Introduction to nonsmooth dynamical systems

Lecture 2. Complementarity systems

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Contents

- Complementarity systems
- Existence and uniqueness of $C^1$ solutions.
- Extension of complementarity systems
- Computation of equilibria
- Lyapunov stability
Outline

Complementarity Systems (CS)

Computations of equilibria for LCS

Stability of Linear Complementarity Systems
  Linear Time Invariant (LTI) passive systems
  Lyapunov stability of LCS
Linear Complementarity Systems (LCS)

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B\lambda(t) + a, \quad x(0) = x_0 \\
y(t) &= Cx(t) + D\lambda(t) + b \\
0 &\leq y(t) \perp \lambda(t) \geq 0.
\end{align*}
\]  

Concept of solutions

- The solution to the LCS (1) depends strongly on the quadruplet \((A, B, C, D)\) and the initial conditions
- We will review the simplest cases
  - \(D\) is a P-matrix
    - \(C^1\) solutions.
  - \(D = 0, \ CB \geq 0\) and consistent initial solutions
    - Absolutely Continuous \((AC)\) solutions
Mathematical nature of the solutions

In order to say more on the mathematical properties of the LCS, we need to characterize the solution $\lambda$ of

$$\begin{cases} 
y = Cx + D\lambda + b \\
0 \leq y \perp \lambda \geq 0
\end{cases}$$

(2)

of its equivalent formulation in terms of inclusion into a subdifferential

$$-(Cx + D\lambda + b) \in \partial \Psi_{\mathbb{R}_+^m}(\lambda)$$

(3)
Linear Complementarity Problem

Definition (Linear Complementarity Problem (LCP))
A Linear complementarity problem (LCP) is to find a vector $\lambda \in \mathbb{R}^m$ that satisfies

$$0 \leq \lambda \perp M\lambda + q \geq 0$$

(4)

for a given matrix $M \in \mathbb{R}^{m \times m}$ and a vector $q \in \mathbb{R}^m$.

Comments
▶ A LCP is often formulated as:

$$\begin{cases} 
    w = M\lambda + q, \\
    0 \leq w \perp \lambda \geq 0.
\end{cases}$$

(5)
Linear Complementarity Problem

Link with quadratic programming (QP)

If $M = M^\top \succ 0$, the LCP is the necessary and sufficient optimality condition to the following quadratic problem

$$
\begin{align*}
\min_{\lambda} & \quad \frac{1}{2} \lambda^\top M \lambda + \lambda^\top q \\
\text{s.t.} & \quad \lambda \geq 0
\end{align*}
$$

or equivalently

$$
\begin{align*}
\min_{\lambda} & \quad \frac{1}{2} \lambda^\top M \lambda + \lambda^\top q + \psi_{\mathbb{R}_+}(\lambda)
\end{align*}
$$

Hints: Write the optimality condition of a convex QP
Linear Complementarity Problem

**Theorem (Fundamental result of complementarity theory)**

The LCP

\[ 0 \leq \lambda \perp M\lambda + q \geq 0 \]

has a unique solution \( \lambda^* \) for any \( q \in \mathbb{R}^m \) if and only if \( M \) is a P-matrix. In this case the solution \( \lambda^* \) is a piecewise linear function of \( q \) (with a finite number of pieces).

**Remarks**

- A P-matrix has all its principal minors positive. A positive definite matrix is a P-matrix.
- A symmetric P-matrix is a positive definite matrix.
- There exist non-symmetric P-matrices which are not positive definite. And there exist positive definite matrices which are not symmetric!
Solutions as continuously differentiable functions ($C^1$ solutions)

**ODE with Lipschitz right-hand–side**
The substitution of $\lambda(x)$ yields a Ordinary Differential Equation (ODE) with a Lipschitz right-hand–side.

