Introduction to nonsmooth dynamical systems
Lecture 3. Differential Inclusions

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Contents

Lecture 3. Maximal Monotone differential inclusions
- Maximal Monotone operators.
- Existence and uniqueness results.
- LCS as maximal monotone differential inclusion.
- Filippov’s Differential inclusion.
- Computation of equilibria
- Lyapunov stability of maximal monotone differential inclusions.
Outline

Differential inclusion

Maximal monotone operators

Maximal monotone differential inclusion

LCS as Maximal Monotone Differential Inclusion

Filippov differential inclusions

Equilibria for differential inclusions

Lyapunov stability of monotone differential inclusions
  Absolutely continuous functions
  Lyapunov stability of monotone differential inclusions
General differential inclusion

**Concept of differential inclusions**
Differential inclusions are a generalization of the concept of differential equations of the form

\[ \dot{x}(t) \in A(x(t), t) \]  

where \((x, t) \mapsto A(x, t)\) is a multi-valued map, *i.e.* \(A(x, t)\) is a set, possibly empty, rather than a single point.

**A very general concept**
Differential inclusions is a very general concept.

▶ It contains Ordinary Differential Equations (ODE)
▶ There are many types of differential inclusions [1].

We will focus on Maximal Monotone Differential Inclusions and Filippov’s differential inclusions
LCS as differential inclusion

Complementarity condition as a subdifferential inclusion

\[ 0 \leq y \perp \lambda \geq 0 \iff -y \in \partial \Psi_{\mathbb{R}_+^m}(\lambda) \iff -\lambda \in \partial \Psi_{\mathbb{R}_+^m}(y) \quad (2) \]

LCS as a differential inclusion with \( D = 0 \) and \( b = 0 \)

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B\lambda(t) + a \\
y(t) &= Cx(t) \\
0 &\leq y(t) \perp \lambda(t) \geq 0 \\
x(0) &= x_0.
\end{align*}
\]

\[
\iff \quad \begin{cases}
-(\dot{x}(t) - Ax(t) - a) \in B\partial \Psi_{\mathbb{R}_+^m}(Cx(t)), \\
x(0) = x_0
\end{cases}
\quad (3)
\]
Outline

Differential inclusion

Maximal monotone operators

Maximal monotone differential inclusion

LCS as Maximal Monotone Differential Inclusion

Filippov differential inclusions

Equilibria for differential inclusions

Lyapunov stability of monotone differential inclusions
  Absolutely continuous functions
  Lyapunov stability of monotone differential inclusions
Maximal monotone operators

Let $2^{\mathbb{R}^n}$ be the set of the subsets of $\mathbb{R}^n$

**Definition (Monotone multi-valued operator)**

A operator $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is monotone if

\[
\forall y_1 \in T(x_1), \quad \forall y_2 \in T(x_2), \quad (y_2 - y_1)^T(x_2 - x_1) \geq 0
\]  

(4)

**Definition (Graph)**

Let $T$ multi–valued operator $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$. The graph of $T$ is defined by

\[
Gr(T) = \{(x, y) \mid y \in T(x)\}
\]

(5)

**Definition (Maximal Monotone multi-valued operator)**

A operator $T$ is maximal monotone if it is maximal for all the monotone operators for the inclusion of graphs.

In other words, $T$ is monotone and for all other monotone operator $S$ then $Gr(T) \subset Gr(S) \implies T = S$
Maximal monotone operators

Definition (Domain)
The domain of an operator $T$ is defined by $D(T) = \{x \mid T(x) \neq \emptyset\}$

Definition (Range of $T$)
Let $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be an operator. The range of $T$ is defined by

$$R(T) = \bigcup_{x \in \mathbb{R}^n} \{y \mid y \in T(x)\} \quad (6)$$

Definition (Inverse of $T$)
Let $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a maximal monotone operator. Its inverse $T^{-1}$ is defined by

$$y \in T(x) \iff x \in T^{-1}(y) \quad (7)$$

and we have $D(T^{-1}) = R(T)$ and $R(T^{-1}) = D(T)$

Its inverse is defined by the symmetry of its graph with respect to $y = x$
Maximal monotone operators
Maximal monotone operators
Maximal monotone operators

\[ \text{sgn}(x) = \partial|x| \]
Outline

Differential inclusion

Maximal monotone operators

Maximal monotone differential inclusion

LCS as Maximal Monotone Differential Inclusion

Filippov differential inclusions

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Maximal monotone differential inclusion

Let $T : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a maximal monotone operator.

Definition (Maximal monotone differential inclusion)
A maximal monotone differential inclusion is defined by

$$-\dot{x}(t) \in T(x(t)).$$  \hspace{1cm} (6)

Definition (Perturbed maximal monotone differential inclusion)
A perturbed maximal monotone differential inclusion is defined by

$$-(\dot{x}(t) + f(x, t)) \in T(x(t)),$$ \hspace{1cm} (7)

where $f$ is a Lipschitz continuous map w.r.t $x$. 
Maximal monotone differential inclusion

The main instance of maximal monotone operators:
The subdifferential of (some) convex functions.

