth generalized α scheme

Splitting the dynamics between smooth and nonsmooth part

$$\dot{\mathbf{q}} = \mathbf{v} \tag{43a}$$

$$\mathrm{d}\mathbf{v} = \mathrm{d}\mathbf{w} + \dot{\tilde{\mathbf{v}}}\,\mathrm{d}t \tag{43b}$$

$$\mathbf{M}(\mathbf{q})\,\dot{\tilde{\mathbf{v}}} - \mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}},T}\,\tilde{\boldsymbol{\lambda}}^{\overline{\mathcal{U}}} = \mathbf{f}(\mathbf{q},\mathbf{v},t) \tag{43c}$$

$$\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}\,\tilde{\mathbf{v}} = \mathbf{0} \tag{43d}$$

$$\tilde{\boldsymbol{\lambda}}^{\mathcal{U}} = \mathbf{0} \tag{43e}$$

$$\mathbf{M}(\mathbf{q})\,\mathrm{d}\mathbf{w} - \mathbf{g}_{\mathbf{q}}^{T}\,(\mathrm{d}\mathbf{i} - \tilde{\boldsymbol{\lambda}}\,\mathrm{d}t) = \mathbf{0} \tag{43f}$$

$$\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}\mathbf{v} = \mathbf{0} \tag{43g}$$

$$\text{if } g^j(\mathbf{q}) \leq 0 \text{ then } 0 \leq g_{\mathbf{q}}^j\,\mathbf{v} + e\,g_{\mathbf{q}}^j\,\mathbf{v}^- \perp \mathrm{d}i^j \geq 0, \quad \forall j \in \mathcal{U} \tag{43h}$$

The nonsmooth generalized α scheme

GGL approach to stabilize the constraints at the position level

The equations of motion become

$$\begin{aligned}
\mathbf{M}(\mathbf{q})\,\dot{\mathbf{q}} - \mathbf{g}_{\mathbf{q}}^T\,\boldsymbol{\mu} &= \mathbf{M}(\mathbf{q})\,\mathbf{v} && (44a)\\
\cancel{\dot{\mathbf{q}} = \mathbf{v}} \rightarrow \qquad \mathbf{g}^{\overline{\mathcal{U}}}(\mathbf{q}) &= \mathbf{0} && (44b)\\
\mathbf{0} \leq \mathbf{g}^{\mathcal{U}}(\mathbf{q}) &\perp \boldsymbol{\mu}^{\mathcal{U}} \geq \mathbf{0} && (44c)\\
\mathrm{d}\mathbf{v} &= \mathrm{d}\mathbf{w} + \dot{\tilde{\mathbf{v}}}\,\mathrm{d}t && (44d)\\
\mathbf{M}(\mathbf{q})\,\dot{\tilde{\mathbf{v}}} - \mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}},T}\,\tilde{\boldsymbol{\lambda}}^{\overline{\mathcal{U}}} &= \mathbf{f}(\mathbf{q},\mathbf{v},t) && (44e)\\
\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}\,\tilde{\mathbf{v}} &= \mathbf{0} && (44f)\\
\tilde{\boldsymbol{\lambda}}^{\mathcal{U}} &= \mathbf{0} && (44g)\\
\mathbf{M}(\mathbf{q})\,\mathrm{d}\mathbf{w} - \mathbf{g}_{\mathbf{q}}^T\,(\mathrm{d}\mathbf{i} - \tilde{\boldsymbol{\lambda}}\,\mathrm{d}t) &= \mathbf{0} && (44h)\\
\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}\mathbf{v} &= \mathbf{0} && (44i)\\
\text{if } g^j(\mathbf{q}) \leq 0 \text{ then } 0 \leq g_{\mathbf{q}}^j\,\mathbf{v} + e\,g_{\mathbf{q}}^j\,\mathbf{v}^- &\perp \mathrm{d}i^j \geq 0, \quad \forall j \in \mathcal{U} && (44j)
\end{aligned}$$

