ON THE NUMBER OF LIMIT CYCLES IN PIECEWISE LINEAR LIÉNARD SYSTEMS

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Abstract

In a previous paper [Tonnelier, 2002] we conjectured that a Liénard system of the form $\dot{x} = p(x) - y$, $\dot{y} = x$ where p is piecewise linear on $n + 1$ intervals has up to $2n$ limit cycles. We construct here a general class of functions p satisfying this conjecture. Limit cycles are obtained from the bifurcation of the linear center.

Keywords: Limit cycles, piecewise linear systems.

1   Introduction and Main Result

A number of physical systems controlled by switching actions cannot be described with smooth dynamical systems. Many applications in different part of sciences are modeled by piecewise linear systems of ordinary differential equations. Examples come from control engineering, design of electrical circuits, embedded software, mechanical systems. The modeling of systems that operate in different linear modes with a transition between modes idealized as an instantaneous one leads to piecewise linear differential equations. In mechanical systems, for instance, dry friction, stick-slip oscillations and impact are described with non-smooth dynamical systems (see e.g. [Leine *et al.*, 2000]).

On the other hand, piecewise linear functions are introduced as an approximation or a global linearization of a nonlinear dynamics. Taking the advantage of their linear parts, piecewise linear systems lead to analytic calculations and provide a better understanding of the richness of the nonlinear

dynamics. Starting from linear systems, nonlinear dynamics are captured quite naturally by working with piecewise linear equations. In neuroscience, for instance, the subthreshold and superthreshold behavior of the neuron is well approximate by a piecewise linear regime introduced as a generalization of the integrate-and-fire model [Tonnelier & Gerstner, 2003].

The theory of smooth dynamical systems is the subject of numerous works, see for instance [Guckenheimer and Holmes, 1983], [Kuznetsov, 1998]. Since the pioneering works of Filippov [1964] and Andronov *et al.* [1966], the mathematical study of non-smooth dynamical systems have been developed. More recent results can be found in [Aubin & Cellina, 1984; Kunze, 2000]. One of the most difficult problem in the qualitative study of dynamical systems is the question of the number and the location of limit cycles. One expects that results from piecewise linear systems can lead to insights into this problem. Most studies deal with planar systems with a limited number of zones. The bifurcation set of continuous piecewise linear systems with two zones and three zones have been studied in details [Freire *et al.*, 1998; Freire *et al.*, 1999; Freire *et al.*, 2002]. A complete study of systems with a line of discontinuity is performed in [Giannakopoulos & Pliete, 2001]. In [Leine *et al.*, 2000] authors studied bifurcations in discontinuous dynamical systems. There are few analytical results on systems with a large number of zones and little is known about their dynamics. During the writing of this paper, two works on this topic have been published [Llibre *et al.*, 2003; Llibre & Ponce, 2003]. In the former, the Poincaré-Bendixson theory is used and in the latter averaging theorem is used that seems computationally equivalent to the approach followed in the present paper. Given the difficulty of the general problem, we will restrict our attention to the so-called Liénard system:

$$
\begin{aligned}
\dot{x} &= \epsilon p(x) - y , \\
\dot{y} &= x ,
\end{aligned}
\tag{1}
$$

where $(x, y) \in R^2$, p is a piecewise linear function and ϵ is a parameter that we consider for convenience. The phase plane is divided into zones so that the restriction of the system into each zone is a linear system with constant coefficients. The boundaries between zones are parallel straight lines given by the discontinuity points of the piecewise linear function p. One expects that the number of limit cycles is closely related to the number of zones, that is the number of parts of the phase plane where system (1) is linear. In [Tonnelier, 2002] we proposed the two conjectures

Conjecture 1 *The piecewise linear Liénard system (1) with $n+1$ zones has up to $2n$ limit cycles.*

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Conjecture 2 *The continuous piecewise linear Liénard system (1) with $n+1$ zones has up to n limit cycles.*

Let us discuss the simplest case where $n = 1$. The proof of conjecture 1 goes back to Andronov *et al.* (see [Giannakopoulos & Pliete, 2001] for a complete treatment) and conjecture 2 is already stated in [Lum & Chua, 1992] (see [Freire *et al.*, 1998] for a proof). Note also that the case $n = 2$ is related to the McKean caricature of the FitzHugh-Nagumo equation [McKean, 1970]. For $n = 2$, conjecture 2 is treated in [Freire *et al.*, 2002]. For a large number of zones, a piecewise continuous Liénard system with $2n + 3$ zones and n limit cycles has been proposed [Llibre & Ponce, 2003].

