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Concurrent multiple impacts modelling: Case study of a 3-ball chain

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Abstract

The aim of this work is to exhibit a multiple impact law for rigid body dynamical systems which meets the properties of closing the non-smooth dynamical equations and of corroborating experiments. This law is based on the *impulse correlation ratio* which is computed from an equivalent regularized model with compliant contact. A case-study on a 3-ball chain and n-ball chain are delineated and results on finite dimensional system are stated.

Keywords: Non-smooth dynamics; Multiple impacts; Unilateral contact; Numerical modelling; Newton's cradle

1. Introduction and motivations

Roughly speaking, a multiple impact can be defined as the occurrence of several shocks at the same time on various points of a mechanical system of rigid bodies. A chain of balls or the Newton's Cradle are academic examples of systems where concurrent multiple impacts occur.

When a rigid body mechanical system with perfect unilateral constraints is subjected to impact, the definition of an *impact law* allows one to compute the post-impact velocity [1]. An impact law must possess the following properties:

(1) It closes the system of non-smooth equations of motion in the sense that it provides the post-impact velocities and the percussions for any pre-impact conditions. The fact that the dynamical system associated with an impact law is mathematically well-posed is an additional interesting feature.

(2) It corroborates the experimental observations, and the set of parameters which enter the law must be measurable and physically justified. Particularly, the law must describe an energetic behavior which is compatible with the basic principles of thermodynamics, and must provide post-impact velocities in agreement with the experiments. Better, the parameters of the law may be correlated with the geometrical and material characteristics of the bodies in impact.

The aim of this work is to exhibit an impact law which meets both the preceding conditions.

When multiple impacts occur, most of the classical formulations do not respect both requirements 1 and 2. The algorithm of Han and Gilmore [2] provides a good energetic treatment but the existence of a solution is not guaranteed [3]. Moreau [4] proposes an impact law, numerically efficient, which always provides a solution, but the post-impact velocities are not always satisfying from an experimental point of view. Frémond [5] presents an elegant and rigorous framework to add internal constraints in mechanical systems, which are consistent with thermodynamic principles. Motivated by an experimental work on Newton's cradle, Ceanga and Hurmuzlu [6] postulate the existence of an *impulse correlation ratio* (ICR) α for a triplet of balls. With the help of energetic restitution coefficients, the post-impact velocities are experimentally shown to be well approximated. However, in the last two works a precise physical definition of the parameters of such laws somewhat lacks.

In this paper, we shed new light on the ICR by studying the regularized system of a 3-ball chain with elastic contact springs. The physical justification of this choice may be found in the work of Falcon et al. [7] on one-dimensional columns of beads. The industrial application of this work is led through a fruitful collaboration with Abadie [8] from Schneider Electric, concerning the virtual prototyping of circuit breaker mechanisms, where a fine modelling of impact is an essential step.

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2. Case study of a 3-ball chain regularized with elastic springs

In this section, we focus our attention on 3-ball chains, which are very interesting examples of systems with multiple impacts. A hard ball behaves as a rigid body with massless springs at contact. In other words, the impact process between hard balls does not excite the natural modes of each ball. Furthermore, Hertz theory of contact is very well correlated with the experiments at low velocity range [7].

2.1. Rigid body model of a 3-ball chain

A dynamical system of three rigid balls of equal mass m, described by their center of mass positions q_1, q_2, q_3 and velocities v_1, v_2, v_3 is considered. Each ball slides without friction on a straight line and the dynamics at the instant of impact is:

$$\begin{cases} m(v_1^+ - v_1) = -p_1 \\ m(v_2^+ - v_2) = p_1 - p_2 \\ m(v_3^+ - v_3) = p_2 \end{cases} \tag{1}$$

where v_i, v_i^+ are respectively the pre-impact and the post-impact velocities and p_i, the impulses. Without loss of generality, the pre-impact velocity of the middle ball is chosen equal to zero ($v_2 = 0$). An additional law is given to address the energetic behaviour at impact. For the conservative case, we have:

$$v_1^2 + v_3^2 = (v_1^+)^2 + (v_2^+)^2 + (v_3^+)^2 \tag{2}$$

If a multiple impact occurs (i.e. the three balls are in contact at the same instant), this system is not mathematically well-posed. Indeed, for $[v_1, v_3] = [1, 0]$, one can easily check that $[v_1^+, v_2^+, v_3^+] = [0, 0, 1]$ and $[v_1^+, v_2^+, v_3^+] = [-1/3, 2/3, 2/3]$ can be solution of this system in applying conservative Newton laws sequentially to the first or the second pairs of balls [3].