Cauchy-Lipschitz Theorem $\Rightarrow$ Existence and uniqueness of a solution as continuously differentiable functions ($C^1$ solutions)

**The LCS case**
The solution $\lambda(x)$ of the following linear complementarity system

$$0 \leq \lambda \perp D\lambda + Cx + b \geq 0$$

(6)

is unique for all $Cx + b$ if and only if $D$ is a P-Matrix and moreover $\lambda(x)$ is a Lipschitz function of $x$.

see the example of the RLCD circuit
Solutions as absolutely continuous functions (AC solutions)

The LCS case with \( D = 0 \) and \( b = 0 \)
If we consider the LCS (1) with \( D = 0 \) and \( b = 0 \), we get
\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B\lambda(t) + a, \quad x(0) = x_0 \\
y(t) &= Cx(t) \\
0 &\leq y(t) \perp \lambda(t) \geq 0.
\end{align*}
\] (7)

Regularity: What should we expect ?
The time-derivative of the state \( \dot{x}(t) \) and \( \lambda(t) \) are expected to be, in this case, discontinuous functions of time.
Indeed, if the output \( y(t) \) reaches the boundary of the feasible domain at time \( t_* \), i.e., \( y(t_*) = 0 \), the time–derivative \( \dot{y}(t) \) needs to jump if \( \dot{y}(t_*) < 0 \)
Solutions as absolutely continuous functions (AC solutions)

Example (Scalar LCS with $D = 0$)

Let us search for a continuous solution $x(t)$ to

\[
\begin{cases}
  x(0) = x_0 > 0 \\
  \dot{x}(t) = -x(t) - 1 + \lambda(t) \\
  0 \leq x(t) \perp \lambda(t) \geq 0
\end{cases}
\]

Two modes:

▶ free dynamics for $0 < t < t_\ast$ with $x(t) > 0$ and $x(t_\ast) = 0$:

\[
\begin{cases}
  x(0) = x_0 > 0 \\
  \dot{x}(t) = -x(t) - 1
\end{cases}
\]

Solution:

\[x(t) = \exp(-t)x_0 + \exp(-t) - 1\]

\[x(t_\ast) = 0 \implies t_\ast = -\ln\left(\frac{1}{1+x_0}\right) > 0\]

▶ dynamics for $t \geq t_\ast$

\[
\begin{cases}
  x(t_\ast) = 0, \\
  \dot{x}(t) + 1 = \lambda(t) \geq 0
\end{cases}
\]
Solutions as absolutely continuous functions (AC solutions)

Example (Scalar LCS with $D = 0$)

Solving the dynamics for $t_\star \leq t < T$:

$$
\begin{aligned}
& x(t_\star) = 0 \\
& \dot{x}(t) + 1 = \lambda(t) \geq 0 
\end{aligned}
$$

(8)

if we are looking for an abs. continuous solution $x(t)$, the abs. continuity and $x(t_\star) = 0$ implies that $\dot{x}(t) \geq 0$, $t \in [t_\star, t_\star + \varepsilon)$, $\varepsilon > 0$, otherwise $x(t_\star + \varepsilon) < 0$.

1. $\dot{x}(t) > 0$, $t \in [t_\star, t_\star + \varepsilon)$, $\varepsilon > 0$.
   By continuity, $x(t + \varepsilon) > 0$, $\lambda(t + \varepsilon) = 0$ then

$$
\dot{x}(t + \varepsilon) = -x(t + \varepsilon) - 1 < 0
$$

(9)

No solution.

2. $\dot{x}(t) = 0$, $\lambda(t) = 1$, $x(t) = 0$ \quad $\forall t \geq t_\star$ ( $T = +\infty$ )
   The only possible continuous solution.
Solutions as absolutely continuous functions (AC solutions)

Example (Scalar LCS with $D = 0$)

Conclusion: A unique continuous $x(t)$ has been computed for all $t \in [0, +\infty)$. The time derivative of the solution $\dot{x}(t)$ jumps at from $t_*$ from $x(t_-^*) = -1$ to $x(t_+^*) = 0$. 