In this paper we provide a general class of systems satisfying the first conjecture. Moreover, we give a simple recursive algorithm to generate this class of functions. The main result of our paper is the following one:

Theorem 1 *Let n be a positive integer and p the piecewise linear function (see Fig. 1)*

$$p(x) = -x + \mu_i \ \text{if} \ x \in (a_i, a_{i+1}), \ i \in \{0, \dots, n\}, \tag{2}$$

where $(a_0 = -\infty) < 0 < a_1 < \dots < a_n(< a_{n+1} = +\infty)$ and $(\mu_i)_{i=0..n}$, are two finite sequences of real numbers. For $\epsilon \neq 0$ sufficiently small, there exist sequences (a_i) and (μ_i) such that system (1) admits $2n$ limit cycles.

The change of variable $\tilde{y} = y - \mu_0 \epsilon$ enables us to consider the case $\mu_0 = 0$. Note that p is discontinuous at the points (a_i) and has a jump of discontinuity equal to $\mu_i - \mu_{i-1}$. It should be noticed that ordinary differential equations with discontinuous right-hand side can lead to trajectories that intersect the set of discontinuity and remain on it. These so-called sliding solutions are possible when $\mu_{i+1} < \mu_i$. However, as we will show, the existence of $2n$ limit cycles requires that the μ_i determine a non-decreasing sequence. Then sliding solutions do not appear in our study and solutions of (1) are defined in a classical manner.

The paper is organized as follows. In section 2 we introduce the mathematical tools needed to state and prove Theorem 1. We also provide a recursive algorithm to generate a family of functions satisfying Theorem 1. Section 3 is devoted to a discussion.

2 A Piecewise Linear Perturbation of the Linear Center

Melnikov's theory has proved to be a powerful tool for predicting the number, positions and multiplicities of limit cycles that bifurcate from homoclinic and

heteroclinic orbits under perturbations [Guckenheimer & Holmes, 1983]. It has been successfully applied for the occurrence of bifurcations in physical systems (see references in [Sanjuán, 1998]). Basically, the theory associates a function to a given dynamical system. The roots of this function are related to the existence and location of limit cycles. Then, the differential problem is reduced to an algebraic one that corresponds to a search for roots of a function.

2.1   Melnikov theory

The Melnikov theory have been developed for the study of planar systems of the form

$$\dot{u} \;=\; f(u) + \epsilon g(u) \tag{3}$$

where $u \in R^2$, $\epsilon \ll 1$ and f, g are sufficiently smooth functions. It is assumed that the unperturbed system, i.e. $\epsilon = 0$, has a one-parameter family of periodic solutions Γ_r: $\gamma_r(t)$, $r > 0$ with period T_r. The Melnikov function for system (3) is given by

$$M(r) = \int_0^{T_r} e^{-\int_0^t \nabla f(\gamma_r(s))ds} f \wedge g(\gamma_r(t))dt \tag{4}$$

where the wedge product of $u, v \in R^2$ is defined by $u \wedge v = u_1 v_2 - v_1 u_2$. The following theorem is proved in [Blows & Perko, 1994] (see also Guckenheimer & Holmes, 1983),

Theorem 2 *If there exists $r_1, r_2, \ldots, r_n$ such that $M(r_j) = 0$ and $M'(r_j) \neq 0$, $j = 1, \ldots, n$, then for all sufficiently small $\epsilon \neq 0$ system (3) has n hyperbolic limit cycles in an $O(\epsilon)$ neighborhood of Γ_{r_j} that bifurcate from the periodic orbits $\gamma_{r_j}(t)$. Furthermore if $M(r_0) \neq 0$, then for all sufficiently small $\epsilon \neq 0$, system (3) has no cycle in an $O(\epsilon)$ neighborhood of Γ_{r_0}.*