The nonsmooth generalized α scheme

Velocity jumps and position correction

The multipliers $\mathbf{\Lambda}(t_n;t)$ and $\boldsymbol{\nu}(t_n;t)$ are defined as

$$\begin{aligned}\mathbf{\Lambda}(t_n;t) &= \int_{(t_n,t]} (\mathrm{d}\mathbf{i} - \tilde{\boldsymbol{\lambda}}(\tau)\,\mathrm{d}\tau) && \text{(45a)}\\ \boldsymbol{\nu}(t_n;t) &= \int_{t_n}^{t} (\boldsymbol{\mu}(\tau) + \mathbf{\Lambda}(t_n;\tau))\,\mathrm{d}\tau && \text{(45b)}\end{aligned}$$

with $\mathbf{\Lambda}(t_n;t_n) = \boldsymbol{\nu}(t_n;t_n) = \mathbf{0}$.
The velocity jump and position correction variables

$$\begin{aligned}\mathbf{W}(t_n;t) &= \int_{(t_n,t]} \mathrm{d}\mathbf{w} = \mathbf{v}(t) - \tilde{\mathbf{v}}(t) && \text{(46a)}\\ \mathbf{U}(t_n;t) &= \int_{t_n}^{t} (\dot{\mathbf{q}} - \tilde{\mathbf{v}})\mathrm{d}t = \mathbf{q}(t) - \tilde{\mathbf{q}}(t) && \text{(46b)}\end{aligned}$$

➜ Low-order approximation of impulsive terms.
➜ Higher–order approximation of non impulsive terms.

The nonsmooth generalized α scheme

$$
\begin{aligned}
\mathbf{M}(\mathbf{q}_{n+1})\mathbf{U}_{n+1} - \mathbf{g}^{T}_{\mathbf{q},n+1}\,\boldsymbol{\nu}_{n+1} &= \mathbf{0} && (47a)\\
\mathbf{g}^{\overline{\mathcal{U}}}(\mathbf{q}_{n+1}) &= \mathbf{0} && (47b)\\
\mathbf{0} \leq \mathbf{g}^{\mathcal{U}}(\mathbf{q}_{n+1}) \perp \boldsymbol{\nu}^{\mathcal{U}}_{n+1} &\geq \mathbf{0} && (47c)\\
\mathbf{M}(\mathbf{q}_{n+1})\dot{\tilde{\mathbf{v}}}_{n+1} - \mathbf{f}(\mathbf{q}_{n+1}, \mathbf{v}_{n+1}, t_{n+1}) - \mathbf{g}^{\overline{\mathcal{U}},T}_{\mathbf{q},n+1}\,\tilde{\boldsymbol{\lambda}}^{\overline{\mathcal{U}}}_{n+1} &= \mathbf{0} && (47d)\\
\mathbf{g}^{\overline{\mathcal{U}}}_{\mathbf{q},n+1}\,\tilde{\mathbf{v}}_{n+1} &= \mathbf{0} && (47e)\\
\mathbf{M}(\mathbf{q}_{n+1})\mathbf{W}_{n+1} - \mathbf{g}^{T}_{\mathbf{q},n+1}\boldsymbol{\Lambda}_{n+1} &= \mathbf{0} && (47f)\\
\mathbf{g}^{\overline{\mathcal{U}}}_{\mathbf{q},n+1}\mathbf{v}_{n+1} &= \mathbf{0} && (47g)\\
\text{if } g^{j}(\mathbf{q}^{*}_{n+1}) \leq 0 \text{ then } 0 \leq g^{j}_{\mathbf{q},n+1}\,\mathbf{v}_{n+1} + e\,g^{j}_{\mathbf{q},n}\,\mathbf{v}_{n} \perp \Lambda^{j}_{n+1} &\geq 0, \forall j \in \mathcal{U}
\end{aligned}
$$

The nonsmooth generalized α scheme

Nonsmooth generalized α-scheme

$$\begin{aligned}
\tilde{\mathbf{q}}_{n+1} &= \mathbf{q}_n + h\mathbf{v}_n + h^2(0.5-\beta)\mathbf{a}_n + h^2\beta\mathbf{a}_{n+1} && \text{(48a)}\\
\mathbf{q}_{n+1} &= \tilde{\mathbf{q}}_{n+1} + \mathbf{U}_{n+1} && \text{(48b)}\\
\tilde{\mathbf{v}}_{n+1} &= \mathbf{v}_n + h(1-\gamma)\mathbf{a}_n + h\gamma\mathbf{a}_{n+1} && \text{(48c)}\\
\mathbf{v}_{n+1} &= \tilde{\mathbf{v}}_{n+1} + \mathbf{W}_{n+1} && \text{(48d)}\\
(1-\alpha_m)\mathbf{a}_{n+1} + \alpha_m\mathbf{a}_n &= (1-\alpha_f)\dot{\tilde{\mathbf{v}}}_{n+1} + \alpha_f\dot{\tilde{\mathbf{v}}}_n && \text{(48e)}
\end{aligned}$$