![trajectory x(t), x_0 = 1](image)
Solutions as absolutely continuous functions (AC solutions)

Example (Scalar LCS with $D = 0$)
Solutions as absolutely continuous functions (AC solutions)

Example (Scalar LCS with $D = 0$)
Solutions as absolutely continuous functions (AC solutions)

Example (Scalar LCS with $D = 0$)
Let us search for a continuous solution $x(t)$ to

$$
\begin{aligned}
&x(0) = x_0 > 0 \\
&\dot{x}(t) = -x(t) + 1 - \lambda(t) \\
&0 \leq x(t) \perp \lambda(t) \geq 0
\end{aligned}
$$

$x(t) > 0$ for $0 < t < t_*$ (free dynamics):

$$
\begin{aligned}
&x(0) = x_0 > 0 \\
&\dot{x}(t) = -x(t) + 1
\end{aligned}
$$

Solution:

$$
x(t) = \exp(-t)(x_0 - 1) + 1 > 0
$$

solution for all $t \in [0; +\infty]$
Solutions as absolutely continuous functions (AC solutions)

Example (Scalar LCS with $D = 0$)

Let us search for a continuous solution $x(t)$ to

\[
\begin{cases}
  x(0) = x_0 = 0 \\
  \dot{x}(t) = -x(t) + 1 - \lambda(t) \\
  0 \leq x(t) \perp \lambda(t) \geq 0
\end{cases}
\]

$x(t) > 0$ for $0 < t < t_*$ (free dynamics):

\[
\begin{cases}
  x(0) = x_0 = 0 \\
  \dot{x}(t) = -x(t) + 1
\end{cases}
\]

Solution:

$x(t) = \exp(-t)(x_0 - 1) + 1 > 0$, for all $t \in [0; +\infty]$  

$x(t) = 0$ for $0 < t < t_*$ (constrained dynamics):

$\dot{x}(t) = 0, \lambda(t) = 1, x(t) = 0$
Solutions as absolutely continuous functions (AC solutions)

Example (Scalar LCS with $D = 0$)

Conclusion

- A unique continuous $x(t)$ has been computed for $x_0 > 0$ for all $t \in [0, +\infty)$.
- Infinitely many continuous $x(t)$ have been computed for $x_0$ for all $t \in [0, +\infty)$. 
Solutions as absolutely continuous functions \((AC\) solutions\)

**Example (Scalar LCS with \(D = 0\))**

Let us search for a continuous solution \(x(t)\) to

\[
\begin{aligned}
    x(0) &= x_0 \geq 0 \\
    \dot{x}(t) &= -x(t) - 1 - \lambda(t) \\
    0 &\leq x(t) \perp \lambda(t) \geq 0
\end{aligned}
\]

**Conclusion**

- A unique maximal continuous \(x(t)\) has been computed for \(x_0 > 0\) for \(t \in [0, t_\star)\).
  - No solution after \(t_\star\)
- No continuous solutions for \(x_0 = 0\).
Solutions as absolutely continuous functions (AC solutions)

Idea of the general statement

If $CB$ is a positive definite matrix (relative degree one) and $C x_0 \geq 0$ (consistent initial condition), the unique solution of (10) is an absolutely continuous function.

Why the condition on $CB$?

Derivation of the output $y(t)$

$$
\begin{align*}
    y(t) &= C x(t) \\
    \dot{y}(t) &= CAx(t) + CB\lambda(t) \text{ if } D = 0
\end{align*}
$$

(8)

If $CB > 0$, we have to solve the following LCP whenever $y(t) = 0$

$$
\begin{aligned}
    \dot{y}(t) &= CAx(t) + CB\lambda(t) \\
    0 \leq \dot{y}(t) \perp \lambda(t) &\geq 0
\end{aligned}
$$

(9)

The LCP (9) is a LCP for the time derivative $\dot{y}(t)$.