If we introduce a value for the ICR, $\alpha = p_1/p_2$, the system becomes well-posed and the unique solution is given by:

$$\begin{cases} v_1^+ = v_1 - \dfrac{\alpha}{\left(1-\alpha+\alpha^2\right)}[\alpha v_1 - v_3] \\ v_2^+ = \dfrac{\alpha - 1}{\left(1-\alpha+\alpha^2\right)}[\alpha v_1 - v_3] \\ v_3^+ = v_3 + \dfrac{1}{\left(1-\alpha+\alpha^2\right)}[\alpha v_1 - v_3] \end{cases} \tag{3}$$

2.2. Numerical experiments

Let us consider an equivalent regularized system for the 3-ball chain. The interaction between two balls is no longer rigid but realized through a Hertzian spring model. We are interested in relative motion between the balls, therefore we choose to write down the dynamical system in terms of indentations, $\delta_i = q_{i+1} - q_i$, as:

$$\begin{cases} m\ddot{\delta}_1 = -2f_1(\delta_1) + f_2(\delta_2) \\ m\ddot{\delta}_2 = -2f_2(\delta_2) + f_1(\delta_1) \\ 0 \le \boldsymbol{f} \perp \boldsymbol{f} - \boldsymbol{K}(\boldsymbol{\delta})\boldsymbol{\delta} \ge 0 \end{cases} \tag{4}$$

where $\boldsymbol{f} = [f_1, f_2]^T$ represents the efforts between balls, $\boldsymbol{\delta} = [\delta_1, \delta_2]^T$ the vector of collected indentations and $\boldsymbol{K}(\boldsymbol{\delta})$ is the stiffness matrix. For Hertzian contact, the stiffness matrix takes the form:

$$K(\boldsymbol{\delta}) = \begin{bmatrix} k_1(\delta_1)^{1/2} & 0 \\ 0 & k_2(\delta_2)^{1/2} \end{bmatrix} \tag{5}$$

where $k_1 = k$ and $k_2 = \kappa k, \kappa \in \mathbb{R}_+$ are the coefficients of stiffness related to material and geometrical parameters.

The integration, which is intractable analytically, is performed with Scilab© for various initial relative velocities (choosing $v_2 = 0$). Actually, the solution is sufficiently smooth to allow the use of a traditional numerical ODE solver.

On Fig. 1, some curves are given which draw the forces between balls versus time. One can remark that the process of collision is not trivial: several periods of contact may occur before the balls separate definitively (see Fig. 1b,c), or the contact period between two balls may not begin at the first instant of contact (see Fig. 1d).

If we define a multiple impact in regularized systems as the existence of a time interval where both contact forces are different from zero, all of these processes lead to multiple impacts. Naturally, the rigid limit in a mathematical sense requires additional care.

2.3. Analytical results for linear springs

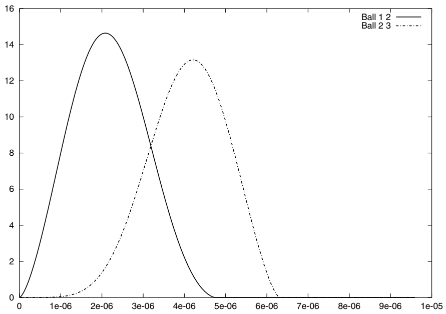
Let us now analyze the 3-ball chain with linear springs. This model is not consistent with the contact mechanics between two balls, but it is useful if we want to perform some analytical developments which are intractable with the Hertz model.

