Local analysis of dynamical systems and application to nonlinear waves

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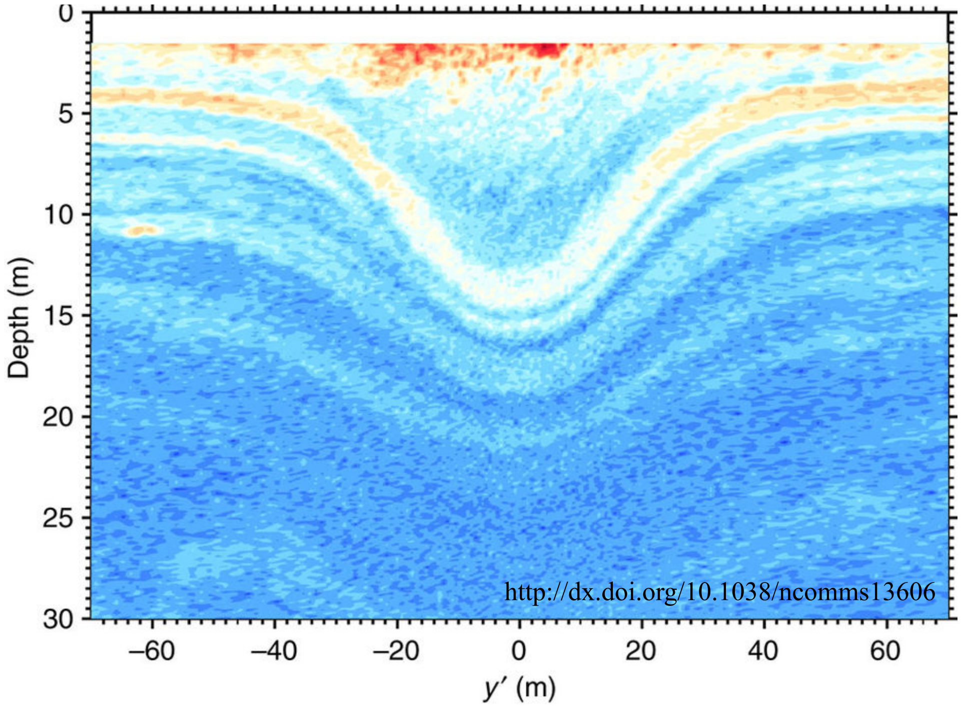
Méthodes de dynamique non linéaire pour l'ingénierie des structures

Solitary waves : spatially localized traveling waves

typical rest state perturbation $\approx \dfrac{1}{\cosh^2(x - c\,t)}$ or $\dfrac{\cos\left[q(x - c\,t)\right]}{\cosh(x - c\,t)}$ c: wave velocity

= balance between dispersion and nonlinearity

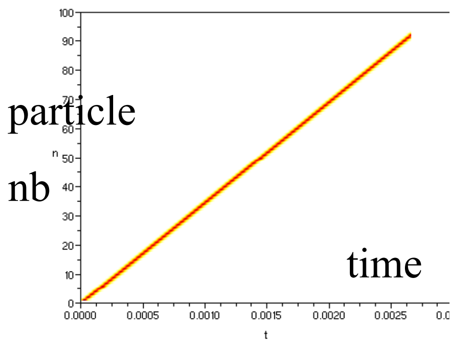
Internal solitary wave in a density-stratified fluid (Saguenay Fjord), Bourgault et al, Nature Com.'16



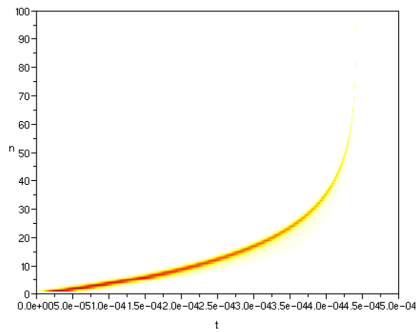
Impact in a chain of beads

Melo et al, Phys Rev E 73, '06

Numerical computation of contact forces :



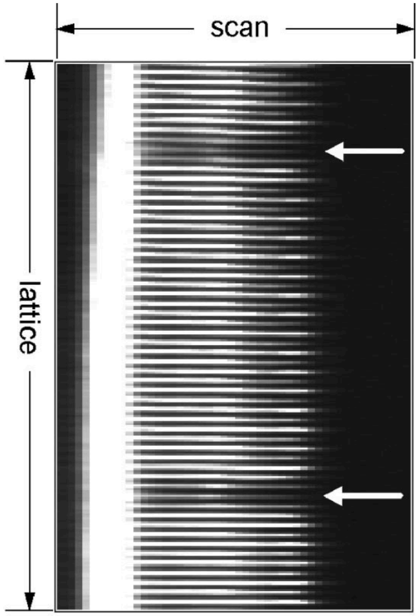
homogeneous chain

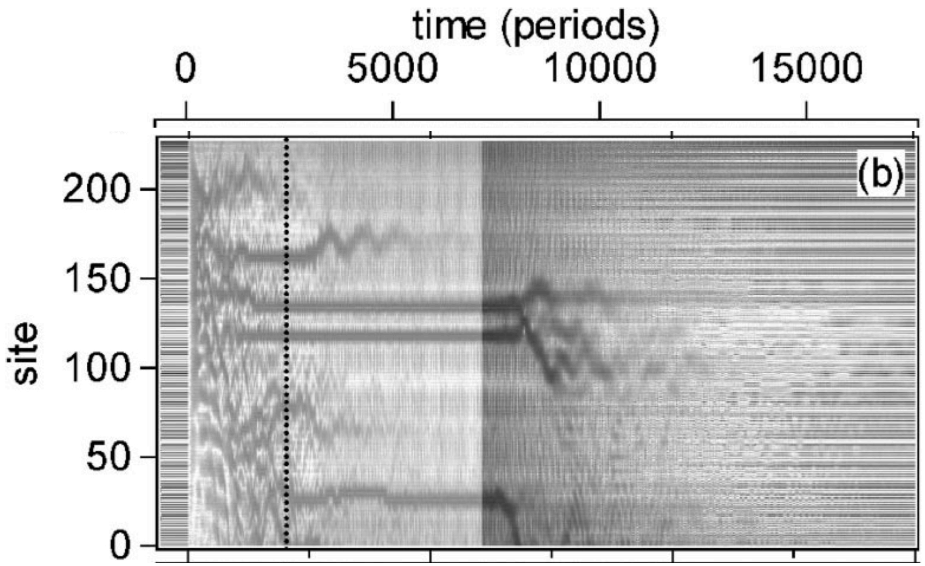


tapered chain

Breathers : spatially localized oscillations

Micromechanical oscillator array : (Sato et al, Chaos 13, ‘03)





typically : field $\approx$ rest state + $\dfrac{\cos[q(x-c\,t)-\Omega t]}{\cosh(x-c\,t)}$ x: discrete or continuous space coordinate

$c=0 \Rightarrow$ static breather (time-periodic, or more general types of oscillations)

$c \neq 0 \Rightarrow$ traveling breather ($\approx$ time-periodic in moving frame)

Static breathers common in nonlinear lattices (large discrete systems) :

bounded phonon band, spectral gaps, nonresonant breather frequencies

An application : nonlinear granular metamaterials

Localized nonlinear defect mode / breather in a granular chain :

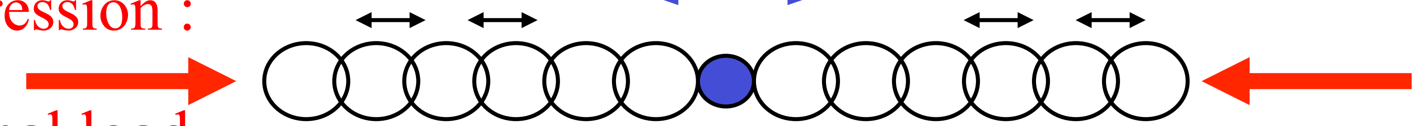
small oscillations

large periodic oscillations

(exponential spatial decay)

Precompression :

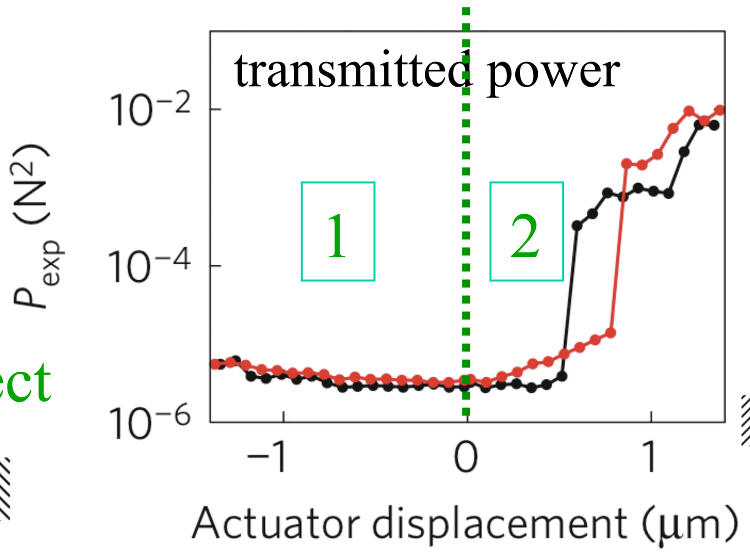
External load



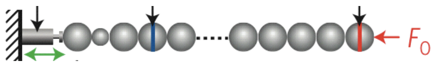
smaller bead

Observation of « acoustic diode » behavior induced by defect mode :

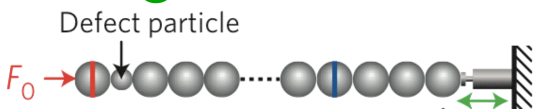
Boechler, Theocharis and Daraio, Nature Materials 10 (2011), 665-668.