The good framework is the differential inclusion framework (see Lecture 3)
Existence and uniqueness results for LCS. Summary

Linear Complementarity Systems (LCS)

\begin{align}
\dot{x}(t) &= A x(t) + B \lambda(t) + a, \quad x(0) = x_0 \\
y(t) &= C x(t) + D \lambda(t) + b \\
0 &\leq y(t) \perp \lambda(t) \geq 0.
\end{align} \tag{10}

LCS with $D$ a P-matrix
ODE with Lipschitz continuous right-hand side.
Cauchy–Lipschitz Theorem \implies existence and uniqueness of solutions.

LCS with $D = 0$
Existence and uniqueness results based on

- Local (or nonzeno) solution based on the leading Markov parameters assumptions ($D, CB, CAB, CA^2 B, \ldots$)

- or maximal monotone differential inclusion
Extensions of complementarity problems

Let $C$ be a nonempty closed convex set. The subdifferential inclusion continues to hold

$$-y \in \partial \Psi_C(\lambda)$$

(11)

The complementarity relation is no longer valid for a set convex that is not a cone, but we can define the following dynamics

$$\begin{cases}
\dot{x}(t) = Ax(t) + B\lambda(t) + u(t) \\
y(t) = Cx(t) + D\lambda(t) + a(t) \\
-y(t) \in \partial \Psi_C(\lambda(t))
\end{cases}$$

(12)
Extensions of complementarity problems

Relay systems

\[ C = [-1, 1] \]

\[
\partial \Psi_{[-1,1]}(\lambda) = \begin{cases} 
\mathbb{R}^- & \text{if } \lambda = -1 \\
0 & \text{if } -1 < \lambda < 1 \\
\mathbb{R}^+ & \text{if } \lambda = 1 
\end{cases}
\] (13)

Equivalent formulations

\[ y \in \partial \Psi_{[-1,1]}(\lambda) \iff \lambda \in \text{sgn}(y) \]

Definition (Relay systems)

\[
\begin{cases} 
\dot{x}(t) = Ax(t) + B\lambda(t) + u(t) \\
y(t) = Cx(t) + D\lambda(t) + a(t) \\
\lambda(t) \in \text{sgn}(y(t))
\end{cases}
\] (14)

Application in sliding mode control, zener diode modeling or friction in mechanical systems
Introduction to nonsmooth dynamical systems

Complementarity Systems (CS)

Piecewise linear systems with monotone graphs
Extensions of complementarity problems

Cone complementarity condition
Let $K$ be a closed non empty convex cone. We can define

$$ K^* \ni y \perp \lambda \in K \iff -y \in \partial \Psi_K(\lambda) \iff -\lambda \in \partial \Psi_{K^*}(y) $$

(15)

where $K^*$ is the dual cone:

$$ K^* = \{ x \in \mathbb{R}^m \mid x^\top y \geq 0 \text{ for all } y \in K \}. $$

(16)

Definition (Cone Linear complementarity systems (CLCS))

$$ \begin{cases} 
\dot{x}(t) = Ax(t) + B\lambda(t) + u(t) \\
y(t) = Cx(t) + D\lambda(t) + a(t) \\
K^* \ni y(t) \perp \lambda(t) \in K,
\end{cases} $$

(17)
Outline

Complementarity Systems (CS)

Computations of equilibria for LCS

Stability of Linear Complementarity Systems
  Linear Time Invariant (LTI) passive systems
  Lyapunov stability of LCS
Equilibria for LCS

Linear Complementarity Systems (LCS)

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B\lambda(t) + a, \quad x \in \mathbb{R}^n, \lambda \in \mathbb{R}^m \\
y(t) &= Cx(t) + D\lambda(t) + b \\
0 &\leq y(t) \perp \lambda(t) \geq 0
\end{align*}
\]

Mixed Linear Complementarity Problem (MLCP)