2.2   The Melnikov function of the piecewise linear Liénard system

For small ϵ, Eq. (1) is a perturbation of the linear center $\dot{x} = -y$, $\dot{y} = x$ that has a one-parameter family of periodic orbits Γ_r: $(r \cos t, r \sin t)$, $r > 0$, with a period $T_r = 2\pi$. Note that system (1) is invariant under the change of variables $(t, x, y, \epsilon) \rightarrow (-t, x, -y, -\epsilon)$. Therefore the stability of limit cycles changes with the sign of ϵ. Note also that the origin is the only fixed point of the system and it is easy to check that the origin is an unstable focus when $\epsilon < 0$.

Using $\nabla f(u) = 0$, the Melnikov function (4) of the Liénard system (1) is given by

$$M(r) = \int_0^{2\pi} -r\cos t \; p(r\cos t) \; dt.$$

Using the change of variable $t \to 2\pi - t$ in the interval $[\pi, 2\pi]$, we have

$$M(r) = 2\int_0^{\pi} -r\cos t \; p(r\cos t) \; dt.$$

Let $r > 0$ and $q \in \{0, \ldots, n\}$ be the integer such that $a_q < r < a_{q+1}$. We define t_i, $i \in \{0, \ldots, q+1\}$, the decreasing finite sequence given by $t_0 = \pi$,

$$\cos t_i = \frac{a_i}{r}$$

for $i = 1, \ldots, q$ and $t_{q+1} = 0$. On (t_{i+1}, t_i), we have $p(r\cos t) = -r\cos t + \mu_i$. Therefore we obtain

$$M(r) = 2r^2 \int_0^{\pi} \cos^2 t \; dt - 2r \sum_{i=0}^{q} \mu_i \int_{t_{i+1}}^{t_i} \cos t \; dt.$$

Computing the integrals, we obtain the following:

$$M(r) = \pi r^2 - 2\sum_{i=1}^{q-1} \mu_i\left(\sqrt{r^2 - a_i^2} - \sqrt{r^2 - a_{i+1}^2}\right) - 2\mu_q\sqrt{r^2 - a_q^2}.$$

Let us recall that the previous expression is derived for $a_q < r < a_{q+1}$. Let $r > 0$, the Melnikov function M is written

$$M(r) = \pi r^2 - 2\sum_{i=1}^{i=n} d_i\sqrt{r^2 - a_i^2} \; H(r - a_i) \tag{5}$$

where $d_i = \mu_i - \mu_{i-1}$ for $i \in \{1, \ldots, n\}$ and H is the Heaviside step function $H(x) = 0$ if $x < 0$ and $H(x) = 1$ if $x > 0$. We note that $M(r)$ is continuous but not $C^1(R^+)$.

2.3 Proof of Theorem 1

Checking the proof of Theorem 2 in [Blows and Perko, 1994], we can see that smoothness is required for f but the discontinuity of g at isolated values is not a restriction for the use of Theorem 2. From Theorem 2 and the expression (5) of the Melnikov function associated with the Liénard system (1),(2), the following proposition holds.

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Proposition 1 *The Liénard system (1),(2) has exactly p hyperbolic limit cycles asymptotic to circles of radii r_j as $\epsilon \to 0$ if and only if the function*

$$f_n(r^2) = \pi r^2 - 2\sum_{i=1}^{i=n} d_i \sqrt{r^2 - a_i^2} \; H(r^2 - a_i^2)$$

has p positives roots $r = r_j$ such that $f_n'(r_j^2) \neq 0$.

More specifically, to prove Theorem 1, we will show in the following that f_n has up to $2n$ roots. For $n = 3$, the typical shape of the Melnikov function with three pairs of roots is shown in Fig. 2A.