2- driving near defect



1- driving far from defect



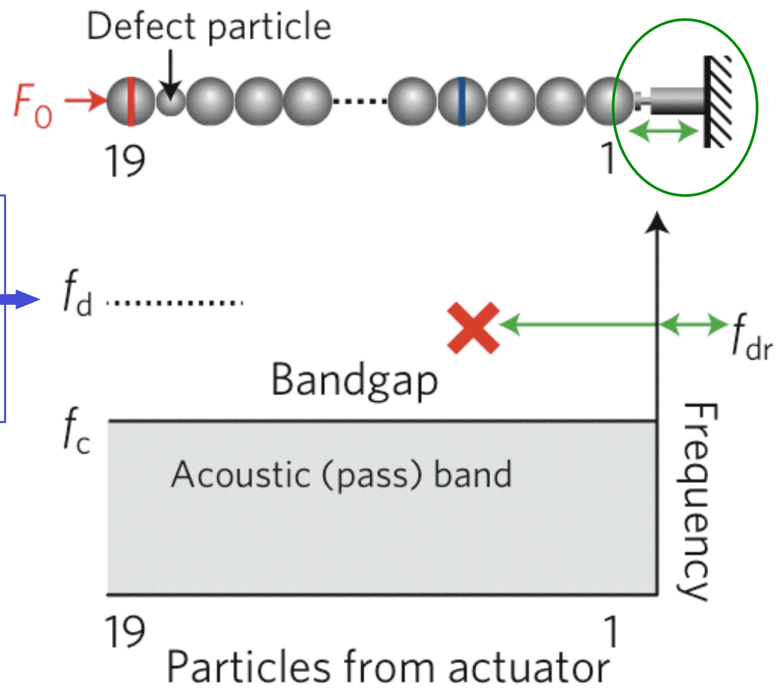
Link between defect mode and acoustic diode behavior :

defect mode

linear

frequency

1- driving far from defect

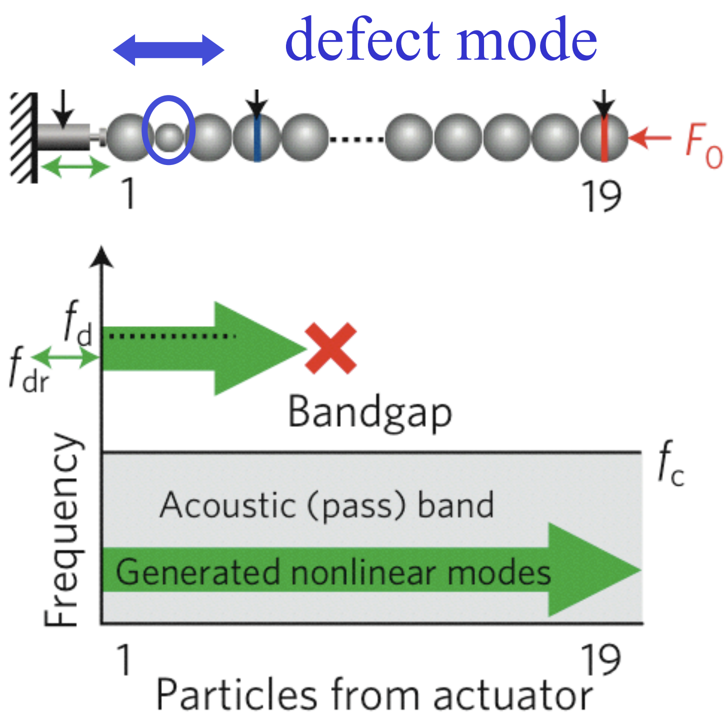


No periodic wavetrains, and

defect mode not excited

➜ low energy transmission

2- driving near defect



defect mode unstable through

bifurcations (quasi-period. and chaos)

➜ high energy transmission

Module outline :

I – introduction to nonlinear localized waves in lattices :

models supporting solitary waves, breathers

approx via amplitude PDE (application to precompressed granular chains)

II - center manifolds for maps (finite and infinite dim)

application : bifurcations of time-periodic breathers in lattices

III - center manifolds for differential equations (finite and infinite dim)

bifurcations of propagating localized waves in lattices

IV - modulation equations for strongly nonlinear spatial couplings

-- strongly nonlinear discrete / continuous NLS equations
-- application : breathers in uncompressed granular chains

I – Introduction to nonlinear localized waves in lattices

Outline :

1 -- Fermi-Pasta-Ulam model, application to granular chains

2 -- solitary waves, Korteweg-de Vries (KdV) approximation

3 -- breathers, nonlinear Schrödinger (NLS) approximation,
application to oscillators chains

1 - Fermi-Pasta-Ulam model and applic. to granular chains

Fermi-Pasta-Ulam (FPU) lattice (E.Fermi, J.Pasta and S.Ulam, 1955) :

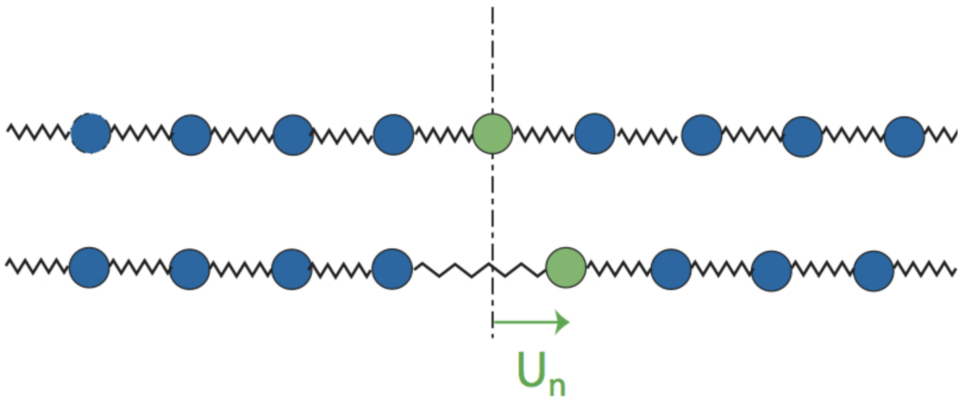
$$\frac{d^2 u_n}{dt^2} = V'(u_{n+1} - u_n) - V'(u_n - u_{n-1}), \; n \in \mathbb{Z}$$

Infinite-dimensional hamiltonian system

$$H = \sum_{n=-\infty}^{+\infty} \frac{1}{2}\left(\frac{du_n}{dt}\right)^2 + V(u_{n+1} - u_n), \qquad u_n(t) \in \mathbb{R}.$$

Anharmonic interaction potential V : $V'(0) = 0, V''(0) > 0.$

- mechanical mass-spring system :

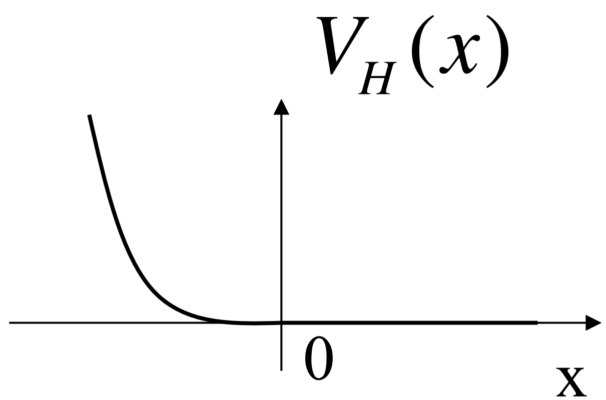


- ionic crystals (Sievers and Takeno 1988)

FPU models for granular chains

Model 1 : fully nonlinear Hertz potential

$$\ddot{x}_n = V_H'(x_{n+1} - x_n) - V_H'(x_n - x_{n-1})$$



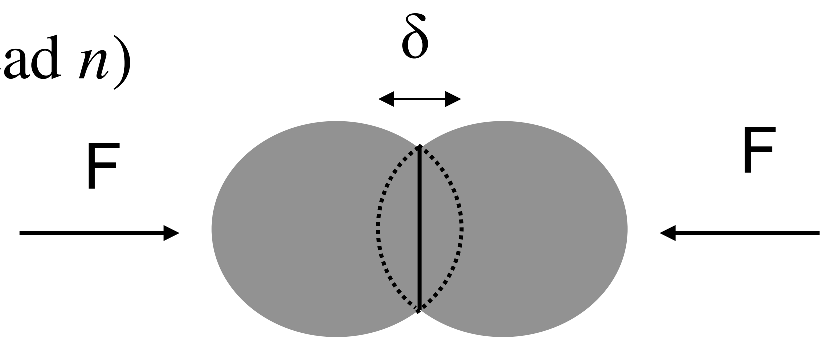
$$V_H(x) = \frac{1}{\alpha+1}(-x)_+^{\alpha+1}, \quad \alpha > 1, \quad (a)_+ = \mathrm{Max}(a,0)$$

$$V_H''(0) = 0$$

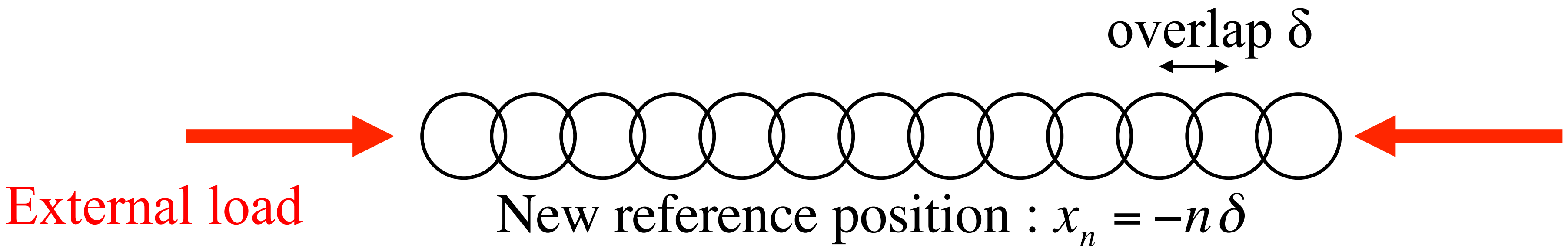
Hertz potential for α=3/2 :

contact force between two spherical beads : $F \approx \delta^{3/2}$

granular chain : (x_n = displacement of bead n)