We have to solve a Mixed Linear Complementarity Problem :

\[
\begin{align*}
0 &= A\tilde{x} + B\lambda + a, \\
y &= C\tilde{x} + D\lambda + b \\
0 &\leq y \perp \lambda \geq 0
\end{align*}
\]
Equilibria for LCS

Existence of solutions to MLCP

- Trivial case $a = 0, b = 0$. $\tilde{x} = 0$ is an equilibrium.
- If $A$ invertible, then we can substitute $\tilde{x} = -A^{-1}(B\lambda + a)$ to get a LCP

$$0 \leq (D - CA^{-1}B)\lambda + A^{-1}a + b \perp \lambda \geq 0$$  \hspace{1cm} (20)

If $(D - CA^{-1}B)$ is a P-matrix, it exists a unique solution $\lambda$ for all $a$ and $b$. The equilibrium is obtained with $\tilde{x} = -A^{-1}(B\lambda + a)$
Equilibria for LCS

Existence of solutions to MLCP
Reformulation into inclusion

\[- \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \lambda \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix} \right) \in \partial \psi_{\mathbb{R}^n \times \mathbb{R}^m} \left( \begin{bmatrix} \tilde{x} \\ \lambda \end{bmatrix} \right) \] (21)

as

\[- (Mz + q) \in \partial \psi_{\mathbb{R}^n \times \mathbb{R}^m} (z) \] (22)

Theorem
If \( M \) is a semi–definite positive matrix, then the inclusion (22) is solvable if and only if it is feasible, that is

\[ \exists z, \quad z \in \mathbb{R}^n \times \mathbb{R}^m \text{ and } Mz + q \in 0^n \times \mathbb{R}^m \] (23)

Application of a more general Theorem 2.4.7 of [? ].

Example
Trivial case \( a = 0, \ b \geq 0 \).
Outline

Complementarity Systems (CS)

Computations of equilibria for LCS

Stability of Linear Complementarity Systems
  Linear Time Invariant (LTI) passive systems
  Lyapunov stability of LCS
Lyapunov stability (Recap.)

Definition (Lyapunov stability)
The equilibrium \( \tilde{x} \) is said to be stable in the sense of Lyapunov if

for every \( \varepsilon > 0, \exists \delta > 0, \) such that \( \|x(0) - \tilde{x}\| < \delta \) then \( \|x(t) - \tilde{x}\| < \varepsilon, \forall t \geq 0. \) (24)

Definition (Asymptotic Lyapunov stability)
The equilibrium \( \tilde{x} \) is said to be asymptotically stable in the sense of Lyapunov if

► it is stable and
► for every \( \varepsilon > 0, \exists \delta > 0, \) such that \( \|x(0) - \tilde{x}\| < \delta \) then \( \lim_{t \to +\infty} \|x(t) - \tilde{x}\| = 0 \)

Definition (Exponential Lyapunov stability)
The equilibrium \( \tilde{x} \) is said to be asymptotically stable in the sense of Lyapunov if

► it is asymptotically stable and
► \( \exists \alpha, \beta, \delta > 0, \) such that \( \|x(0) - \tilde{x}\| < \delta \) then
  \[ \|x(t) - \tilde{x}\| \leq \alpha \|x(0) - \tilde{x}\| e^{-\beta t}, \forall t \geq 0 \]
Linear Time Invariant (LTI) systems

Let us consider the following system:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B\lambda(t) \\
y(t) &= Cx(t) + D\lambda(t)
\end{align*}
\]  

(25)

with a quadratic function \( V(x) = \frac{1}{2}x^T P x \) with \( P = P^T \).