Theorem
For a lower semi–continuous convex proper function \( \Phi \), the subdifferential \( \partial \Phi(x) \) is a maximal monotone operator

Remarks
- Obvious in the regular case: \( \phi(x) : \mathbb{R} \to \mathbb{R} \) a convex potential \( C^2 \)
  \( \phi''(x) \geq 0 \) and \( \phi'(x) \) is monotone (increasing single–valued function)
- For a maximal monotone operator in \( \mathbb{R} \), i.e. \( T : \mathbb{R} \to 2^{\mathbb{R}} \) it exists a lower semi–continuous convex proper function \( \Phi \) such that \( T = \partial \Phi \)
Maximal monotone differential inclusion

Definition (lower semi-continuity)

A function $\Phi : \mathbb{R}^n \to \mathbb{R} \cup +\infty$ is lower semi-continuous if one of the following equivalent assertions is satisfied:

- $\liminf_{x \to x_0} \Phi(x) \geq \Phi(x_0)$
- Its epigraph is closed

Remarks

- $\liminf_{x \to x_0} \Phi(x) = \lim_{\varepsilon \to 0} \left( \inf \{ \Phi(x), x \in B(x_0, \varepsilon) \setminus \{x_0\} \} \right)$
- Continuity implies semi-continuity.
Maximal monotone differential inclusion

For a convex proper function $\Phi$, the semi–continuity property has only to be checked on the boundary of the domain of definition

$$\partial D(\Phi) = \overline{D(\Phi)} \setminus D(\Phi).$$

Examples

\begin{align*}
    y &= \Phi(x) \\
    y &= \Phi(x) \\
    y &= \Phi(x)
\end{align*}
Maximal monotone differential inclusion

Counter-examples

\[ y = \Phi(x) \]

\[ \begin{array}{c}
\begin{array}{c}
\infty \\
\infty \\
x
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\infty \\
\infty \\
x
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{c}
\infty \\
\infty \\
x
\end{array}
\end{array} \]
Maximal monotone differential inclusion

Examples

- $\Phi(x) = 0 = \Psi_\mathbb{R}, \ T(x) = 0$
  \[-\dot{x} + f(x, t) = 0\]  \hspace{1cm} (8)

- $\Phi(x) = \psi_c(x), \ T(x) = \partial \psi_C(x), \ C$ a closed non empty convex set
  \[-\dot{x} + f(x, t) \in \partial \psi_C(x)\]  \hspace{1cm} (9)

- relay or sign function $\Phi(x) = |x|, \ T(x) = \partial |x|$
  \[-\dot{x} \in \partial |x| \iff -\dot{x} \in \text{sgn}(x)\]  \hspace{1cm} (10)

- 2-norm $\Phi(x) = \|x\|, \ T(x) = \partial \|x\| = \begin{cases} \{ \frac{x}{\|x\|} \} & \text{if } x \neq 0 \\ \{ s \mid \|s\| \leq 1 \} & \text{if } x = 0 \end{cases}$
Maximal monotone differential inclusion

Examples

▶ relay with dead zone

\[
\Phi(x) = \begin{cases} 
-x + 1, & \text{if } x \leq -1 \\
0, & \text{if } -1 \leq x \leq 1 \\
x - 1, & \text{if } x \geq 1
\end{cases}
\]  

(8)

▶ sum of convex functions. \( \Phi(x) = \frac{1}{2} \cdot ax^2 + |x|, T(x) = ax + \text{sgn}(x) \)

\[-\dot{x} \in ax + \partial|x| \iff -\dot{x} - ax \in \text{sgn}(x)\]

(9)

1. \( a > 0 \). \( \Phi(x) \) is convex and \( T(x) \) is maximal monotone.
2. \( a < 0 \). \( \Phi(x) \) is not convex and \( T(x) \) is not monotone.
Maximal monotone differential inclusion

\[ \Phi(x) = \frac{1}{2}x^2 + |x| \]

\[ \Phi(x) = -\frac{1}{2}x^2 + |x| \]
Maximal monotone differential inclusion

\[ y \in \text{sgn}(x) + ax, \quad a > 0 \]

\[ y \in \text{sgn}(x) + ax, \quad a < 0 \]
Maximal monotone differential inclusion

Link with gradient systems with convex potentials

1. $\phi(x): \mathbb{R} \rightarrow \mathbb{R}$ a convex potential $C^2$
   $\phi''(x) \geq 0$ and $\phi'(x)$ is monotone (increasing function)
   $$-\dot{x} = \phi'(x)$$  \hspace{1cm} \text{(8)}

2. $\Phi(x): \mathbb{R} \rightarrow \mathbb{R}$ a convex potential not necessarily differentiable, but proper and lower semi-continuous $\partial \Phi(x)$ is a maximal monotone operator.
   $$-\dot{x} = \partial \Phi(x)$$  \hspace{1cm} \text{(9)}
Existence and uniqueness results

Theorem (Brézis 1973)

Let \( T : \mathbb{R}^n \to 2^{\mathbb{R}^n} \) be a maximal monotone operator such that \( D(T) \neq \emptyset \). Let a function \( f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) such that

1. the function \( f(x, \cdot) \) is Lipschitz continuous on \( D(T) \) that is

\[
\exists L \geq 0, \forall t \in [0, t_{\text{max}}], \forall x_1, x_2 \in \overline{D(T)}, \quad \|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\| \quad (10)
\]

2. \( \forall x \in \overline{D(T)} \), the mapping \( t \mapsto f(x, t) \) belongs to \( L^\infty(0, t_{\text{max}}; \mathbb{R}^n) \)

Then, for all \( x_0 \in \overline{D(T)} \), it exists a unique solution \( x(t) \) which is absolutely continuous such that

\[
\begin{cases}
-(\dot{x}(t) + f(x(t), t)) \in T(x(t)), \text{ almost everywhere on } [0, t_{\text{max}}] \\
x(0) = x_0
\end{cases}
\quad (11)
\]
Existence and uniqueness results

Existence. Ideas of proof

Option 1 By using the Moreau-Yosida regularization of $T$

$$T_\lambda(x) = \frac{1}{\lambda}(I - J_\lambda)(x), \lambda > 0, \quad (12)$$

with $J_\lambda(x)$ the resolvent of $T(x)$ given by

$$J_\lambda(x) = (I + \lambda T)^{-1}(x). \quad (13)$$

For a maximal monotone operator $T$ or $\mathbb{R}$, $J_\lambda$ is defined over $\mathbb{R}$ and is contracting. The mapping $T_\lambda$ is a maximal monotone operator and Lipschitz continuous with a Lipschitz constant of $\frac{1}{\lambda}$. We consider that ODE with Lipschitz r.h.s.

$$-(\dot{x}_\lambda(t) + f(x_\lambda(t), t)) = T_\lambda(x_\lambda(t)) \quad (14)$$

and then the limit $\lambda \to 0$ of the sequence of solutions $x_\lambda$.