Model 2 : Hertzian interaction potential including precompression



New reference position : $x_n = -n\delta$

displacement $u_n(t)$
from reference position

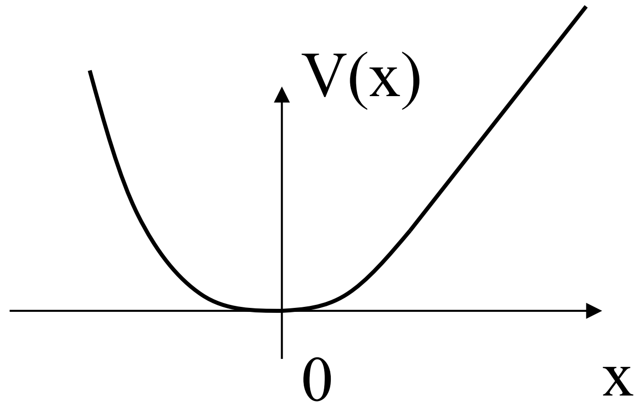
$$\ddot{u}_n = V'(u_{n+1} - u_n) - V'(u_n - u_{n-1}) \qquad \text{(FPU)}$$

Renormalized potential:

$$V(x) = V_H(x-\delta) - V_H'(-\delta)\,x - V_H(-\delta)$$

$$V''(0) = \alpha\,\delta^{\alpha-1} := c_s^2 > 0$$

$$c_s = \text{"sound velocity"}$$



Linear approximation for small amplitude oscillations :

$$V(x) \approx \frac{c_s^2}{2} x^2$$

$$\ddot{u}_n = c_s^2 (u_{n+1} - 2u_n + u_{n-1}) \qquad n \in \mathbb{Z}$$

Linear periodic traveling waves ("Phonons") :

$$u_n(t) = a \cos(qn - \omega_q t + \varphi)$$

$$\omega_q = \pm 2 c_s |\sin(q/2)| \qquad \text{(dispersion relation)}$$

Dispersive equation : the group velocity $\dfrac{d\omega_q}{dq}$ varies with q

→ localized initial conditions disperse : $\|(u(t),\dot{u}(t))\|_\infty \leq \dfrac{C}{(1+|t|)^{1/3}} \|(u(0),\dot{u}(0))\|_1$

(Mielke and Patz, Appl. Anal. 89, 2010)

$$u(t) = (u_n(t))_n$$

➜ no robust localized waves (solitary waves, breathers) in homogeneous (or periodic) linear chains !

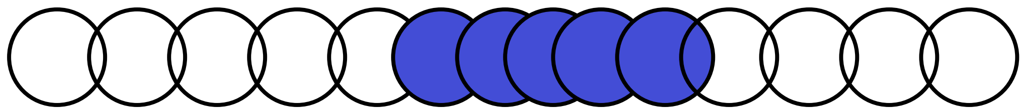
2 – Solitary waves and KdV approximation

In FPU: nonlinearity compensates linear dispersion ⟹ solitary waves

granular chain

with precompression :

solitary wave : c → $c > c_s = \sqrt{\alpha\, \delta^{\alpha-1}}$



At small amplitudes :

$$u_{n+1}(t) - u_n(t) \approx -\frac{c - c_s}{\mathrm{ch}^2[\sqrt{c-c_s}\,(n - c\,t)]}$$

Mathematical theory of FPU solitary waves :

existence theorems, stability, continuum limit (KdV, error bounds), two-soliton solutions, generalized solitary waves with dispersive « tails »…

Kalyakin (‘89), Friesecke and Wattis (‘94), Smets and Willem (‘97), Friesecke and Pego (‘99,’02,’04), Schneider and Wayne (‘00), Iooss (‘00), Pankov and Pfügler (‘00), Friesecke and Matthies (‘02), Treschev(‘04), Iooss and G.J. (‘05), Bambusi and Ponno (‘05,’06), Hoffman and Wayne (‘08,’09), G.J. and Pelinovsky (‘14)…

Approximate solitary wave solutions in the KdV continuum limit

$$\ddot{u}_n = V'(u_{n+1} - u_n) - V'(u_n - u_{n-1}) \qquad \text{(FPU)}$$

Assume $V''(0) > 0,\ V^{(3)}(0) \neq 0$

Renormalization (rescaling of t and u) $\Rightarrow$ $V''(0) = 1,\ V^{(3)}(0) = -1/2$

Ex: rescaled dynamical equations for a granular chain with precompression

(relative displacements small wrt precompression)

$$\ddot{u}_n = \frac{2}{3}\left((1 + u_{n-1} - u_n)^{3/2} - (1 + u_n - u_{n+1})^{3/2}\right)$$

Taylor expansion of V and truncation at order 3 :

$$\ddot{u}_n - \Delta u_n = -\frac{1}{4}\left((u_{n+1} - u_n)^2 - (u_n - u_{n-1})^2\right)$$

$$\Delta u_n = u_{n+1} - 2u_n + u_{n-1}$$

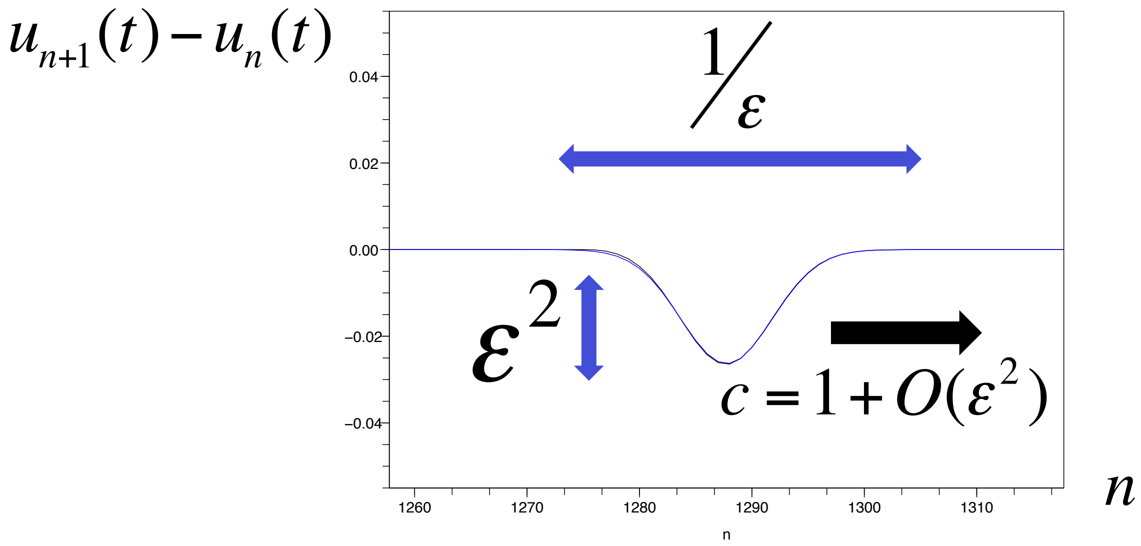
Approximate solitary wave solutions in the KdV continuum limit

$$\ddot{u}_n - \Delta u_n = -\frac{1}{4}\left((u_{n+1} - u_n)^2 - (u_n - u_{n-1})^2\right)$$

$$\Delta u_n = u_{n+1} - 2u_n + u_{n-1}$$

⇨ Ansatz for small amplitude long waves : ε : small parameter

$$u_n(t) = \varepsilon\, u(\xi, T) + \varepsilon^2 R(\xi, T) \qquad \xi = \varepsilon(n - t) \qquad T = \varepsilon^3 t$$



$$\left.\begin{aligned}\partial_t^2 &= \varepsilon^2\partial_\xi^2 - 2\varepsilon^4\partial_{\xi T}^2 + O(\varepsilon^6)\\ \Delta &= \varepsilon^2\partial_\xi^2 + \frac{1}{12}\varepsilon^4\partial_\xi^4 + O(\varepsilon^6)\end{aligned}\right\}(\partial_t^2-\Delta)u_n = -2\varepsilon^5\partial_{\xi T}^2 u - \frac{1}{12}\varepsilon^5\partial_\xi^4 u + O(\varepsilon^6)$$

$$(u_{n+1}-u_n)^2-(u_n-u_{n-1})^2 = \varepsilon^5\partial_\xi\left[(\partial_\xi u)^2\right]+O(\varepsilon^6)$$

Neglect $O(\varepsilon^6)$ terms in :

$$\ddot{u}_n - \Delta u_n = -\frac{1}{4}\left((u_{n+1}-u_n)^2-(u_n-u_{n-1})^2\right)$$

$\rightarrow$ PDE satisfied by $y(\xi,\tau) = -\partial_\xi u$ $\quad\left(\tau = \frac{T}{24}\right)$

Korteweg-de Vries (KdV) equation :

nonlinearity (Burgers) + dispersion (Airy)

$$\partial_\tau y + 6y\partial_\xi y + \partial_\xi^3 y = 0$$

Traveling wave solutions of KdV : $y(\xi,\tau) = z(\eta) \qquad \eta = \xi - v\tau$