Special cases

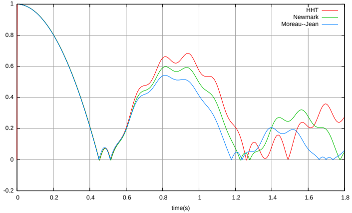
- $\alpha_m = \alpha_f = 0$ ➜ Nonsmooth Newmark
- $\alpha_m = 0, \alpha_f \in [0, 1/3]$ ➜ Nonsmooth Hilber-Hughes–Taylor (HHT)

Spectral radius at infinity $\rho_\infty \in [0, 1]$

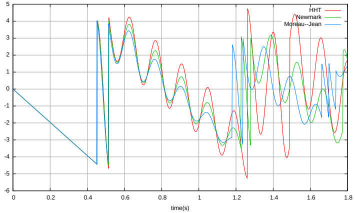
$$\alpha_m = \frac{2\rho_\infty - 1}{\rho_\infty + 1}, \quad \alpha_f = \frac{\rho_\infty}{\rho_\infty + 1}, \quad \beta = \frac{1}{4}(\gamma + \frac{1}{2})^2. \qquad (49)$$

Numerical Illustrations

Two ball oscillator with impact.

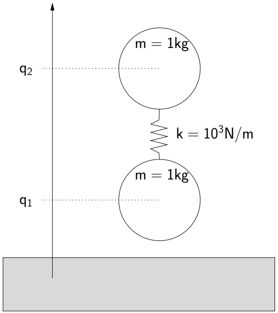


Position of the first ball



Velocity of the first ball

Time–step : $h = 2e-3$.
Moreau ($\theta = 1.0$).
Newmark ($\gamma = 1.0, \beta = 0.5, \alpha_m = \alpha_f = 0$).
HHT ($\gamma = 1.0, \beta = 0.5, \alpha_f = 0.1, \alpha_m = 0$)



Numerical Illustrations

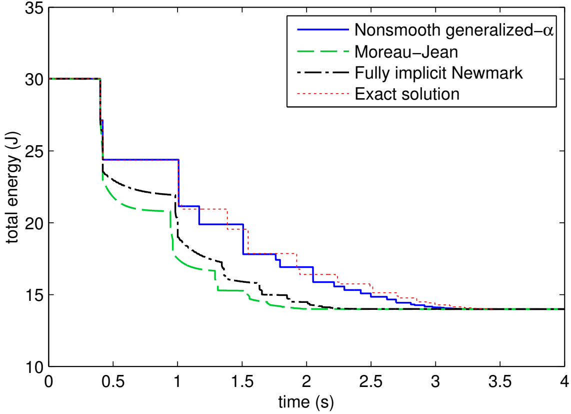
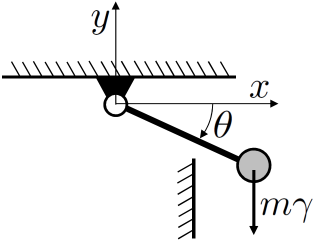


Figure 7. Numerical results for the total energy of the bouncing oscillator.

Numerical Illustrations

Bouncing Pendulum

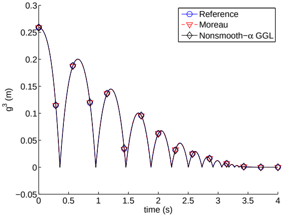
$$\begin{aligned} \mathbf{q} &= [x, y, \theta]^T \\ g_1(\mathbf{q}) &= x - l\cos\theta = 0 \\ g_2(\mathbf{q}) &= y - l\sin\theta = 0 \\ g_3(\mathbf{q}) &= x - \sqrt{2}/2 \geq 0 \end{aligned}$$

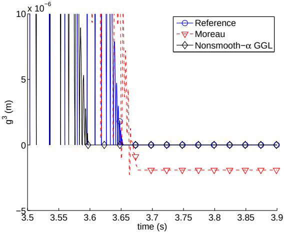


Time–step : $h = 2e-3$.
Moreau ($\theta = 1/1.8$).
α-schemes ($\rho_\infty = 0.8$)