Let us define the composition:

\[
V(t) : \mathbb{R} \rightarrow \mathbb{R} \\
t \mapsto V(x(t))
\]  

(26)
LTI passive systems

Derivation of $\mathcal{V}(t)$

$\dot{\mathcal{V}}(t) = x^T(t)P\dot{x}(t)$ \hspace{1cm} (25)

$x^T(t)P\dot{x}(t) = x^T(t)PAx(t) + x^T(t)B\lambda(t)$

$x^T(t)P\dot{x}(t) - \lambda^T(t)y(t) = x^T(t)PAx(t) + x^T(t)PB\lambda(t) - \lambda^T(t)y(t)$

$x^T(t)P\dot{x}(t) - \lambda^T(t)y(t) = x^T(t)PAx(t) + \lambda^T(t)B^TPx(t) - \lambda^T(t)(Cx(t) + D\lambda(t))$

$x^T(t)P\dot{x}(t) - \lambda^T(t)y(t) = x^T(t)PAx(t) + \lambda^T(t)(B^TP - C)x(t) - \lambda^T(t)D\lambda(t)$ \hspace{1cm} (26)
Derivation of $\mathcal{V}(t)$

\begin{align*}
\mathcal{V}(x(T) - V(x(0)) & = \int_0^T \lambda^T(t)y(t)dt \\
& = \int_0^T x^T(t)PAx(t) + \lambda^T(t)(B^TP - C)x(t) - \lambda^T(t)(D\lambda(t))dt \\
& = \frac{1}{2} \int_0^T \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}^T \begin{bmatrix} A^TP + PA & PB - C^T \\ B^TP - C & -(D + D^T) \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} dt \\
& \quad (25)
\end{align*}
LTI passive systems

Linear Time Invariant (LTI) systems

\[ \begin{aligned}
\dot{x}(t) &= A x(t) + B \lambda(t) \\
y(t) &= C x(t) + D \lambda(t)
\end{aligned} \]  
(26)

Definition

The system \( \Sigma(A, B, C, D) \) given in (26) is said to be passive (dissipative with respect to the supply rate \( \lambda^T y \)) if there exists a function \( V : \mathbb{R}^n \to \mathbb{R}_+ \) (a storage function) such that

\[ V(x(t_0)) + \int_{t_0}^{t} \lambda^T(t)y(t)dt \geq V(x(t)) \]  
(27)

holds for all \( t_0 \) and \( t \) with \( t \geq t_0 \) and for all \( L^2 \)-solutions \( (x, y, \lambda) \).
LTI passive systems

Theorem
The system $\Sigma(A, B, C, D)$ is passive if and only if the following linear matrix inequality (LMI)

$$P = P^T > 0 \text{ and } \begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -(D + D^T) \end{bmatrix} \leq 0$$

(28)

has a solution.

In this case, $V(x) = \frac{1}{2} x^T P x$ is the corresponding energy storage function.
LTI passive systems

Theorem

The system \( \Sigma(A, B, C, D) \) is passive if there exist matrices \( L \in \mathbb{R}^{n \times m} \) and \( W \in \mathbb{R}^{m \times m} \) and a symmetric positive semi-definite matrix \( P \in \mathbb{R}^{n \times n} \), such that:

\[
\begin{align*}
A^T P + PA &= -LL^T \quad \text{(29)} \\
B^T P - C &= -W^T L^T \quad \text{(30)} \\
-D - D^T &= -W^T W. \quad \text{(31)}
\end{align*}
\]
LTI passive systems

Reformulation

\[
\begin{bmatrix}
A^T P + PA & PB - C^T \\
B^T P - C & -(D + D^T)
\end{bmatrix} = - \begin{bmatrix}
LL^T & LW \\
W^T L^T & W^T W
\end{bmatrix} = - \begin{bmatrix}
L \\
W
\end{bmatrix}^T \begin{bmatrix}
L \\
W
\end{bmatrix} \triangleq -Q
\]
LTI passive systems