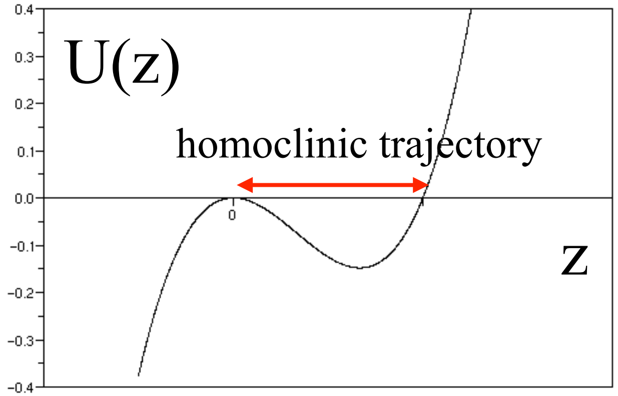
⇨ Stationary KdV equation (integrable ODE):

$$-v\partial_\eta z + 6z\partial_\eta z + \partial_\eta^3 z = 0$$

By integrating once (and canceling the integration constant) :

$$\partial_\eta^2 z + U'(z) = 0, \qquad U(z) = z^3 - \frac{v}{2} z^2$$

case v>0



KdV soliton solution :

$$z(\eta) = \frac{v}{2}\ \mathrm{sech}^2\left(\sqrt{\frac{v}{4}}\ \eta\right)$$

We introduce the wave velocity : $c = 1 + \frac{v}{24}\varepsilon^2$

The KdV soliton $z(\eta) = \frac{v}{2}\ \mathrm{sech}^2\left(\sqrt{\frac{v}{4}}\ \eta\right)$ yields the following approximate solitary wave solutions for the granular chain, parameterized by $c > 1$:

$$u_{n+1}(t) - u_n(t) = -\frac{12(c-1)}{\mathrm{ch}^2[\sqrt{6(c-1)}\ (n - c\,t)]} \qquad (c_s = 1)$$

Qualitative properties : small amplitude « long waves »

→ supersonic wave velocity $c > c_s$

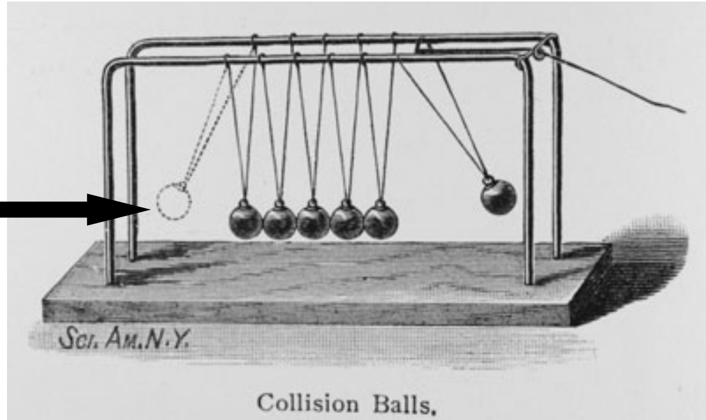
→ wave amplitude $\approx\ c - c_s$

→ exponential decay, wave width $\approx\ 1/\sqrt{c - c_s}$

Solitary waves without precompression

Solitary wave in Newton's cradle :

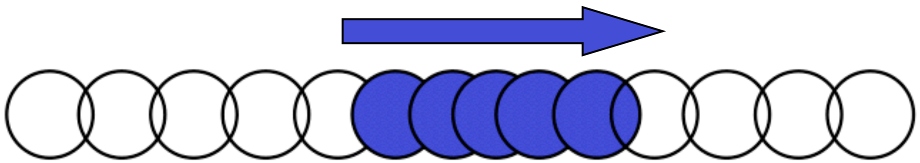
Impacting bead



bead ejection

In fact, multiple impacts !

Nesterenko (84) :
propagation of a solitary wave

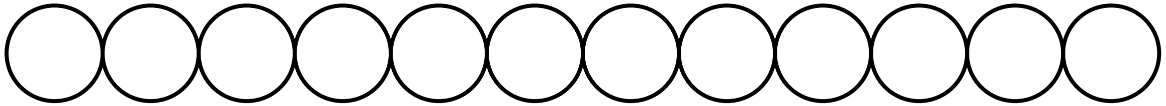


$$\text{contact forces} = f(n - c\,t), \qquad \lim_{|\xi|\to\infty} f(\xi) = 0$$

(n: bead index)

Experimental studies of Nesterenko's solitary wave :
Lazaridi and Nesterenko (85), Coste, Falcon and Fauve (97), Falcon et al (98)

Solitary waves without precompression



←Reference configuration

$$\ddot{x}_n = (x_{n-1} - x_n)_+^{3/2} - (x_n - x_{n+1})_+^{3/2}$$

Nesterenko's solitary wave (84) : formal continuum limit

Approximate solution with compact support :

$$x_n(t) = y(n - c\,t), \quad y'(\xi) \approx -c^4 \sin^4\left(\sqrt{\frac{2}{5}}\,\xi\right) \text{ for } 0 \le \xi \le \pi\sqrt{\frac{5}{2}}, \quad y'(\xi) = 0 \text{ elsewhere}$$

Solitary wave width = 5 balls ➔ not a long wave

Mathematical results on solitary waves: existence, doubly-exponential decay

Friesecke and Wattis '94, MacKay '99, Ji and Hong '99,

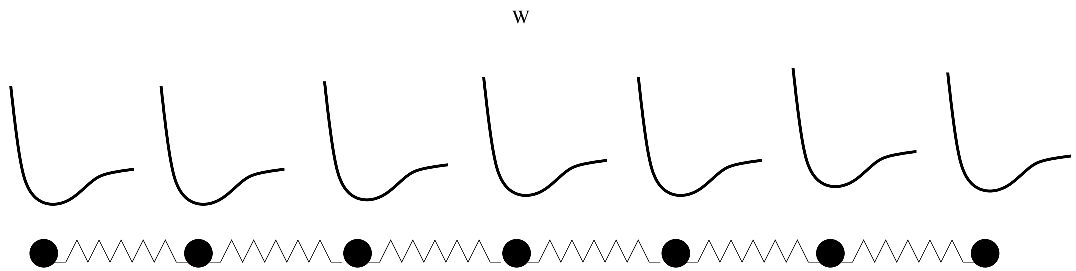
English and Pego '05, Herrmann '10, Stefanov and Kevrekidis '12

3 – Breathers in oscillator chains, continuum NLS approx

The excitation of discrete breathers can be enhanced by

local confining potentials :

$$\ddot{x}_n + W'(x_n) = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1})$$



$$V'(0) = W'(0) = 0,$$
$$V''(0) \geq 0, \quad W''(0) > 0$$

➢V harmonic (linear discrete Laplacian) $\Rightarrow$ Klein-Gordon lattice

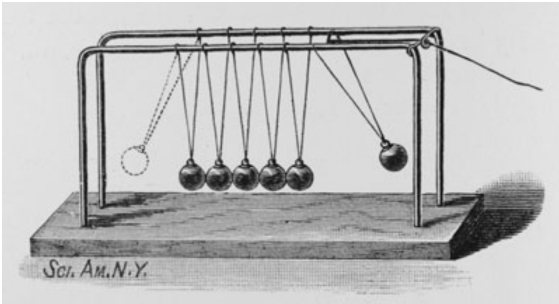
➢V anharmonic $\Rightarrow$ mixed FPU / Klein-Gordon lattice

Example 1 : granular chain with local potential

$$\ddot{x}_n + \omega^2 x_n = (x_{n-1} - x_n)_+^{3/2} - (x_n - x_{n+1})_+^{3/2}$$

strongly nonlinear coupling : $V''(0) = 0$

Classical Newton's cradle :



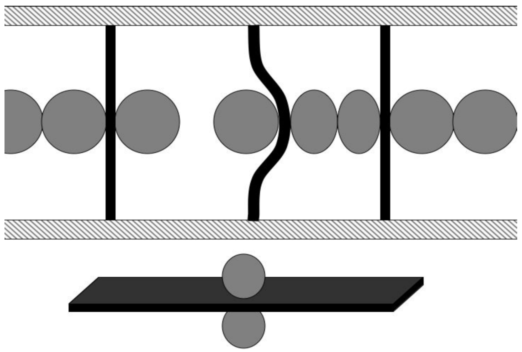
Collision Balls.

$\Rightarrow$ but $\omega \sim \dfrac{\text{bead collision time}}{\text{local oscillation period}} \ll 1$

$\omega \sim 10^{-4}$ for impact velocity $\approx 1\text{m/s}$

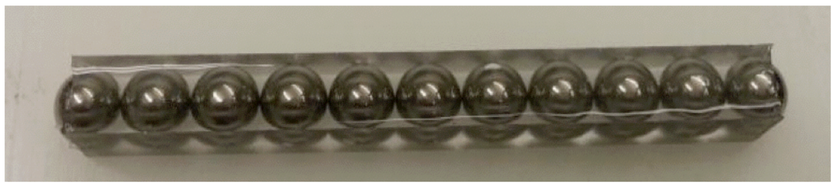
Stiff attachements (plates) : ω~1

G.J., Kevrekidis, Cuevas '13

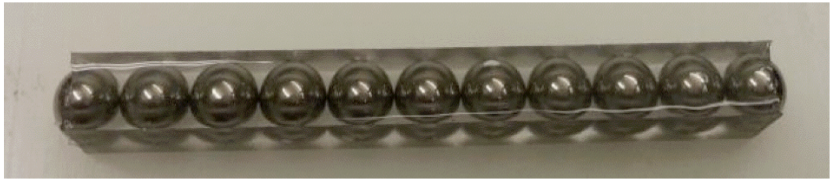


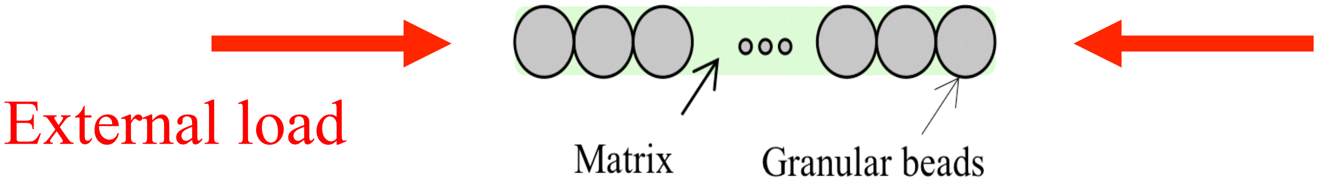
Beads in an elastic matrix :