Proof. We note $x = r^2$ and we define the family of functions:

$$f_k(x) \;\; = \;\; \pi x - 2\sum_{i=1}^{i=k} d_i \sqrt{x - a_i^2} H(x - a_i^2), \quad k = 1, \ldots, n$$

Note that f_k and f_{k+1} coincide on $(0, a_{k+1})$. We construct recursively the function f_k, and thus p, choosing appropriate values for μ_k and a_k in order to ensure, at each step, the existence of two additional roots of f_k in $(a_k, +\infty)$.

Let us start with $k = 1$. We have

$$f_1(x) = \pi x - 2d_1 \sqrt{x - a_1^2}$$

for $x > a_1^2$. It is easy to show that the equation $f_1(x) = 0$ has two real solutions if and only if $d_1 > \pi a_1$. We easily check that these solutions are greater than a_1^2. Note that $a_1 > 0$ is arbitrary chosen.

In order to start the recurrence, we consider the case $k = 2$. We choose a_2 sufficiently big such that a_2^2 is greater than the two roots of f_1. This requirement ensures that the two roots of f_1 are also roots of f_2. Note that we have $f_2(a_2^2) = f_1(a_2^2) > 0$. We take d_2 such that $\exists x^* > a_2^2, \; f_2(x^*) < 0$. Notice that this choice is always possible since for $x > a_2^2$ and fixed, $\lim_{d_2 \to \infty} f_2(x) < 0$ (note that $f_2(a_2^2)$ does not depend on d_2). Using $\lim_{x \to \infty} f_2(x) = +\infty$ and the continuity of f_2 on $(a_2^2, +\infty)$ it is clear that f_2 has two additional roots in $(a_2^2, +\infty)$.

We assume that we have constructed two finite sequences $a_1, \ldots, a_{k-1}$ and $\mu_1, \ldots, \mu_{k-1}$ such that f_{k-1} has $2k - 2$ roots on $(a_1^2, +\infty)$. Then we choose a_k^2 sufficiently big to be greater than the roots of f_{k-1}. Then f_k has $2k - 2$ roots in $(0, a_k)$. Moreover we require $f_k(a_k^2) > 0$. We choose $d_k > 0$ sufficiently big to ensure the existence of $x^* > a_k^2$ such that $f_k(x^*) < 0$. Since $\lim_{x \to \infty} f_k(x) = +\infty$ it follows that f_k has two additional roots in $(a_k, +\infty)$.
∎

If we study the variations of f_k, $k \in \{1, \ldots, n\}$, we can refine the proposition and show that $2n$ is the maximal number of limit cycles. Moreover, if we note indifferently x_k the two roots of f_k in (a_k, a_{k+1}), by construction we have $f_n'(x_k) = f_k'(x_k) \neq 0$ and then the limit cycles are hyperbolic.

2.4 Some specific examples

In this section we derive explicit sufficient conditions on (a_k) and (μ_k) for the existence of $2n$ limit cycles. Note that shifting the y-axis in (1), we can take $\mu_0 = 0$. A recursive construction of the function p is provided using the following proposition

Proposition 2 *If p is such that (a_k) and (μ_k) satisfy*

$$a_1 > 0, \ \mu_1 > \pi a_1, \ a_k \geq \frac{2\mu_{k-1}}{\pi}, \ \mu_k \geq \pi a_k, \ k = \{2, \ldots, n\}$$

then system (1),(2) has $2n$ hyperbolic limit cycles.

Proof. Let us recall that the equation $f_1(x) = \pi x - 2\mu_1\sqrt{x - a_1^2} = 0$ has two distinct real solutions on $(a_1^2, +\infty)$ if and only if $\mu_1 > \pi a_1$ (i.e. $d_1 > \pi a_1$). Moreover these two solutions are given by

$$x_{\pm}^2 = \frac{2\mu_1^2 \pm 2\mu_1\sqrt{\mu_1^2 - \pi^2 a_1^2}}{\pi^2}.$$

Hence, it is sufficient to choose a_2 such that $a_2 \geq 2\mu_1/\pi$. For $d_2 > 0$, the study of the derivative of f_2 shows that there exists x_2^* such that f_2 is decreasing on (a_2^2, x_2^*) and increasing on $(x_2^*, +\infty)$ (note that $d_2 < 0$, i.e. $\mu_2 < \mu_1$, does not allow the existence of roots of f_2 in $(a_2^2, +\infty)$). Then f_2 has at most two solutions on (a_2^2, a_3^2). We have