Dissipation inequality

The *dissipation equality*

\[
V(x(T)) - V(x(0)) = \frac{1}{2} \int_0^T \lambda^T(t)y(t) + \frac{1}{2} \int_0^T \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}^T Q \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} dt, \quad \forall \ T \geq 0 \tag{29}
\]

in terms of the positive semi-definite matrix

\[
Q \triangleq \begin{pmatrix} LL^T & W^T L^T \\ LW & W^T W \end{pmatrix}, \tag{30}
\]

then implies that

\[
V(x(T)) - V(x(0)) - \frac{1}{2} \int_0^T \lambda^T(t)y(t) \leq 0. \tag{31}
\]

Strictly passive LTI systems

The system is said to be *strictly passive* when \( Q \) is positive definite.
LTI passive systems

Remarks

- $(D + D^T) \succeq 0$ implies that $D$ is a semi-definite positive matrix.
- if $D = 0$, then $(D + D^T) = W^T W = 0 \implies W = 0$ and we get

$$B^T P - C = -W^T L^T = 0 \implies C = B^T P \implies CB = B^T PB \succeq 0 \quad (32)$$

The matrix $CB$ is a semi-definite positive matrix.
Passive LCS

Assumption
The trajectory \( x(t) \) of the LCS is continuous.

Definition (Passive LCS)
The LCS
\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B\lambda(t) \\
y(t) &= Cx(t) + D\lambda(t) \\
0 &\leq y(t) \perp \lambda(t) \geq 0
\end{align*}
\] (33)
is said to be (strictly) passive if the system \( \Sigma(A, B, C, D) \) is (strictly) passive.

Supply rate
The complementarity condition implies that \( \lambda^T(t)y(t) = 0 \) for all \( t \geq 0 \). Then the dissipation inequality reduces to
\[
V(x(T)) - V(x(0)) \leq 0
\] (34)
Lyapunov stability of LCS

Theorem

- If the LCS is passive, then the LCS is Lyapunov stable.
- If the LCS is strictly passive, then the LCS is globally exponentially stable.

The energy storage function plays the role of a Lyapunov function.
If the LCS is passive, then $D$ is a semi-definite positive matrix.
Lyapunov stability of LCS

Example (The RLC circuit with a diode. A half wave rectifier)
A LC oscillator supplying a load resistor through a half-wave rectifier.

**Figure:** Electrical oscillator with half-wave rectifier
Lyapunov stability of LCS

Example (The RLC circuit with a diode. A half wave rectifier)

The following linear complementarity system is obtained:

\[
\begin{pmatrix}
\dot{v}_C \\
\dot{i}_L
\end{pmatrix} =
\begin{pmatrix}
0 & \frac{-1}{C} \\
\frac{1}{L} & 0
\end{pmatrix} \cdot \begin{pmatrix} v_C \\
\dot{i}_L
\end{pmatrix} + \begin{pmatrix} \frac{-1}{C} \\
0
\end{pmatrix} \cdot i_D
\]

together with a state variable \( x \) and one of the complementary variables \( \lambda \):

\[
x = \begin{pmatrix} v_C \\
\dot{i}_L
\end{pmatrix}, \quad \lambda = i_D, \quad y = -v_D
\]

and

\[
y = -v_D = \begin{pmatrix} -1 & 0 \end{pmatrix} x + \begin{pmatrix} R \end{pmatrix} \lambda,
\]

Standard form for LCS

\[
\begin{cases}
\dot{x} = Ax + B\lambda \\
y = Cx + D\lambda \\
0 \leq y \perp \lambda \geq 0
\end{cases}
\]
Lyapunov stability of LCS

Example (The RLC circuit with a diode. A half wave rectifier)

- $D = R$ so $D^T + D = 2R > 0$
- We choose $P = \begin{bmatrix} C & 0 \\ 0 & L \end{bmatrix}$

$$V(x) = \frac{1}{2} C v_C^2 + \frac{1}{2} L i_L^2$$ (35)

we get

$$PB - C^T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A^T P + PA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ (36)

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2R \end{bmatrix}$$ (37)