Hasan et al, Granular Matter '15



Example 2 : precompressed granular chain with local potential



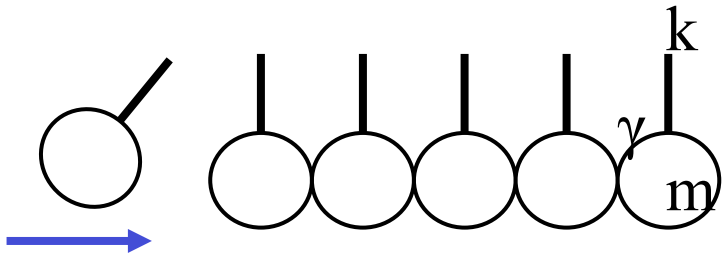


$$\ddot{u}_n + \omega^2 u_n = (1 + u_{n-1} - u_n)^{3/2} - (1 + u_n - u_{n+1})^{3/2}$$

$u_n(t)$ = displacement from reference position

The interaction potential involves a harmonic part : $V''(0) > 0$

Waves generated by an impact (G.J., Kevrekidis, Cuevas '13)

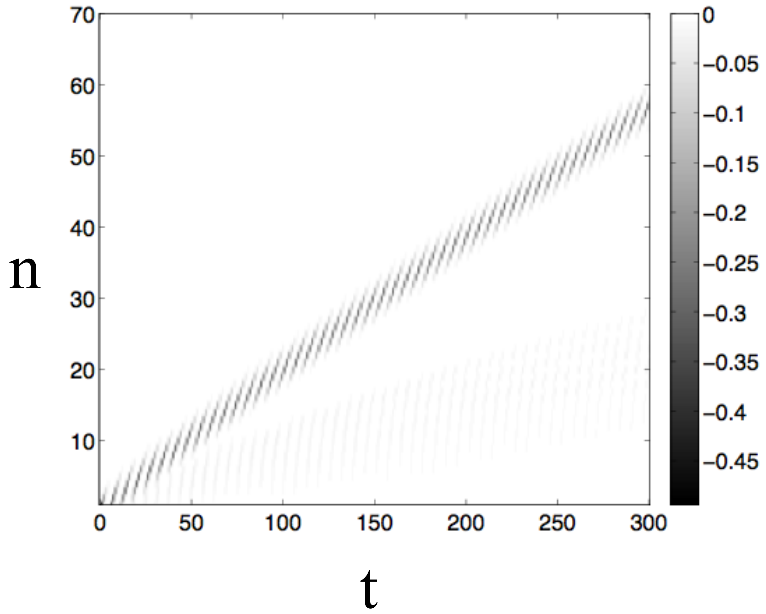


initial velocity $\lesssim k^{5/2} m^{-1/2} \gamma^{-2}$

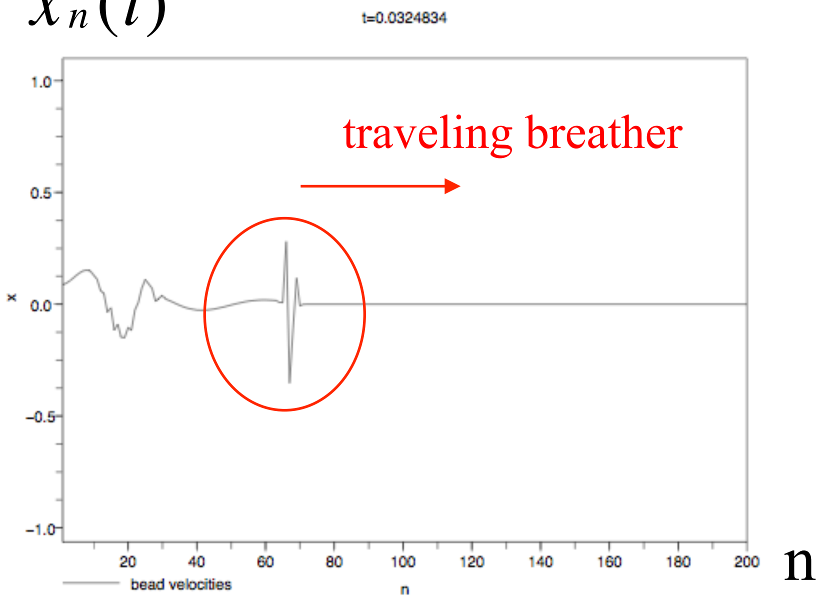
Case 1 : harmonic local potential, no precompression

$$m\ddot{x}_n + k\,x_n$$
$$= \gamma(x_{n-1} - x_n)_+^{3/2} - \gamma(x_n - x_{n+1})_+^{3/2}$$

Contact forces $= -(x_n - x_{n+1})_+^{3/2}$



$\dot{x}_n(t)$



Waves generated by an impact

Case 2 : soft anharmonic local potential

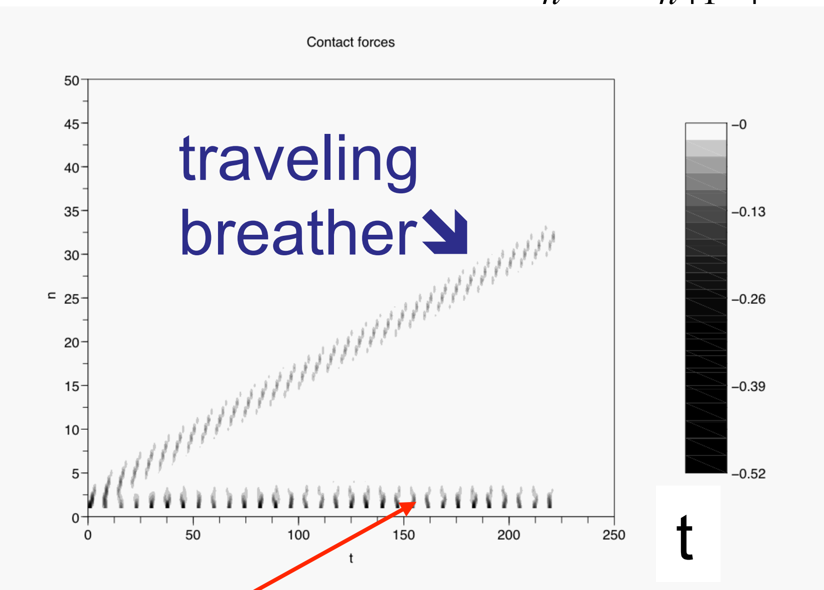
$$\ddot{x}_n + x_n + s\,x_n^3 = (x_{n-1} - x_n)_+^{3/2} - (x_n - x_{n+1})_+^{3/2}$$

s<0

Initial condition : $x_n(0) = 0,\quad \dot{x}_0(0) > 0,\quad \dot{x}_n(0) = 0 \text{ for } n \geq 1$

Contact forces $= -(x_n - x_{n+1})_+^{3/2}$

n



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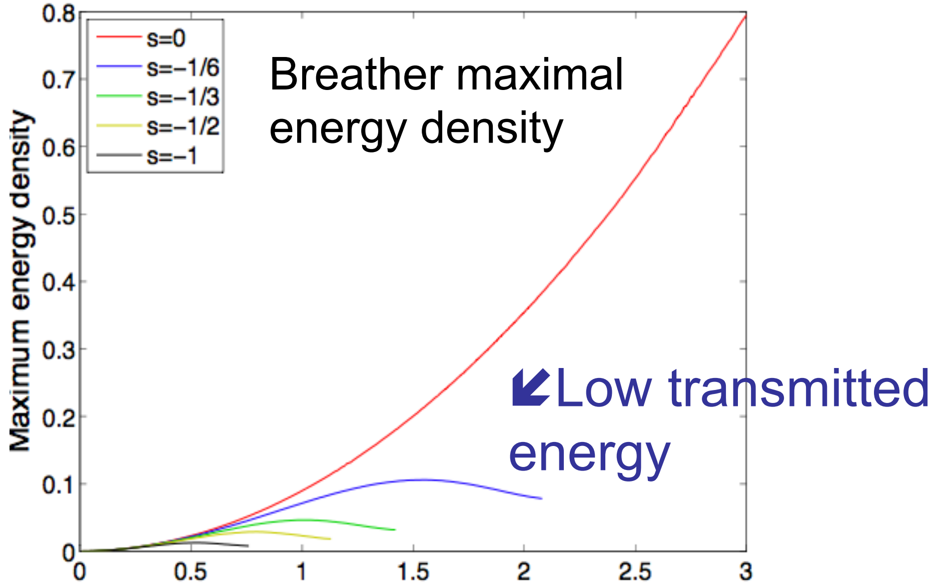
Energy trapped as a surface mode