$e = 0.8$

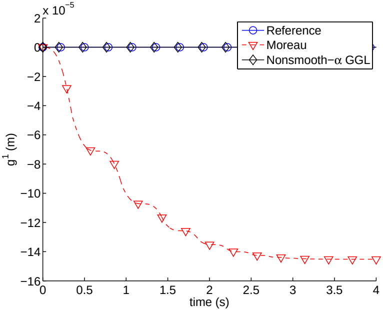
Unilateral constraint

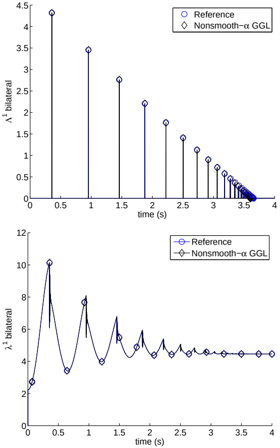




Numerical Illustrations

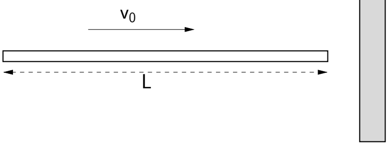
Bouncing Pendulum





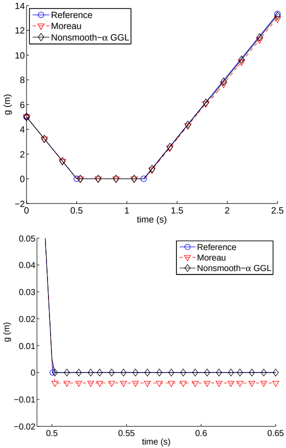
Numerical Illustrations

Impacting elastic bar



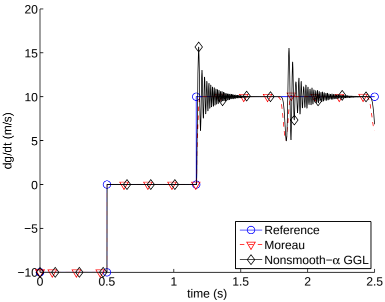
$g_3(\mathbf{q}) = x_1 \geq 0$
$e = 0.0$
200 finite elements
Time–step : $h = 2e - 3$.
Moreau ($\theta = 1/1.8$).
α-schemes ($\rho_\infty = 0.8$)

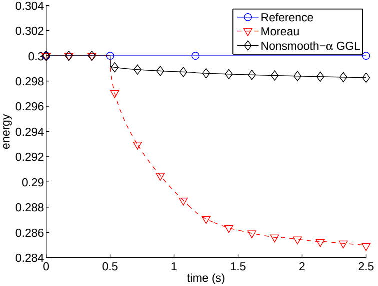
Unilateral constraint



Numerical Illustrations

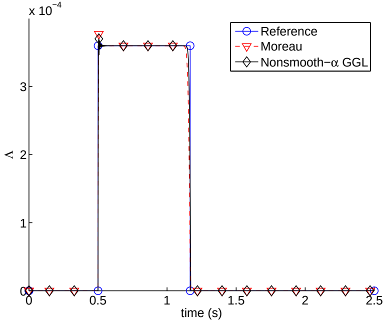
Impacting elastic bar

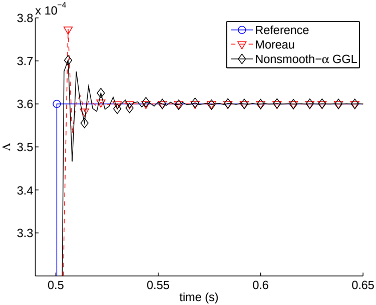




Numerical Illustrations

Impacting elastic bar





Perspectives

1. Rolling friction and fracture for rock-fall trajectory
   - Numerical algorithms for second order cones.
   - Cohesive zone modeling of interfaces with damage, contact and friction (Frémond-like).
2. Rock interaction with elasto-plastic obstacles
   - Plasticity and damage as nonsmooth behavior law (complementarity and differential inclusions)
   - Numerical methods based on modern optimization techniques
3. Debris flows with rigid bodies and obstacles
   - Debris Flows with large objects and accumulation and contact.
   - Non Newtonian fluids with non-associated plasticity (Bingham, Drucker-Prager, Mohr-Coulomb)
   - Material Point Method with behavior laws based in second order cones for elastic domains.
4. High performance computing

Thank you for your attention.