Option 2 By approximation using a discretization scheme, and then compactness results.
Existence and uniqueness results

Uniqueness. Simple case – $\dot{x}(t) \in T(x(t)), \ x \in \mathbb{R}$

Let us consider two solution $x_1$ and $x_2$

Since $T(x)$ is monotone, we have

$$(\dot{x}_1(s) - \dot{x}_2(s))^T(x_1(s) - x_2(s)) \leq 0 \text{ almost everywhere on } [0, T] \quad (12)$$

By integrating over $[0, t]$, we get

$$\frac{1}{2}(x_2(t) - x_1(t))^2 - \frac{1}{2}(x_2(0) - x_1(0))^2 \leq 0 \quad (13)$$

If $x_1(0) = x_2(0)$, we have

$$\frac{1}{2}(x_2(t) - x_1(t))^2 \leq 0 \implies x_2 = x_1 \quad (14)$$
Existence and uniqueness results

Uniqueness. Perturbed case \((-\dot{x}(t) + f(x, t)) \in \mathcal{T}(x(t))\)

Let us consider two solution \(x_1\) and \(x_2\)

Since \(\mathcal{T}(x)\) is monotone, we have

\[
(\dot{x}_1(s) + f(x_1(s), s) - \dot{x}_2(s) - f(x_2(s), s))^T(x_1(s) - x_2(s)) \leq 0
\]  \quad (12)

almost everywhere on \([0, T]\).

By integrating over \([0, t]\), we get

\[
\frac{1}{2}(x_2(t) - x_1(t))^2 \leq \int_0^t (f(x_2(s), s) - f(x_1(s), s))^T(x_1(s) - x_2(s))ds
\]  \quad (13)

Since \(f\) is lipschitz, we have

\[
(x_2(t) - x_1(t))^2 \leq 2L \int_0^t \|x_1(s) - x_2(s)\|^2 ds
\]  \quad (14)
Existence and uniqueness results

**Gronwall Lemma**

Let $a$ a positive constant and $m$ a integrable function, nonnegative almost everywhere on $(0, t_{\text{max}})$ and a function $\phi$ a continuous function on $[0, t_{\text{max}}]$. If

$$\forall t \in [0, t_{\text{max}}], \phi(t) \leq a + \int_0^t m(s)\phi(s) \, ds \quad (12)$$

then

$$\forall t \in [0, t_{\text{max}}], \phi(t) \leq a \exp\left(\int_0^t m(s) \, ds\right) \quad (13)$$

Applying the Gronwall Lemma, for $a = 0$ and $m(s) = 2L$ and $\phi(s) = \|x_1(s) - x_2(s)\|^2$, we get

$$\|x_2(t) - x_1(t)\|^2 \leq 0 \implies x_2 = x_1 \quad (14)$$
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Differential inclusion

Maximal monotone operators

Maximal monotone differential inclusion

LCS as Maximal Monotone Differential Inclusion

Filippov differential inclusions

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Come back to LCS with $D = 0$ but $B \neq I_d \neq C$

**Theorem (LCS as maximal monotone differential inclusion)**

Let us consider the following LCS

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B\lambda(t) + a(t), \quad x(0) = x_0 \\
y(t) &= Cx(t) \\
0 &\leq y(t) \perp \lambda(t) \geq 0.
\end{align*}
\] (15)

If there exists $P$ a symmetric definite positive matrix such that

\[PB = C^T\] (16)

then we can perform a change of variable $z = Rx$ with $R^2 = P$, $R \geq 0$, $R = R^T$

\[-(\dot{z}(t) - RAR^{-1}z(t) - Ra(t)) \in RB \partial_{\mathbb{R}^m_+} (CR^{-1}z(t))\] (17)

such that (17) is a maximal monotone differential inclusion.
Come back to LCS with $D = 0$ but $B \neq I_d \neq C$

We have the following equivalence

\[
\begin{cases}
\dot{x}(t) = Ax(t) + B\lambda(t) + a(t) \\
y(t) = Cx(t) \\
0 \leq y(t) \perp \lambda(t) \geq 0, \\
x(0) = x_0
\end{cases} \iff \begin{cases}
-(\dot{x}(t) - Ax(t) - a(t)) \in B\partial\Psi_{\mathbb{R}^m_+}(Cx(t)), \\
x(0) = x_0
\end{cases}
\]

We can perform a change of variable $z = Rx$ with $R^2 = P$, $R \succeq 0$, $R = R^T$

\[
-(\dot{z}(t) - RAR^{-1}z(t) - Ra(t)) \in RB \partial\Psi_{\mathbb{R}^m_+}(CR^{-1}z(t))
\]
Come back to LCS with $D = 0$ but $B \neq I_d \neq C$

For a matrix $E$, the function $\phi(x) = \Psi_{\mathbb{R}^m_+}(Ex)$ is a proper convex function and its subdifferential is given by

$$\partial \phi(x) = E^T \partial \Psi_{\mathbb{R}^m_+}(Ex) \quad (15)$$

($\text{Im}(E)$ contains a point of $\text{ri}(D(\partial \Psi_{\mathbb{R}^m_+}))$) (Chain rule)