$\dot{x}_0(0)$

Waves generated by an impact

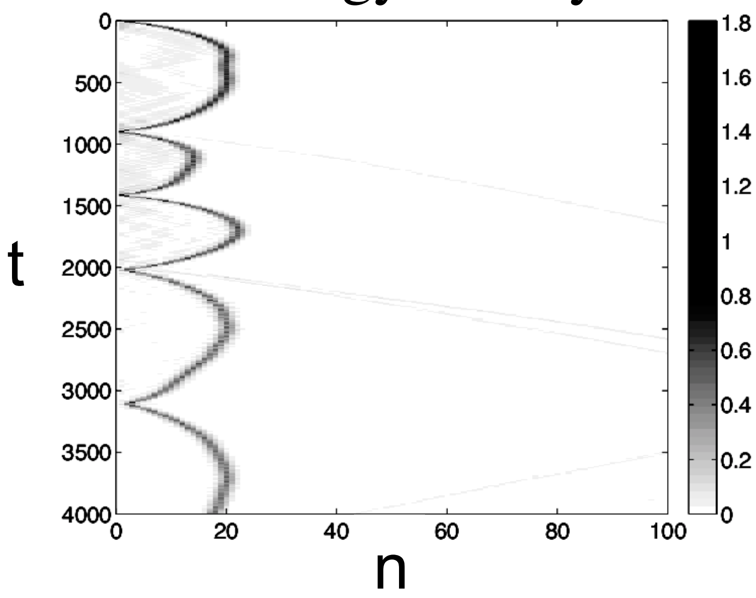
Case 3 :
hard local potential

$$\ddot{x}_n + x_n + x_n^3 = (x_{n-1} - x_n)_+^{3/2} - (x_n - x_{n+1})_+^{3/2}$$

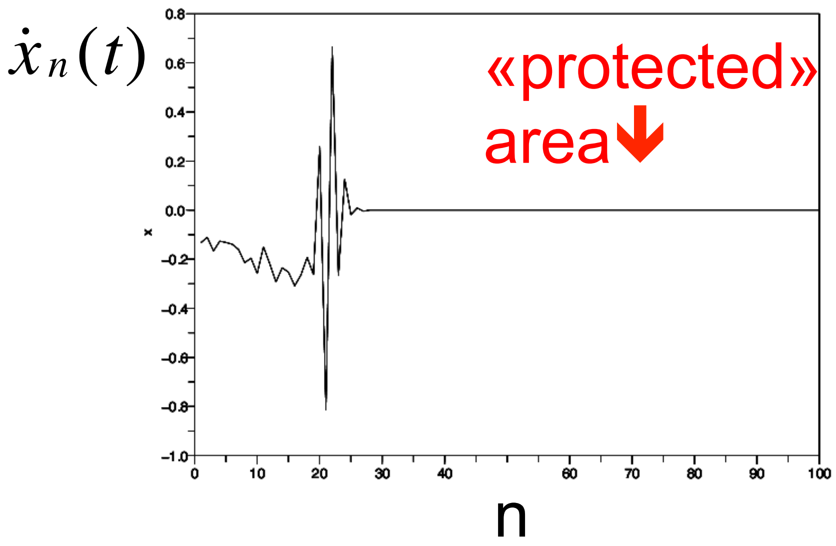
Initial impact : $x_n(0) = 0, \quad \dot{x}_0(0)=1.9, \quad \dot{x}_n(0)=0$ for $n \geq 1$

➔traveling breather
➔breather pinning
➔direction-reversal

energy density



onset of direction reversal :

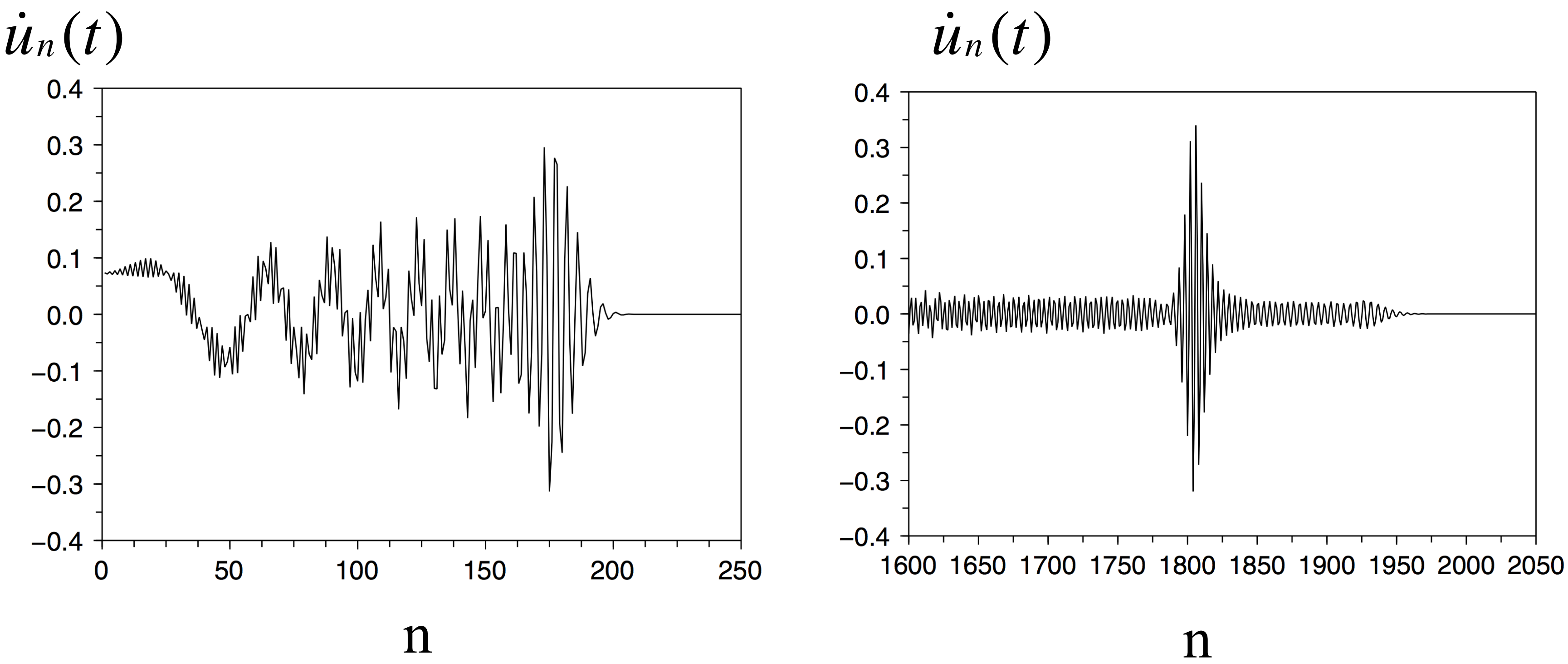


Waves generated by an impact

Case 4 : hard anharmonic local potential, precompression

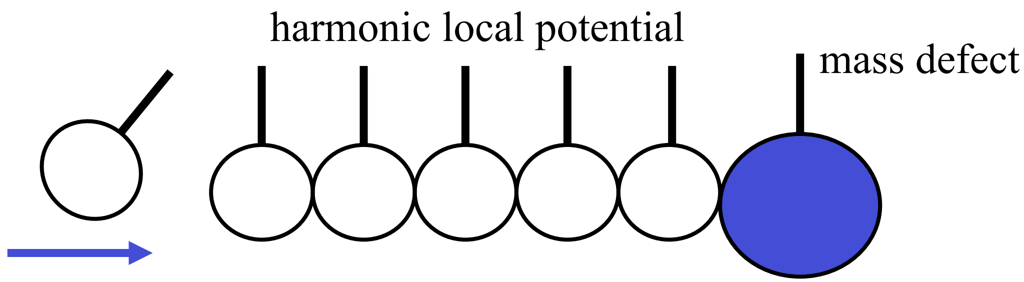
$$\ddot{u}_n + u_n + u_n^3 = (\delta + u_{n-1} - u_n)_+^{3/2} - (\delta + u_n - u_{n+1})_+^{3/2} \qquad \delta = 1/2$$

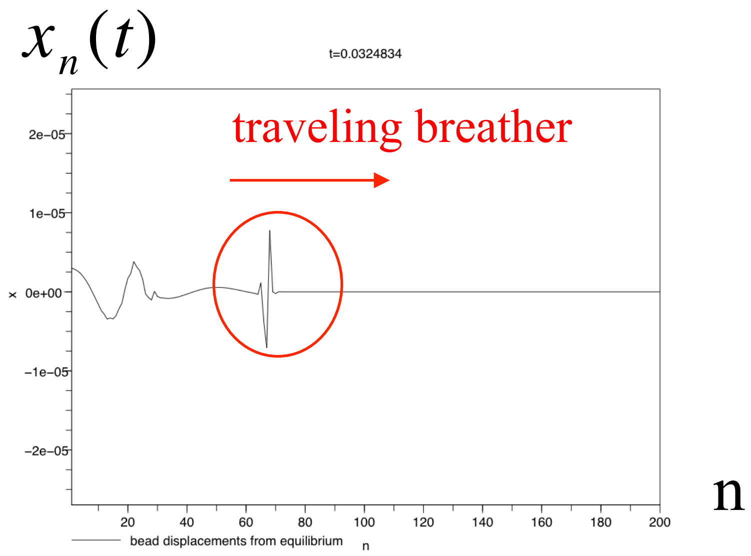
⇨ traveling breather with oscillatory tail



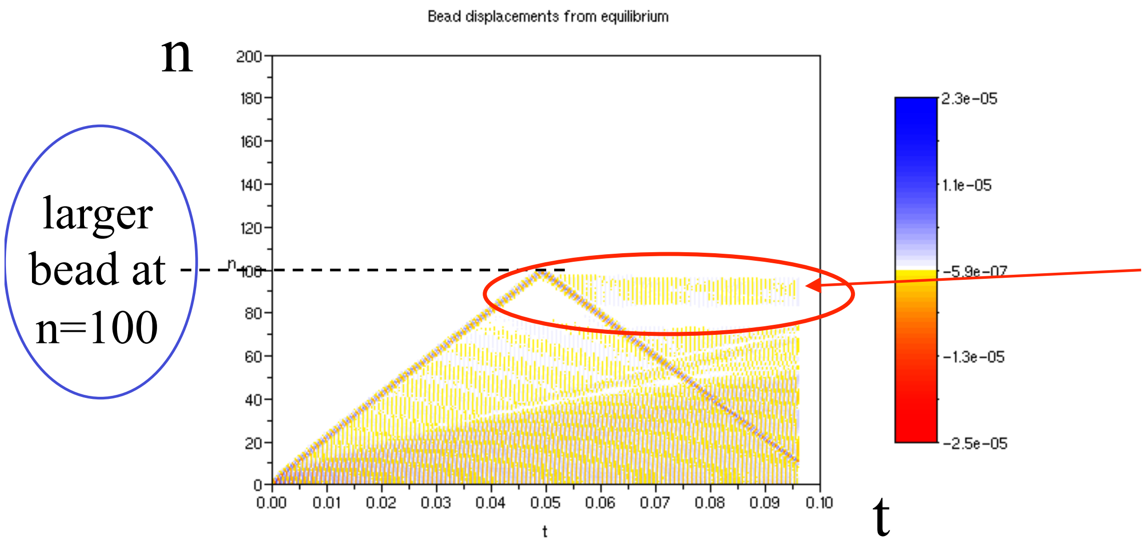
Mechanisms for static breather generation

1- Traveling breather-defect interaction





mass displacement $x_n(t)$



Some vibrational energy

remains trapped !