$$f_2(x) = \pi x - 2\mu_1\sqrt{x - a_1^2} - 2(\mu_2 - \mu_1)\sqrt{x - a_2^2}$$

$$< \pi x - 2\mu_1\sqrt{x - a_2^2} - 2(\mu_2 - \mu_1)\sqrt{x - a_2^2}.$$

Then the following (strict) inequality holds

$$f_2(x) < \pi x - 2\mu_2\sqrt{x - a_2^2}$$

Since $\pi x - 2\mu_2\sqrt{x - a_2^2}$ has two real roots (greater than a_2^2) if $\mu_2 > \pi a_2$, we conclude that a sufficient condition to have two roots in $(a_2^2, +\infty)$ is $\mu_2 \geq \pi a_2$

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(in particular, $\mu_2 > \mu_1$). Now, the next step is to choose a_3^2 greater than the roots of f_2. Noting that

$$f_2(x) > \pi x - 2\mu_1 \sqrt{x - a_1^2} - 2(\mu_2 - \mu_1)\sqrt{x - a_1^2} = \pi x - 2\mu_2 \sqrt{x - a_1^2}$$

it is clear that the two roots of f_2 on $(a_2^2, +\infty)$ are between the roots of $\pi x - 2\mu_2 \sqrt{x - a_1^2}$. Thus, we choose $a_3 \geq 2\mu_2/\pi$, i.e. a_3^2 is greater than the roots of $\pi x - 2\mu_2 \sqrt{x - a_1^2}$. Note that $f_4(a_3^2) > 0$.
The study on the intervals $(a_3^2, a_4^2), \ldots, (a_{k-1}^2, a_k^2)$ is very similar and it is not difficult to prove that a sufficient condition for the existence of $2k$ roots for f_k is given by $a_k \geq 2\mu_{k-1}/\pi$ and $\mu_k \geq \pi a_k$. $\blacksquare$
The following sequence is an example that fulfilled the conditions of Proposition 2 :

$$a_1 > 0, \ \alpha > 1, \ a_k = ra_1 2^{k-1}, \ \mu_k = \alpha a_1 \pi 2^{k-1}.$$

In table 1 we numerically compute the location of the limit cycles for $a_1 = 1$ and $\alpha = 2$, i.e. $a_k = 2^k$, $\mu_k = \pi 2^k$. Figure 3 shows, in the phase plane, three pairs of limit cycles of the Liénard system obtained for $\epsilon = 0.01$.

3   Discussion

Since they frequently appear in physical and biological systems, limit cycles have attracted a wide attention. Limit cycles model self sustained oscillations that are involved in numerous applications. The problem of finding the number and the location of limit cycles is related to the already unsolved Hilbert's 16th problem. Piecewise linear systems capture the richness of the dynamical behavior of general nonlinear systems, including limit cycles. Taking the advantage of their linear parts, one expects to obtain insights into the problem of the existence of limit cycles.

In this paper we have studied the number and the location of limit cycles of a piecewise linear Liénard system. Given an even integer $2n$, we construct a planar piecewise linear system with $2n$ limit cycles. A piecewise linear differential system with an arbitrary number of limit cycles has already been proposed [Llibre & Ponce, 2003]. We give here a general class of functions and present a recursive algorithm to generate it. Moreover we claim that our system is minimal in the sense that a system with lesser linear parts, i.e. lesser than $n+1$, has lesser limit cycles. Moreover, based on numerical simulations, we find that we also provide an example for Conjecture 2 taking a continuous version of the function p with sufficiently small matching zones.