For example, let us consider, $v_1 > 0$, $v_2 = v_3 = 0$ with $\kappa > 1$. We can demonstrate that there exists a non-zero interval $[0, t^\star]$ in which the system behaves as the following bilateral system:

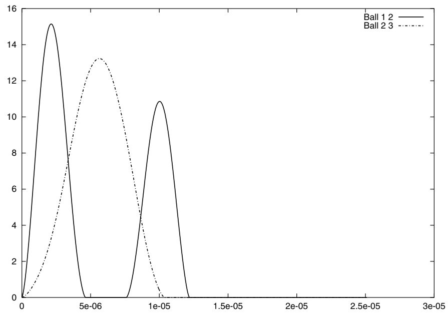
$$\begin{cases} m\ddot{\delta}_1 = -2k(\delta_1) + \kappa k(\delta_2) \\ m\ddot{\delta}_2 = -2\kappa k(\delta_2) + k(\delta_1) \\ \delta_1(0) = \delta_2(0) = 0, \quad \dot{\delta}_1(0) = -v_1, \quad \dot{\delta}_2(0) = v_3 \end{cases} \tag{6}$$

On $[0, t^\star]$, the solution of (6) is:

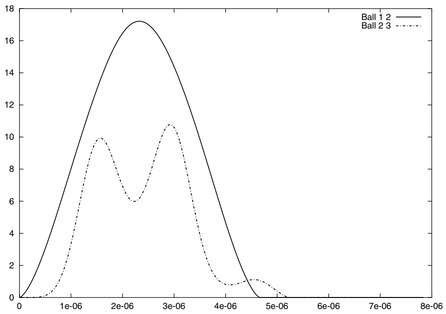
$$\begin{cases} \delta_1(t) = \dfrac{-v_1}{\beta - \gamma}\left(\dfrac{\beta}{\omega_1}\sin(\omega_1 t) - \dfrac{\gamma}{\omega_2}\sin(\omega_2 t)\right) \\ \delta_2(t) = \dfrac{-\beta\gamma v_1}{\beta - \gamma}\left(\dfrac{1}{\omega_2}\sin(\omega_2 t) - \dfrac{1}{\omega_1}\sin(\omega_1 t)\right) \end{cases} \tag{7}$$