= static breather,

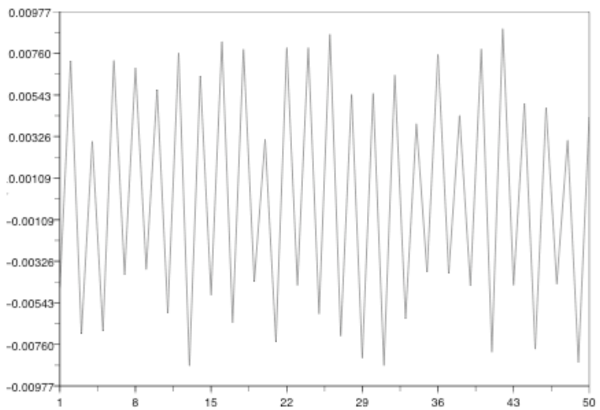
or nonlinear defect mode

Mechanisms for static breather generation

2 - Modulational instability of a nonlinear mode (homogeneous chain)

Perturbed π-mode
(neighbors out of phase)

$x_n(0)$

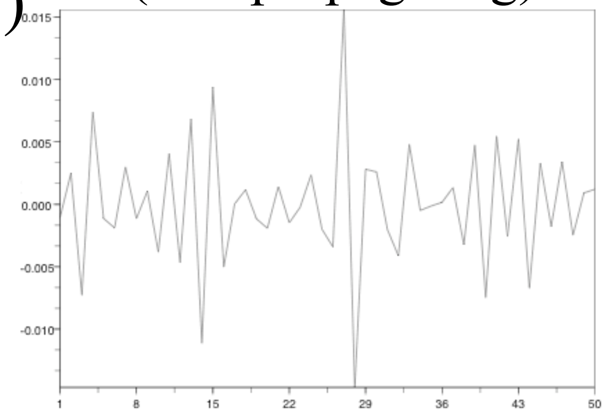


n

modulational instability

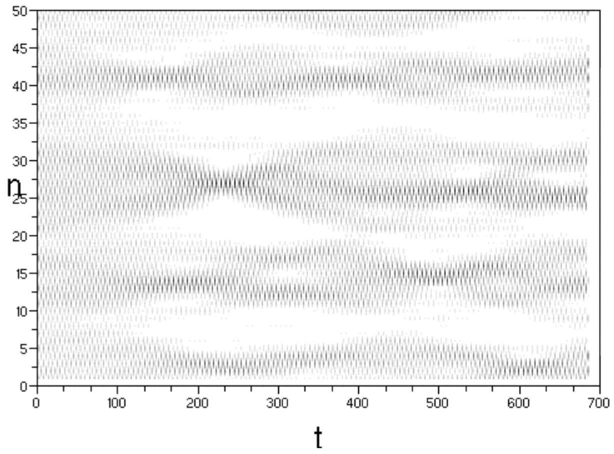
Breathers
(non-propagating)

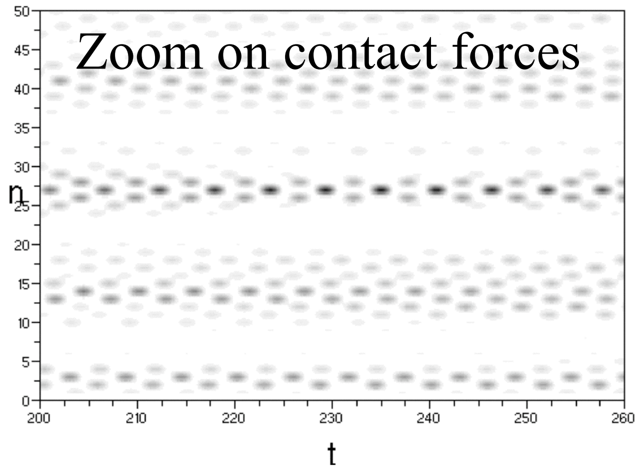
$x_n(t)$



n

Contact forces :
(grey levels)



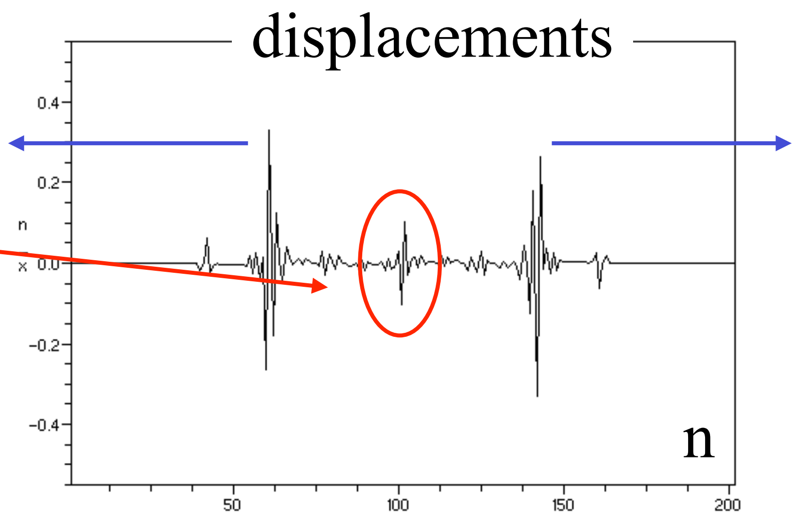


Mechanisms for static breather generation

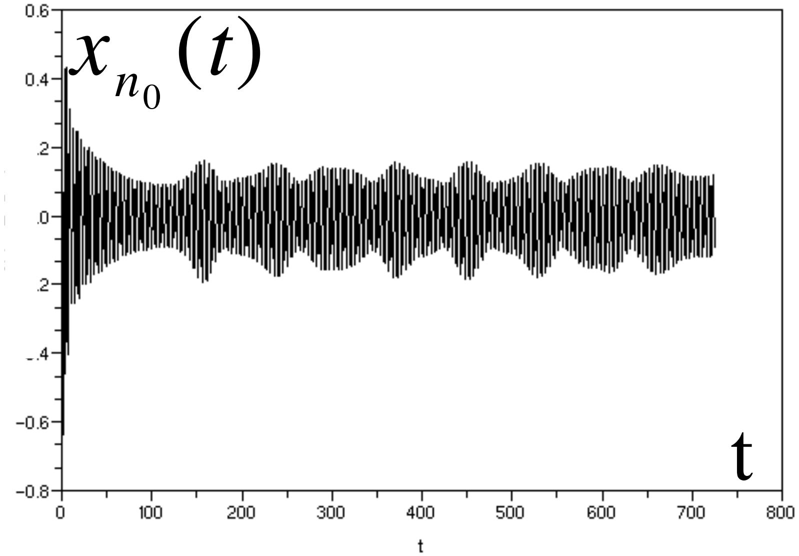
3 - Initial compression of two beads

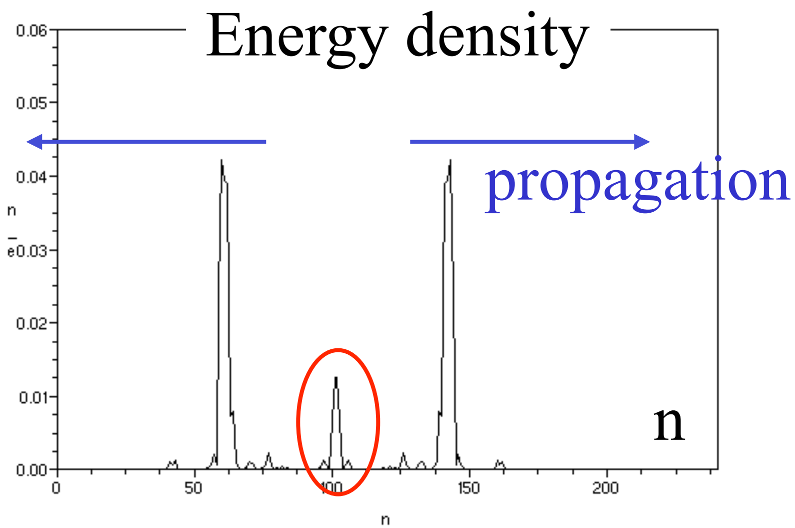
($n = n_0$ and $n = n_0 + 1$)

Breather
(non-propagating)



Initially compressed bead





Analysis of vibrational localization using modulation theory :

$\Longrightarrow$PDE approximating evolution of small well-prepared initial conditions

$\Longrightarrow$ Formal derivation of the nonlinear Schrödinger (NLS) equation

$$\frac{d^2 u_n}{dt^2} + W'(u_n) = V'(u_{n+1} - u_n) - V'(u_n - u_{n-1}), \;\; n \in \mathbb{Z}$$

$$V'(0) = W'(0) = 0, \quad V''(0) > 0, W''(0) > 0$$

Linear case : $V(u) = \frac{v_1}{2} u^2, \quad W(u) = \frac{w_1}{2} u^2$

$$u_n(t) = A\, e^{i(qn - \omega t)} + c.c.$$

$\Rightarrow$ dispersion relation $\omega^2 = 4 v_1 (\sin{(q/2)})^2 + w_1$

Weakly nonlinear case :

NLS limit : approximate solutions $=$ modulated plane waves

Remoissenet (86), Konotop (96)

$$\ddot{u}_n + W'(u_n) = V'(u_{n+1} - u_n) - V'(u_n - u_{n-1})$$

$$u_n(t) = \epsilon\, A(\epsilon^2 t, \epsilon(n - c\,t))\, e^{i(qn-\omega t)} + c.c. + O(\epsilon^2)$$

$$u_n(t) = \sum_{k\geq 1} \epsilon^k \sum_{p=-k}^{k} A_{k,p}(s,\xi)\, e^{ip(qn-\omega t)}$$

Slow variables : $s = \epsilon^2 t, \quad \xi = \epsilon(n - c\,t)$

$\Rightarrow c = \omega'(q)$

$\Rightarrow$ NLS equation for the envelope $A(s,\xi)$

$$i\,\partial_s A = -\frac{1}{2} w''(q)\, \partial_\xi^2 A + h\,|A|^2 A,$$

h depends on q and the derivatives of V, W at 0 up to order 4.