In our application, we set $E = CR^{-1}$ and we have

$$E^T = R^{-T} C^T = R^{-1} R^2 B = RB \quad (16)$$

The obtained inclusion

$$-(\dot{z}(t) - RAR^{-1}z(t) - Ra) \in \partial \Phi(z(t)) = E^T \partial \Psi_{\mathbb{R}^m_+}(Ez(t)), \quad (17)$$

is a maximal monotone differential inclusion
Outline

Differential inclusion

Maximal monotone operators

Maximal monotone differential inclusion

LCS as Maximal Monotone Differential Inclusion

Filippov differential inclusions

Equilibria for differential inclusions

Lyapunov stability of monotone differential inclusions
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Filippov differential inclusions

Ordinary differential equations with discontinuous right hand side

Let us consider the Cauchy problem

\[
\begin{align*}
\dot{x}(t) &= f(x) \\ x(t_0) &= x_0
\end{align*}
\]

(18)

where \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a bounded function.

- The Cauchy problem may have no (Carathéodory) solution, if \( f \) is discontinuous
Example

\[
\begin{cases}
\dot{x}(t) = a - \text{sgn}(x(t)), \text{ with } 0 < a < 1 \\
x(t_0) = x_0 
\end{cases}
\] (19)

with the signum function defined as

\[
\text{sgn}(x) = \begin{cases}
1, & x > 0 \\
0, & x = 0 \\
-1, & x < 0 
\end{cases}
\] (20)

▶ \(x_0 = 0\) no solution
▶ \(x_0 > 0\) no global solution
Filippov differential inclusion

Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a bounded function.

**Definition (Filippov’s concept of solutions[2, 5])**

The Filippov differential inclusion is defined by

\[
\begin{align*}
\dot{x}(t) & \in F(x) \\
x(t_0) & = x_0
\end{align*}
\]

with

\[
F(x) = \bigcap_{\varepsilon > 0} \text{co}\{f(x + \varepsilon B_n)\}
\]

where

- \( B_n \) is the unit ball of \( \mathbb{R}^n \)
- \( \text{co}(X) \) defines the convex hull of \( X \)
- \( \overline{X} \) is the closure of \( X \)
Filippov differential inclusion

Geometrical interpretations

- If \( f \) is continuous in \( x \), then \( F(x) = \{ f(x) \} \)
- The solution does not depend on the values of \( f(x) \) at the discontinuity points
  An equivalent formulation is given by
  \[
  F(x) = \bigcap_{\varepsilon > 0} \bigcap_{\mu(S) = 0} \text{co}\{ f(B(x, \varepsilon) \setminus S) \}
  \]  
  (23)

  where \( \mu \) is the Lebesgue measure.
  More generally, the definition works also for \( f \) essentially bounded (bounded on a bounded neighborhood of every point, excluding sets of measure zero).
Filippov differential inclusion

Geometrical interpretations

- smearing out the vector fields near the discontinuities

\[ f(x) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]
Filippov differential inclusions

Computation of the Filippov right-hand-side in practice.

In practice, we often have the following assumptions:

- the set of discontinuities points of \( f \), denoted as \( \mathcal{M} \) is given by a set of hypersurfaces of co-dimension 1:

\[
\mathcal{M} = \{ x \in \mathbb{R}^n | \varphi_k(x) = 0, k = [1, N] \}
\]  \( (23) \)

where \( \varphi_k : \mathbb{R}^n \to \mathbb{R} \) are differentiable maps that define a finite number of domains \( G_j \subset \mathbb{R}^n, j \in [1, M] \).

- for \( x \in \mathcal{M} \) and a neighborhood \( \Omega \) of \( x \), we have

\[
G_j \cap \Omega \neq \emptyset, \text{ for some } i \in I(x)
\]  \( (24) \)

- for \( i \in I(x) \), the function \( f \) admits a limit for all the domains such that

\[
\lim_{y \to x} f(y, t) = f_i(x, t)
\]  \( (25) \)

- then we get

\[
F(x) = \text{co}\{f_i(x, t), i \in I(x)\}
\]  \( (26) \)
Filippov differential inclusions

Example

\[
\begin{align*}
\dot{x}(t) &= a - \text{sgn}(x(t)), \text{ with } 0 < a < 1 \\
x(t_0) &= x_0
\end{align*}
\] (27)

- $x > 0$, we get $F(x) = \{a - 1\}$
- $x < 0$, we get $F(x) = \{a + 1\}$
- $x = 0$, we get $F(x) = \text{co}\{a - 1, a + 1\} = [a - 1, a + 1]$  

For $x(t_*) = 0$, we have a solution $\dot{x}(t) = 0, t \in [t_*, +\infty)$ since

\[ 0 \in F(x) = [a - 1, a + 1] \] (28)
Filippov differential inclusions

\[ \text{sgn}(x) \]

\[ 1 \]

\[ 0 \]

\[ -1 \]

\[ \text{Sgn}(x) \]

\[ 1 \]

\[ -1 \]
Upper semi-continuous differential inclusion

Definition (Upper semi-continuous set-valued map [5, 4])
A set-valued map $T : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ is upper semi-continuous at $x_0 \in \mathbb{R}^n$ if for any open set $M$ containing $F(x_0)$ there exists a neighborhood $\Omega$ of $x_0$ such that $F(\Omega) \subset M$.

Theorem (Upper semi-continuous differential inclusion)
Let us consider the following differential inclusion

$$\dot{x}(t) \in T(x(t))$$

(27)

where $T : \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ is upper semi-continuous set valued map with closed convex values. Then the differential inclusion (27) has always a solution for any $x(t_0) = x_0$. 
Upper-semi continuous differential inclusion

Idea of the proof

- Approximation of $T$ by a sequence of $T_k$ of Lipschitz set-valued operators.
- Passing to the limit by compactness.
Filippov differential inclusion

Theorem (Filippov [2])

*The Filippov differential inclusion (22) has always a solution for any* \( x(t_0) = x_0 \).
Filippov differential inclusion

Lemma (Closed Graph I)

Let $F$ be an upper semi-continuous multi-valued map with closed values. Then, the graph of $F$ is closed.