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Acknowledgments

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Table

Table 1. Values of the radius of circles that approximate the limit cycles of the Liénard system as $\epsilon \to 0$. These values are obtained as the two roots $r_{k,1}$, $r_{k,2}$ of f_n (see proposition 1) in (a_k, a_{k+1}), $k \in \{1, \ldots, n\}$ for $n = 7$ discontinuities and $a_1 = 1$, $a_k = 2^k$ and $\mu_k = \pi 2^k$.

k	1	2	3	4	5	6	7
a_k	1	4	8	16	32	64	128
$r_{k,1}$	1.1882	4.3147	8.0432	16.0534	32.1003	64.1990	128.3978
$r_{k,2}$	1.8515	6.9042	14.4240	28.9705	57.9707	115.9489	231.8997

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Figure Legends

Figure 1: Graph of the piecewise linear function p with discontinuities at $a_1, a_2, \ldots, a_n$.

Figure 2: (A) Graph of the Melnikov function for the Liénard system (1) where the function p is shown in panel (B). The nonzero roots of the Melnikov function correspond to three pairs (unstable and stable) of limit cycles. Parameters are those of Table 1.

Figure 3: Numerical computation of limit cycles in the phase plane of the Liénard system $\dot{x} = \epsilon p(x) - y$, $\dot{y} = x$ for $\epsilon = 0.01$ and p given in Table 1 (we restrict here to three discontinuities). The locations of the limit cycles are predicted by the roots of the Melnikov function given in Fig.2A. The full lines represent stable limit cycles and the dashed line represent unstable limit cycles.
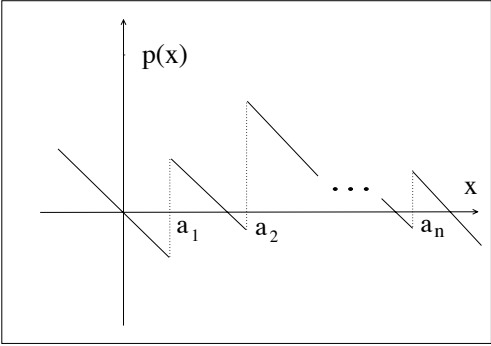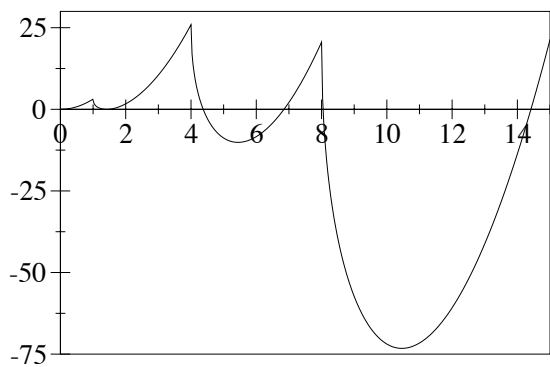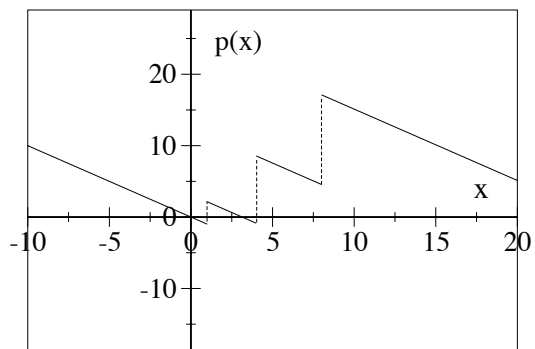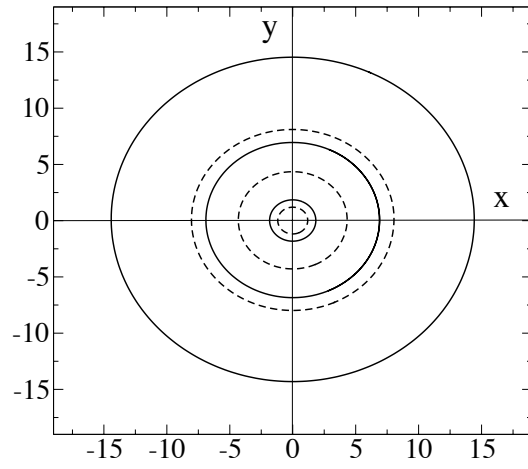
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Figure 1.

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Figure 2.

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Figure 3.

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