Giannoulis and Mielke (2004,2006) :

validity of NLS over times of order $O(1/\epsilon^2)$.

$$\frac{d^2 u_n}{dt^2} + W'(u_n) = V'(u_{n+1} - u_n) - V'(u_n - u_{n-1}),\ \ n \in \mathbb{Z}$$

V, W sufficiently regular, $V'(0) = W'(0) = 0$, $V''(0) > 0$, $W''(0) > 0$

$$i\,\partial_s A = -\frac{1}{2} w''(q)\,\partial_\xi^2 A + h\,|A|^2 A, \qquad (NLS)$$

THM : Let $A\ :\ [0,\tau_0] \times \mathbb{R} \to \mathbb{C}$ be a solution of (NLS) with $A(0,.) \in H^5(\mathbb{R})$ and

$$\{U_\epsilon^A(t)\} = \epsilon\, A(\epsilon^2 t, \epsilon(n - c\,t))\, e^{i(qn-\omega t)} + c.c., \quad c = \omega'(q)$$

For small enough ϵ, if $\|\,(\,\{u(0)\},\{\dot{u}(0)\}\,) - (\,\{U_\epsilon^A(0)\},\{\dot{U}_\epsilon^A(0)\}\,)\,\|_{\ell_2\times\ell_2} \leq \epsilon^{3/2}$

then for all $t \in [0, \tau_0/\epsilon^2]$

$$\|\,(\,\{u(t)\},\{\dot{u}(t)\}\,) - (\,\{U_\epsilon^A(t)\},\{\dot{U}_\epsilon^A(t)\}\,)\,\|_{\ell_2\times\ell_2} \leq C\,\epsilon^{3/2}$$

FPU case (W=0) : Tsurui ‘72 (derivation of NLS), Schneider ’10 (error bounds)

Formal existence of spatially localized oscillations

$$\frac{d^2 u_n}{dt^2} + W'(u_n) = V'(u_{n+1} - u_n) - V'(u_n - u_{n-1}),\ \ n \in \mathbb{Z}$$

$$u_n(t) = \epsilon\, A(\epsilon^2 t, \epsilon(n - c\,t))\, e^{i(qn - \omega t)} + c.c. + O(\epsilon^2).$$

$$i\,\partial_s A = -\frac{1}{2} w''(q)\,\partial_\xi^2 A + h\,|A|^2 A, \qquad c = \omega'(q)$$

Focusing case $w''(q)\,h < 0$:

$$u_n(t) = \epsilon\,\alpha\,\frac{e^{i(qn - (\omega + O(\epsilon^2))t)}}{\cosh\left(\epsilon(n - c\,t)\right)} + c.c. + O(\epsilon^2)$$

$\omega'(\,k\pi\,) = c = 0 \Rightarrow$ breather

$\omega'(q) \neq 0 \Rightarrow$ pulsating solitary wave (travelling breather)

References :

Error bounds for KdV approximation (over long finite times) :

L.A. Kalyakin, Russian Math. Surveys 44 (1989), 3-42.

G.Schneider, C.E.Wayne, in : International Conf. on Diff. Eq., Vol. 1,2 (Berlin, 1999), p. 390-404, World. Scientific, 2000.

D.Bambusi and A.Ponno, Comm. Math. Phys. 264 (2006), 539-561.

Error bounds for NLS approximation :

J.Giannoulis, A.Mielke, Dispersive evolution of pulses in oscillator chains with general interaction potentials, Discr. Cont. Dyn. Syst. B 6 (2006), 493-523.

G.Schneider, Bounds for the NLS approximation of the FPU system, Appl. Anal. 89 (2010).

Error bounds for NLS and KdV approximations in periodic 1D lattices :

M.Chirilus-Bruckner et al, Discr. Cont. Dyn. Syst. S 5 (2012), 879-901.

NLS approximation for granular chains with precompression :

G.J., P.Kevrekidis, J.Cuevas, Physica D 251 (2013), 39-59

Breathers in nonlinear lattices :

S.Flach, A.Gorbach, Physics Reports 467 (2008), 1-116.
S.Flach, C.R.Willis, Physics Reports 295 (1998), 181-264.

Extension 1 : exact breathers (static/traveling) close to NLS approximation

$$\frac{d^2 u_n}{dt^2} + W'(u_n) = V'(u_{n+1} - u_n) - V'(u_n - u_{n-1}), \; n \in \mathbb{Z}$$

Approximate solutions in the NLS limit : $u_n(t) = \epsilon\, \alpha\, \frac{e^{i(qn - \omega t)}}{\cosh\left(\epsilon(n - c\, t)\right)} + c.c. + O(\epsilon^2)$

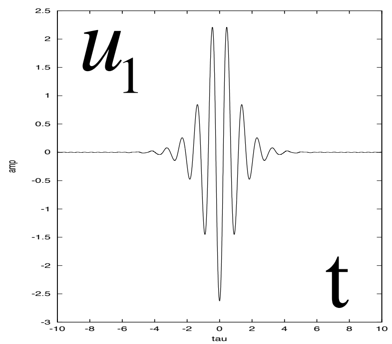
$\Rightarrow$ Exact solutions of the atomic chain close to the NLS approximate solutions ?

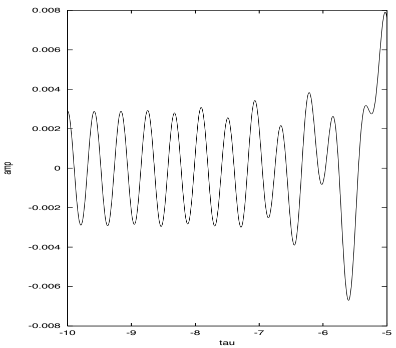
- Not obvious ! Counterexample of the semilinear wave equations on $\mathbb{R}$: no exact breather solution except for special potentials (Kichenassamy 91, Birnir 94)

- numerical computation of a traveling breather :

(Sire, G.J. '05)

W(x)=1-cos(x), V harmonic





in general, small nondecaying oscillatory tail

Exact solutions can be obtained using center manifold reduction and spatial dynamics, in the discrete (part II) or continuous (part III) settings

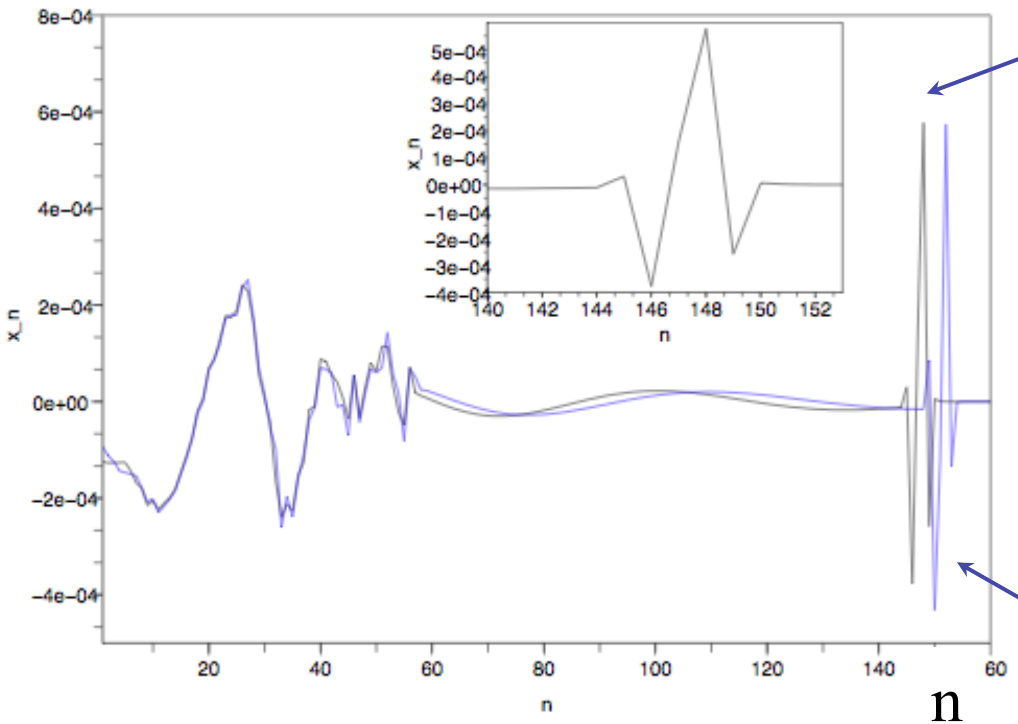
Extension 2 : discrete NLS equations

Example : cubic DNLS $\quad i\,\partial_\tau A_n = \gamma(A_{n+1} - 2A_n + A_{n-1}) + h\,A_n\,|A_n|^2$

Ansatz : $\quad x_n^{A,\varepsilon}(t) = \varepsilon\,\mathrm{Re}(A_n(\varepsilon^2 t)e^{it}) + \text{h.o.t.}$

$\Longrightarrow$approx. general small initial conditions, captures more phenomena (wave interactions, pinning), several continuum approx., inhomogeneities

Newton's cradle : impact at $n = 0$



$x_n(\mathrm{t})$

$x_n^{A,\varepsilon}(\mathrm{t})$

n

$x_n(0) = 0,\ \dot{\mathrm{x}}_\mathrm{n}(0)=0\ \forall\ n \geq 1$

$\dot{\mathrm{x}}_0(0)=\mathrm{O}(\varepsilon),\ \varepsilon = 10^{-3}$

$\varepsilon^{1/2} t = 415$

In part IV : modification of DNLS and NLS for strongly nonlinear interactions

Evolves under discrete p-Schrödinger equation