Lemma (Closed Graph II)

Assume that the graph of $F$ is closed and the set

$$M = \{ F(x) \mid |x - x_0| < \delta \}$$

(28)

with $\delta > 0$, is compact. Then, $F$ is upper semi-continuous at $x_0$

Ideas of the proof for the Filippov DI

▶ The values of $F$ is closed and convex by definition.
▶ The graph of $F(x)$ is closed and $F$ is bounded, by the Lemma Closed Graph II, $F$ is a semi-continuous map.
Filippov differential inclusion

Comments
Uniqueness of solutions is not guaranteed by Filippov convexification.

\[ f(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ f(x) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \]

\[ f(x) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \]
Filippov differential inclusion

Example (Utkin’s example)

\[
\begin{cases}
\dot{x}_1(t) = -\text{sgn}(x_1) + 2\text{sgn}(x_2) \\
\dot{x}_2(t) = -2\text{sgn}(x_1) - \text{sgn}(x_2),
\end{cases}
\] (29)

\[
\dot{x}(t) = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad \text{if} \quad x \in X_1 = \{x_1 > 0, x_2 > 0\},
\]

\[
\dot{x}(t) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \text{if} \quad x \in X_2 = \{x_1 < 0, x_2 > 0\},
\] (30)

\[
\dot{x}(t) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad \text{if} \quad x \in X_3 = \{x_1 < 0, x_2 < 0\},
\]

\[
\dot{x}(t) = \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \quad \text{if} \quad x \in X_4 = \{x_1 > 0, x_2 < 0\}.
\]
Filippov differential inclusion

Example (Utkin’s example)
Finite accumulation of switches in 0.
Filippov differential inclusion

\[
\begin{align*}
\dot{x}_1(t) &= -\text{Sgn}(x_1) + (1 + c) \text{Sgn}(x_2) \\
\dot{x}_2(t) &= -(1 + c) \text{Sgn}(x_1) - \text{Sgn}(x_2),
\end{align*}
\]

with \( c = 25 \) \quad (31)
Filippov differential inclusion

Siconos Platform – INRIA
Introduction to nonsmooth dynamical systems Lecture 3. Equilibria and stability

Equilibria for differential inclusions

Outline

Differential inclusion

Maximal monotone operators

Maximal monotone differential inclusion

LCS as Maximal Monotone Differential Inclusion

Filippov differential inclusions

Equilibria for differential inclusions

Lyapunov stability of monotone differential inclusions
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Equilibria for differential inclusions

Computation of equilibria

The equilibria of

\[-(\dot{x}(t) + f(x)) \in T(x(t))\]  \hspace{1cm} (32)

are given by the following generalized equation

\[-f(\tilde{x}) \in T(\tilde{x})\]  \hspace{1cm} (33)

Generalized Equation

\[0 \in f(\tilde{x}) + T(\tilde{x})\]  \hspace{1cm} (34)
Equilibria for differential inclusion. The simple case $f(\tilde{x}) = 0$

$$0 \in T(\tilde{x}) \iff x \in T^{-1}(0)$$

A condition for $T^{-1}(0) \neq \emptyset$ is $0 \in D(T^{-1}) = R(T)$. 
Equilibria for differential inclusion. The simple case $f(\tilde{x}) = 0$

**Theorem**

Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R} \cup +\infty$ be a proper convex function. Then

$$0 \in \partial \phi(\tilde{x}) \iff \tilde{x} \in \text{argmin}_{z \in \mathbb{R}^n} \phi(z)$$

(35)

A solution to $0 \in \partial f(\tilde{x})$ exists if $\min_{z \in \mathbb{R}^n} \phi(z) > -\infty$

---

**Figure:** Absolute value function
Equilibria for differential inclusion. The simple case $f(\tilde{x}) = 0$

Example ($\phi(x) = \Psi_C(x)$)

$$\arg\min_{z \in \mathbb{R}^n} \Psi_C(z) = C$$ (35)
Equilibria for differential inclusion. The affine case $f(\tilde{x}) = A\tilde{x} + a$

$T(x) = \partial \psi_C(x)$ with $C$ a polyhedral set $C = \{Cx + d \geq 0\}$

$$\partial \psi_C(x) = N_C(x) = \{s \mid s = -C^T \lambda, 0 \leq \lambda \perp Cx + d \geq 0\}$$ (36)

The generalized equation

$$-(A\tilde{x} + a) \in \partial \psi_C(\tilde{x})$$ (37)

is equivalent to the following MLCP

$$\begin{cases} A\tilde{x} + a = C^T \lambda \\ y = Cx + d \\ 0 \leq y \perp \lambda \geq 0 \end{cases}$$ (38)

that can be written in turns as an inclusion

$$-\left( \begin{bmatrix} A & -C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \lambda \end{bmatrix} + \begin{bmatrix} a \\ d \end{bmatrix} \right) \in \partial \psi_{\mathbb{R}^n \times \mathbb{R}^m_+} \left( \begin{bmatrix} \tilde{x} \\ \lambda \end{bmatrix} \right)$$ (39)

If $A$ is semidefinite positive then $\begin{bmatrix} A & -C^T \\ C & 0 \end{bmatrix}$ is semi-definite positive. If the inclusion is feasible, then it is solvable.
Equilibria for differential inclusion. The affine case $f(\tilde{x}) = A\tilde{x} + a$