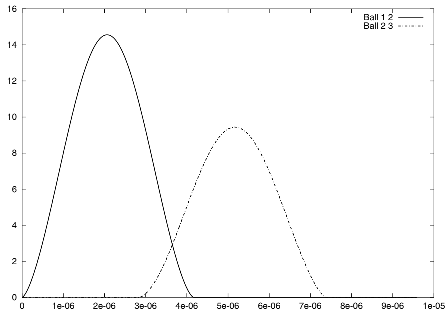
(a) $\kappa = 1, v_1 = 1, v_3 = 0$



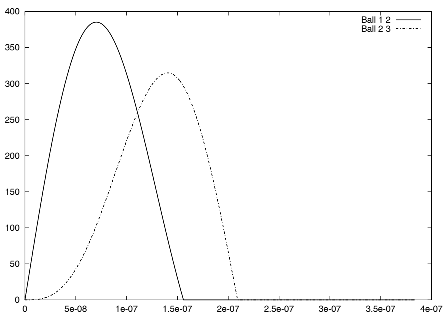
(b) $\kappa = 0.1, v_1 = 1, v_3 = -1$



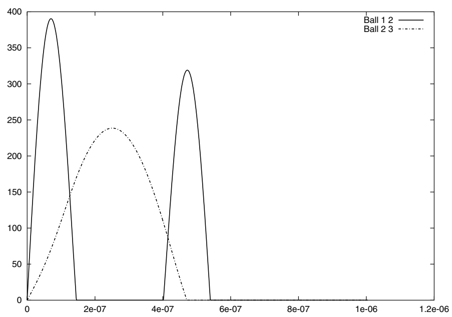
(c) $\kappa = 100, v_1 = 1, v_3 = 0$



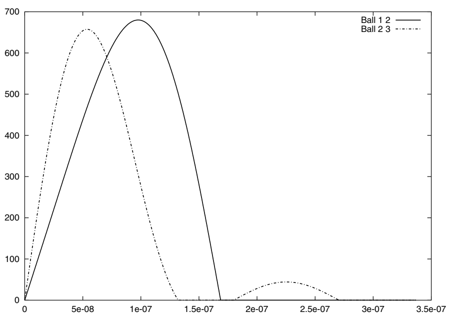
(d) $\kappa = 1, v_1 = 1, v_3 = 0.3$



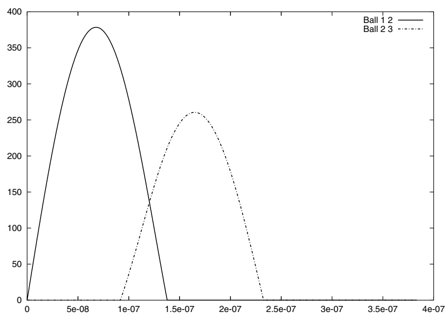
(e) $\kappa = 1, v_1 = 1, v_3 = 0$



(f) $\kappa = 0.1, v_1 = 1, v_3 = -1$



(g) $\kappa = 2.24, v_1 = 1, v_3 = -1$



(h) $\kappa = 1, v_1 = 1, v_3 = 0.3$

Fig. 1. Numerical integration of 3 balls chain. Forces between balls versus time. (a–d) Hertzian spring contact. (e-h) Linear spring.

where (ω_i, ϕ_i) are the natural modes of the system given by:

$$\begin{cases} \omega_1^2 = \dfrac{k}{m}\left(\kappa + 1 - \sqrt{\kappa^2 - \kappa + 1}\right), \\ \qquad \phi_1 = [\beta = \kappa - 1 + \sqrt{\kappa^2 - \kappa + 1}, 1]^T \\ \omega_2^2 = \dfrac{k}{m}\left(\kappa + 1 + \sqrt{\kappa^2 - \kappa + 1}\right), \\ \qquad \phi_2 = [\gamma = \kappa - 1 - \sqrt{\kappa^2 - \kappa + 1}, 1]^T \end{cases} \tag{8}$$

The first time one contact breaks, denoted as $t^\star$, is provided by the smallest positive root of the transcendental equations:

$$t_{12} = \min_{t \in \mathbb{R}^{+*}} \left\{ f_1(t) = 0 \right. \\ \left. \text{with } f_1(t) = \sin(\omega_1 t) - \frac{\omega_1}{\omega_2}\frac{\gamma}{\beta}\sin(\omega_2 t) \right\}$$

(first pair of balls)

$$t_{23} = \min_{t \in \mathbb{R}^{+*}} \left\{ f_2(t) = 0 \text{ with } f_2(t) = \sin(\omega_1 t) - \frac{\omega_1}{\omega_2}\sin(\omega_2 t) \right\}$$

(second pair of balls)

Finding the smallest root with respect to the physical parameters of the system is a painful work. However, for this particular case, the following holds:

Proposition 1
If $\omega_2/\omega_1 = j \in \mathbb{N}^*$ then $t_{12} = t_{23} = t^\star = \pi/\omega_1$.
If $\omega_2/\omega_1 \in (j; j+1), j \in \mathbb{N}^*$ and j odd (resp. even) then $t^\star = t_{12} < t_{23}$ (resp. $t_{12} > t_{23} = t^\star$).

For $t > t^\star$, only two balls are still in contact. The rest of the process is easily integrable up to the final separation at the time t_f. Moreover, one can show that there is no further contact between the balls as illustrated in Fig. 1e.

For $t^\star = t_{12} < t_{23}$, the ICR is calculated as follows:

$$\begin{aligned} \alpha &= \frac{p_1}{p_2} \\ &= \frac{1}{\beta\gamma}\left(\frac{-\beta}{\omega_1^2}\big[\cos(\omega_1 t_{12}) - 1\big] - \frac{-\gamma}{\omega_2^2}\big[\cos(\omega_2 t_{12}) - 1\big]\right) \\ &\quad \Big/ \Bigg[\frac{1}{\omega_2^2}\big(\cos(\omega_2 t_{12}) - 1\big) - \frac{1}{\omega_1^2}\big(\cos(\omega_1 t_{12}) - 1\big) \\ &\quad + \frac{1}{\omega_2'^2}\big(\cos(\omega_2 t_{12}) - \cos(\omega_1 t_{12})\big)\big(\cos(\omega_2' \hat{t}_{23}) - 1\big) \\ &\quad - \frac{1}{\omega_2'}\left(\frac{1}{\omega_2}\sin(\omega_2 t_{12}) - \frac{1}{\omega_1}\sin(\omega_1 t_{12})\right)\big(\sin(\omega_2' \hat{t}_{23})\big) \Bigg] \end{aligned} \tag{9}$$