V. Acary. Projected event-capturing time-stepping schemes for nonsmooth mechanical systems with unilateral contact and Coulomb's friction. Computer Methods in Applied Mechanics and Engineering, 256:224 – 250, 2013. ISSN 0045-7825. doi: 10.1016/j.cma.2012.12.012. URL http://www.sciencedirect.com/science/article/pii/S0045782512003829.

V. Acary. Energy conservation and dissipation properties of time-integration methods for the nonsmooth elastodynamics with contact. Zeitschrift für Angewandte Mathematik und Mechanik, submitted, 2014.

P. Ballard. The dynamics of discrete mechanical systems with perfect unilateral constraints. arma, 154:199–274, 2000.

O. Brüls, V. Acary, and A. Cardona. Simultaneous enforcement of constraints at position and velocity levels in the nonsmooth generalized-α scheme. Computer Methods in Applied Mechanics and Engineering, 281:131–161, Nov. 2014. doi: 10.1016/j.cma.2014.07.025. URL http://hal.inria.fr/hal-01059823.

O. Brüls, V. Acary, and A. Cardona. On the Constraints Formulation in the Nonsmooth Generalized-α Method. In S. I. Publishing, editor, Advanced Topics in Nonsmooth Dynamics. Transactions of the European Network for Nonsmooth Dynamics, pages 335–374. June 2018. doi: 10.1007/978-3-319-75972-2_9. URL https://hal.inria.fr/hal-01878550.

Q. Z. Chen, V. Acary, G. Virlez, and O. Brüls. A nonsmooth generalized-α scheme for flexible multibody systems with unilateral constraints. International Journal for Numerical Methods in Engineering, 96(8):487–511, 2013. ISSN 1097-0207. doi: 10.1002/nme.4563. URL http://dx.doi.org/10.1002/nme.4563.

F. Clarke. Generalized gradients and its applications. Transactions of A.M.S., 205: 247–262, 1975.

F. Clarke. Optimization and Nonsmooth analysis. Wiley, New York, 1983.

G. De Saxcé. Une généralisation de l'inégalité de fenchel et ses applications aux lois constitutives. Comptes Rendus de l'Académie des Sciences, t 314,série II:125–129, 1992.

M. Jean. The non smooth contact dynamics method. Computer Methods in Applied Mechanics and Engineering, 177:235–257, 1999. Special issue on computational modeling of contact and friction, J.A.C. Martins and A. Klarbring, editors.

B. Mordukhovich. Generalized differential calculus for nonsmooth ans set-valued analysis. Journal of Mathematical analysis and applications, 183:250–288, 1994.

J. Moreau. Liaisons unilatérales sans frottement et chocs inélastiques. cras, 296 série II:1473–1476, 1983.

J. Moreau. Unilateral contact and dry friction in finite freedom dynamics. In J. Moreau and P. Panagiotopoulos, editors, Nonsmooth mechanics and applications, number 302 in CISM, Courses and lectures, pages 1–82. CISM 302, Spinger Verlag, Wien- New York, 1988. Formulation mathematiques tire du livre Contacts mechanics.

J. J. Moreau. On unilateral constraints, friction and plasticity. In G. Capriz and e. G. Stampacchia, editors, New Variational Techniques in Mathematical Physics (C.I.M.E. II ciclo 1973, Edizioni Cremonese, Roma, 1974), pages 173–322, 1974.

L. Paoli and M. Schatzman. A numerical scheme for impact problems I: The one-dimensional case. SIAM Journal of Numerical Analysis, 40(2):702–733, 2002.

M. Schatzman. Sur une classe de problèmes hyperboliques non linéaires. Comptes Rendus de l'Académie des Sciences Série A, 277:671–674, 1973.

M. Schatzman. A class of nonlinear differential equations of second order in time. nla, 2(3):355–373, 1978.

T. Schindler and V. Acary. Timestepping schemes for nonsmooth dynamics based on discontinuous galerkin methods: Definition and outlook. Mathematics and Computers in Simulation, 95:180–199, 2013. ISSN 0378-4754. doi: 10.1016/j.matcom.2012.04.012. URL http://www.sciencedirect.com/science/article/pii/S0378475412001231.

T. Schindler, S. Rezaei, J. Kursawe, and V. Acary. Half-explicit timestepping schemes on velocity level based on time-discontinuous galerkin methods. Computer methods in Applied Mechanics in Engineering, 290(15):250–276, 2015.