$T(x) = \partial \Psi_C(x)$ with $C$ a convex set and $A$ symmetric definite positive

We can define a convex function $\Phi(x) = \Psi_C(x) + \frac{1}{2}x^T Ax + a^T x$. Then

$$\min_{z \in \mathbb{R}^n} \Phi(x) = \min_{z \in \mathbb{R}^n} \Psi_C(x) + \frac{1}{2}x^T Ax + a^T x = \min_{z \in C} \frac{1}{2}x^T Ax + a^T x \quad (40)$$

This is a convex minimization problem that possess a solution and the optimality conditions are

$$0 \in \partial \Phi(\tilde{x}) = A\tilde{x} + a + \partial \Psi_C(\tilde{x}) \quad (41)$$

Remark

If a polyhedral set $C = \{Cx + d \geq 0\}$, then the optimality condition are

$$\begin{cases} Ax + a = C^T \lambda \\ y = Cx + d \\ 0 \leq y \perp \lambda \geq 0 \end{cases} \quad (42)$$
Outline

Differential inclusion

Maximal monotone operators

Maximal monotone differential inclusion

LCS as Maximal Monotone Differential Inclusion

Filippov differential inclusions

Equilibria for differential inclusions

Lyapunov stability of monotone differential inclusions
    Absolutely continuous functions
    Lyapunov stability of monotone differential inclusions
Application of standard results for stability and asymptotic behavior

**Sufficient assumptions**

- Existence of absolutely continuous solution.
- Continuity with respect to initial conditions.
- Lyapunov function $V \in C^1$
- Invariants in the interior of the domain of maximal monotone operators

With these assumptions, the main result for smooth systems can be proved

- Lyapunov stability theorems.
- Lasalle invariance principle.

**Relaxed results**

In the literature, a large number of results relax the assumptions that are sometimes not necessary. For the sake of simplicity, we assume that there are valid for our applications. In the sequel, we present more specific results for Maximal Monotone differential inclusions.
Lyapunov stability of monotone differential inclusions

Monotone differential inclusions, $x(0) = x_0$

- **Standard form**
  \[ -\dot{x}(t) \in T(x(t)) \quad (43) \]

- **Standard perturbed form**
  \[ -\dot{x}(t) + f(x(t), t) \in T(x(t)) \quad (44) \]

- **Sub-differential of $\Phi$ convex, proper and lower-semicontinuous**
  \[ -\dot{x}(t) + f(x(t), t) \in \partial \Phi(x(t)) \quad (45) \]

**Solutions**
We assume that there exists an absolutely continuous solution such that one of the previous inclusion is satisfied almost everywhere
Absolutely continuous functions

Definition

Let $I$ be an interval in the real line $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is absolutely continuous on $I$ if for every positive number $\varepsilon$, there exists a positive number $\delta$ such that whenever a finite sequence of pairwise disjoint sub-intervals $(x_k, y_k)$ of $I$ satisfies

$$\sum_k (y_k - x_k) < \delta \quad (46)$$

then

$$\sum_k |f(y_k) - f(x_k)| < \varepsilon \quad (47)$$
Absolutely continuous functions

Proposition

The following conditions on a real-valued function $f$ on a compact interval $[a, b]$ are equivalent:

1. $f$ is absolutely continuous
2. $f$ has derivative almost everywhere, the derivative is Lebesgue integrable, and

\[ f(t) = f(a) + \int_a^t f'(t) \, dt \]  \hspace{1cm} (46)

for all $x$ on $[a, b]$.

3. there exists a Lebesgue integrable function $g$ on $[a, b]$ such that

\[ f(t) = f(a) + \int_a^t g(t) \, dt \]  \hspace{1cm} (47)

for all $x$ on $[a, b]$.

If these equivalent conditions are satisfied then necessarily $g = f'$ almost everywhere. Equivalence between (1) and (3) is known as the fundamental theorem of Lebesgue integral calculus, due to Lebesgue.
Absolutely continuous functions

Properties

- The sum and difference of two absolutely continuous functions are also absolutely continuous.
- If the two functions are defined on a bounded closed interval, then their product is also absolutely continuous.
- If an absolutely continuous function is defined on a bounded closed interval and is nowhere zero then its reciprocal is absolutely continuous.
- Every absolutely continuous function is uniformly continuous and, therefore, continuous. Every Lipschitz-continuous function is absolutely continuous.
- If $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous, then it is of bounded variation on $[a, b]$.
- If $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous, then it can be written as the difference of two monotonic nondecreasing absolutely continuous functions on $[a, b]$.
- If $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous, then it has the Luzin N property (that is, for any $L \subseteq [a, b]$ such that $\lambda(L) = 0$, it holds that $\lambda(f(L)) = 0$, where $\lambda$ stands for the Lebesgue measure on $\mathbb{R}$).
- $f : I \rightarrow \mathbb{R}$ is absolutely continuous if and only if it is continuous, is of bounded variation and has the Luzin N property.
- The composition of two absolutely continuous functions is not necessarily a absolutely continuous function.
Absolutely continuous functions

Proposition

Let $f$ be Lipschitz continuous on $\mathbb{R}$ and $g$ be an absolutely continuous function on $[a, b]$. Then the composition $f \circ g$ is absolutely continuous on $[a, b]$. 
Lyapunov stability of monotone differential inclusions

Differentiability of the Lyapunov function

Let us assume that we have a $C^1$ Lyapunov function, then

$$\mathcal{V}(t) : \mathbb{R} \to \mathbb{R}$$

$$t \mapsto V(x(t))$$

is absolutely continuous if $x(t)$ is also absolutely continuous. This implies that $\dot{\mathcal{V}}(t)$ exists almost everywhere. Furthermore, if $\dot{\mathcal{V}}(t) \leq 0$ almost everywhere then

$$\mathcal{V}(t) - \mathcal{V}(0) = \int_0^t \dot{\mathcal{V}}(t)dt \leq 0 \implies \mathcal{V}(t) \text{ is decreasing}$$
Monotone differential inclusions \(- (\dot{x}(t) + f(x(t))) \in T(x(t))\)