where $\omega_2' = \sqrt{2\kappa k/m}$ is the natural pulsation of two balls in contact and $\hat{t}_{23} = t_{23} - t^\star$.

2.4. Preliminary conclusions

Other cases have been treated in the same way. It is noteworthy that the occurrence of transcendental equations in the resolution creates serious difficulties to integrate analytically the process of collisions. Particularly, the time and the order of interactions are not easily predictable.

Nevertheless, a preliminary conclusion can be stated, on which more general results will be provided in Section 4:

Proposition 2
The instants of changes in the contact interactions, in an adimensional scale of time, for instance, $T = \omega_i t$, and the ratio of impulses, α, do not depend on the absolute values of stiffness k and mass m. Moreover, the impulse correlation ratio α is completely determined by the natural modes of the regularized dynamical system and the pre-impact velocities.

This conclusion outlines two important consequences:
• from a mechanical point of view, the introduction of an impulse ratio enhances the model with some information about the behavior of a dynamical system when it is bound by elastic contact,
• from a numerical modelling point of view, the independence of absolute value of k allows one to consider in a consistent manner its applications to very large stiffnesses, which are generally encountered in applications.

3. Some remarks on impulse correlation ratios in n-ball chains

An important aspect of a correct impact law is that it qualitatively represents the physical phenomena. For the n-ball chain or Newton's cradle, we know that conservation of kinetic energy and momentum is not sufficient to explain that there is no ball at rest after an impact [9]. The introduction of a set of ICR in an n-ball chain as Ceanga and Hurmuzlu [6] have done, describes qualitatively this important phenomenon.

From a quantitative point of view, some remarks must be made. Let us study the values of the ICR obtained by numerical simulation of an n-ball chain made of steel ($E = 210$ Mpa, $\nu = 0.3$, $\rho = 7800$ kg/m^3) regularized with elastic Hertz model, where the first ball is dropped at 1 m/s and the other balls are at rest.

On Fig. 2a, the number of balls of radius 10 mm in the chain ranges from 3 to 21. For n balls, there are $n-1$ impulses and $n-2$ ICR, defined by:

$$\alpha_i = \text{icr}(i) = \frac{p_i}{p_{i+1}} \tag{10}$$

The first remark is that only the ICR which corresponds to the last triplet in the chain (for instance, the point A for 8 balls) is very different from the others. Therefore, the value of ICR measured from an experiment on a triplet cannot be used for the n-ball chain.

On Fig. 2b, we observe the value of ICR in a 21-ball chain where the tenth ball has been changed to a big ball