Let us formulate the autonomous differential inclusion as

\[
\begin{cases}
\dot{x}(t) + f(x(t)) = \lambda(t) \\
-\lambda(t) \in T(x(t))
\end{cases}
\]  

(48)

If \( V \) is \( C^1 \), we want to satisfy

\[
\dot{V}(t) = \nabla_x V(x(t)) \cdot [-f(x(t)) + \lambda(t)] \leq 0 \text{ with } -\lambda(t) \in T(x(t))
\]  

(49)
Lyapunov stability of monotone differential inclusions

Monotone differential inclusions, $-\dot{x}(t) \in T(x(t))$

The case when $f(x(t)) = 0$ and we choose $V(x) = \frac{1}{2} \|x - \bar{x}\|^2$, $\nabla_x V(x) = (x - \bar{x})$ than we get

$$\dot{V}(t) = (x(t) - \bar{x})^T \lambda(t), \text{ with } - \lambda(t) \in T(x(t))$$

(50)

Let us consider and equilibrium point $\bar{x} \in \mathcal{D}(T)$, $0 \in T(\bar{x})$ then the monotony implies

$$(-\lambda(t) - 0)^T (x(t) - \bar{x}) \geq 0$$

(51)

that is

$$\dot{V}(t) = (x(t) - \bar{x})^T \lambda(t) \leq 0$$

(52)

For a monotone differential inclusion $-\dot{x}(t) \in T(x(t))$, a equilibrium with $\bar{x} \in \mathcal{D}(T)$ is Lyapunov stable. If $T$ is strictly monotone, $\bar{x}$ is asymptotically stable.
Lyapunov stability of monotone differential inclusions

Monotone differential inclusions
If $\tilde{x} \in \partial D(T)$, the classical Lyapunov stability theorem does no longer apply immediately, since it is not possible to find a open set $\Omega$ that is a neighborhood of $\tilde{x}$. 


**Assumption 1**

Let us consider the differential inclusion

\[-(\dot{x}(t) + f(x(t)) \in \partial \Phi(x(t)), \quad dt\text{-a.e} \quad (53)\]

with

- $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ a proper lower semi-continuous convex function
- $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ a Lipschitz continuous function
- an equilibrium point in $0 \in D(\partial \Phi)$, that is

\[-f(0) \in \partial \Phi(0).\]

If Assumption 1 holds then we have a unique absolutely continuous solution whatever $x_0 \in D(\partial \Phi)$. 

---

Lyapunov stability of monotone differential inclusions – 47/55
The following theorems are extracted from [3].

**Theorem**

Let us assume the Assumption 1 holds. Suppose that there exist $R > 0$, $a > 0$ and $V \in C^1(R^n, R)$ such that

\[
(\forall x \in D(T), \|x\| = R), \ V(x) \geq a
\]  

(53)

and

\[
\nabla_x V(x) \cdot f(x) + \Phi(x) - \Phi(x - \nabla V(x)) \geq 0.
\]  

(54)

Then, for any $x_0 \in D(\partial \Phi)$ with $\|x_0\| \leq R$ and $V(x_0) < a$, the solution $x(t; t_0, x_0)$ satisfies

\[
\forall t \geq t_0, \|x(t; t_0, x_0)\| < R
\]  

(55)
**Lyapunov stability of monotone differential inclusions**

Idea of the proof:

\[
\dot{V}(t) = \nabla_x V(x(t)) \cdot \dot{x}(t) \quad \text{a.e} \tag{56}
\]

with

\[
\dot{x}(t) + f(x(t)) = \lambda(t) \quad \text{with} \quad -\lambda(t) \in \partial \Phi(x(t)) \quad \text{a.e} \tag{57}
\]

Applying the definition of the sub-differential,

\[
-\lambda(t) \in \partial \Phi(x(t))
\]

\[
\updownarrow
\]

\[
(\lambda(t))^T(v - x(t)) + \Phi(v) - \Phi(x(t)) \geq 0, \forall v \in \mathbb{R}^n \tag{58}
\]

we get

\[
(\dot{x}(t) + f(x(t)))^T(v - x(t)) + \Phi(v) - \Phi(x(t)) \geq 0, \forall v \in \mathbb{R}^n \tag{59}
\]
Lyapunov stability of monotone differential inclusions

Idea of the proof:
Let us choose \( v = x - \nabla_x V(x(t)) \)

\[
- (\dot{x}(t) + f(x(t)))^T \nabla_x V(x(t)) + \Phi(x(t) - \nabla_x V(x(t))) - \Phi(x(t)) \geq 0 \text{ a.e } \tag{56}
\]

thus

\[
\dot{\mathcal{V}}(t) \leq - [f(x(t))^T \nabla_x V(x(t)) + \Phi(x(t)) - \Phi(x(t) - \nabla_x V(t))] \text{ a.e } \tag{57}
\]

from the assumption we get

\[
\dot{\mathcal{V}}(t) \leq 0 \text{ a.e } \tag{58}
\]
Let us denote by $B_\sigma$ the ball of radius $\sigma > 0$, $B_\sigma = \{x \mid \|x\| \leq \sigma\}$

**Theorem (Stability)**

*Let us assume the Assumption 1 holds. Suppose that there exists $\sigma > 0$ and $V \in C^1(\mathbb{R}^n, \mathbb{R})$ such that $V(0) = 0$

$$\forall x \in D(\partial \Phi) \cap B_\sigma, \quad V(x) \geq a(\|x\|)$$  

with $a : [0, \sigma] \to \mathbb{R}$, $a(t) > 0$, $\forall t \in (0, \sigma)$, and

$$\forall x \in D(\partial \Phi) \cap B_\sigma, \quad \nabla_x V(x) \cdot f(x) + \Phi(x) - \Phi(x - \nabla V(x)) \geq 0.$$  

Then 0 is a stable equilibrium.*
Lyapunov stability of monotone differential inclusions

**Example**

Let us consider this example:

\[ f(x) = \begin{bmatrix} x_2 \\ -\sin(x_1) \end{bmatrix}, \quad x \in \mathbb{R}^2 \]  

(61)

and

\[ \Phi(x) = \Psi_{\mathbb{R}_+^2}(x) \]  

(62)

We choose

\[ V(x) = 1 - \cos(x_1) + \frac{x_2^2}{2} \]  

(63)

and we obtain

\[ \nabla_x V(x) = \begin{bmatrix} \sin(x_1) \\ x_2 \end{bmatrix} \]  

(64)

and

\[ \nabla_x V(x) \cdot f(x) = 0 \]  

(65)
Example

There exists $\sigma > 0$ such that

$$\|x\| \geq \sigma \implies 1 - \cos(x_1) \geq \frac{x_1^2}{4}$$

Thus

$$\|x\| \geq \sigma \implies V(x) \geq \frac{x_1^2 + x_2^2}{4}$$

We have also

$$x \in \mathbb{R}_+^2 \implies x - \nabla_x V(x) = \begin{bmatrix} x_1 - \sin(x_1) \\ 0 \end{bmatrix} \in \mathbb{R}_+^2$$

Thus

$$x \in \mathbb{R}_+^2, \|x\| \sigma \implies \nabla_x V(x) \cdot f(x) + \Phi(x) - \Phi(x - \nabla V(x)) = 0.$$  

With the previous theorem, we can conclude to the stability of the equilibrium $\tilde{x} = 0$.  

Lyapunov stability of monotone differential inclusions
Lyapunov stability of monotone differential inclusions

Example

level sets of $V(x) = 1 - \cos(x_1) + \frac{1}{2}x_2^2$
Lyapunov stability of monotone differential inclusions

Theorem (Asymptotic Stability)

Let us assume the Assumption 1 holds. Suppose that there exist $\sigma > 0$, $\lambda > 0$ and $V \in C^1(\mathbb{R}^n, \mathbb{R})$ such that $V(0) = 0$

$$\forall x \in D(\partial \Phi) \cap B_{\sigma}, \ V(x) \geq a(\|x\|)$$

(61)

with $a : [0, \sigma] \to \mathbb{R}, \ a(t) > ct^\tau, \forall t \in (0, \sigma)$ for some $c > 0, \tau > 0$, and

$$\forall x \in D(\partial \Phi) \cap B_{\sigma}, \nabla_x V(x) \cdot f(x) + \Phi(x) - \Phi(x - \nabla V(x)) \geq \lambda V(x).$$

(62)

Then 0 is an asymptotic stable equilibrium.
Lyapunov stability of monotone differential inclusions

Definition (Set of stationary points)

\[ S(F, \Phi) = \{ x \in D(\partial \Phi) \mid -f(x) \in \partial \Phi(x) \} \]  \hspace{1cm} (63)

or equivalently

\[ S(F, \Phi) = \{ x \in D(\partial \Phi) \mid f^T(x)(v - z) + \Phi(v) - \Phi(x), \forall v \in \mathbb{R}^n \} \]  \hspace{1cm} (64)

Definition

Let \( V \in C^1 \). We define

\[ \mathcal{E}(F, \Phi, V) = \{ x \in D(\partial \Phi) \mid \nabla_x V(x) \cdot f(x) + \Phi(x) - \Phi(x - \nabla_x V(x)) = 0 \} \]  \hspace{1cm} (65)
Lyapunov stability of monotone differential inclusions

**Theorem**

*Let us assume the Assumption 1 holds.*

*Let $A$ be a subset of $\mathbb{R}^n$. Suppose that there exists $V \in C^1(\mathbb{R}^n; \mathbb{R})$ such that*

$$
\forall x \in D(\partial \Phi) \cap A, \nabla_x V(x) \cdot f(x) + \Phi(x) - \Phi(x - \nabla V(x)) \geq 0.
$$

*(66)*

*Then*

$$
S(F, \Phi) \cap A \subset E(F, \Phi, V)
$$

*(67)*
Lyapunov stability of monotone differential inclusions

**Theorem**

Let us assume the Assumption 1 holds.

Suppose that there exist $\sigma > 0$ and $V \in C^1(\mathbb{R}^n; \mathbb{R})$ such that

$$\forall x \in D(\partial \Phi) \cap B_\sigma, \nabla_x V(x) \cdot f(x) + \Phi(x) - \Phi(x - \nabla V(x)) \geq 0.$$  \hspace{1cm} (68)

and

$$\mathcal{E}(F, \Phi, V) \cap B_\sigma = \{0\}$$  \hspace{1cm} (69)

Then the stationary solution is isolated in $S(F, \Phi)$.
Assumption

**Theorem**

Let us assume the Assumption 1 holds. Suppose that there exists $V \in C^1(\mathbb{R}^n; \mathbb{R})$ such that

$$\forall x \in D(\partial \Phi), \nabla_x V(x) \cdot f(x) + \Phi(x) - \Phi(x - \nabla V(x)) \geq 0.$$  \hspace{1cm} (70)

and

$$\mathcal{E}(F, \Phi, V) = \{0\}$$  \hspace{1cm} (71)

Then $S(F, \Phi) = \{0\}$ that is the stationary solution is the unique equilibrium point.