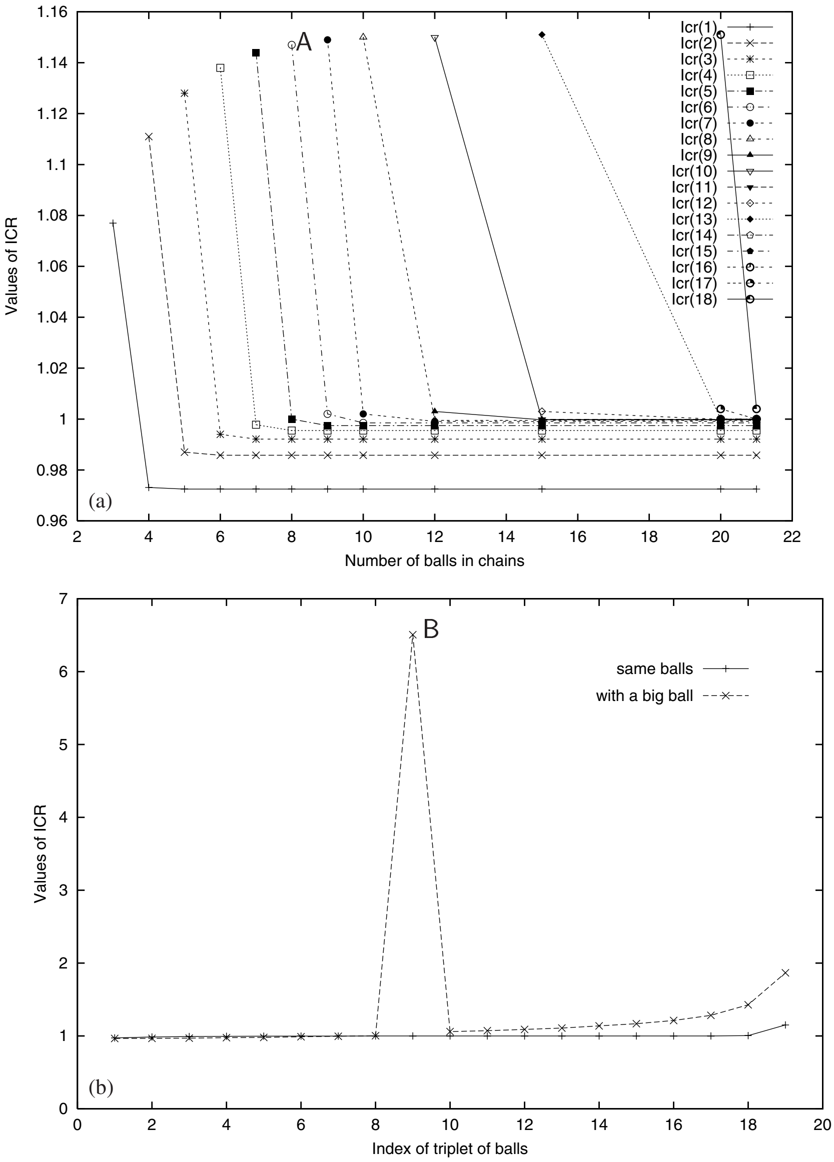


Fig. 2. Impulse correlation ratios in an n-ball chain. (a) ICR versus the number of balls in the chain. (b) ICR versus the index of triplet in the 21-ball chain. Comparison between chain of same balls and a chain with a big ball 10.

of radius 50 mm. The ICR corresponding to the percussion on the big ball is different, but also the value of ICR for the triplets 10 to 19. Moreover, the value of ICR computed for a 3-ball with a middle big wall is about 63.47, which is very different from the value computed in the whole chain (point B). This shows that the ICR depends on the dynamical features of the whole coupled system.

4. Towards an extension to finite dimensional systems – major results and conclusion

The case study of a 3-ball chain is extended to finite dimensional systems subjected to perfect unilateral constraints. The major results are:

(1) The post-impact velocity, computed with the multiple impact law defined by *impulse correlation ratio*, is provided in a unique way and the system becomes mathematically well-posed.
(2) If the perfect constraints are regularized by a general viscoelastic contact model corresponding to a linear

viscoelastic bulk behavior [10,11] i.e.

$$f = K\delta^n + C\delta^{n-1}\dot{\delta} \tag{11}$$

then

(a) the ratio of impulse is finite and the subspace of the state space defined by

$$E = \{\delta \geq 0, \dot{\delta} \geq 0\} \tag{12}$$

is globally attractive. Moreover, the amplitude of the force asymptotically tends towards zero and the relative velocity $\dot{\delta}$ towards a finite constant. This last point is very important from a numerical point of view. Extending these results to finite time convergence is still an issue.

(b) The ICRs are independent of the absolute value of stiffness.

(3) If the perfect constraints are regularized by a linear elastic model, i.e.

$$f = K\delta \tag{13}$$

then the ICRs depend only on natural modes of the system and the pre-impact velocities.

(4) The augmented impact law, which consists of a set of *energetic coefficients* and *impulse correlation ratio* fits within Frémond's thermodynamic framework [5]. It ensures that the principles of thermodynamics are respected.

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