A monolithic computational method for elasto-dynamics with plasticity and contact based on variational approach.

ICCCM2023. July 6th 2023

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Motivations and Objectives

Elasto-dynamics with plasticity and unilateral contact

1. A single differential variational inequality
   ▶ multi-criteria elasto-plastic flow rules with contact constraints.

2. A nonsmooth dynamical framework for finite-dimensional systems
   ▶ dealing with jumps in velocities and impulsive forces (FEM discretized)

3. A Moreau-Jean type time-stepping method,
   ▶ enabling the consistent integration of the nonsmooth dynamics.

4. A discrete energy balance
   ▶ a practically stable scheme with positive dissipation.

5. Variational approach:
   ▶ formulation of a saddle point problem (min-max problem) and a convex quadratic program
   ▶ well-posedness results (existence and uniqueness)
   ▶ numerical optimization methods as an alternative to return-mapping algorithm
Formulation of elastodynamics with contact

Space and time discretization

Well posedness of the discrete problem

Discrete energy balance

Conclusion and perspectives
Plasticity in the generalized standard materials framework

Simplest framework: Small perturbation, associative plasticity and linear hardening

- Small perturbations hypothesis with additive decomposition of the strain

\[ \varepsilon = \varepsilon^e + \varepsilon^p. \]  

- Linear elasticity and hardening laws

\[ \sigma = E : \varepsilon^e \quad \text{and} \quad a = -D \cdot \alpha. \]  

- Generalized standard material (GSM) (associative plasticity)

\[ \left( \begin{array}{c} \dot{\varepsilon}^p \\ \dot{\alpha} \end{array} \right) \in N_C \left( \begin{array}{c} \sigma \\ a \end{array} \right). \]  

\[ C(\sigma, a) \] a convex set of admissible stresses \( \sigma \) and hardening forces \( a \)

- Clausius-Duhem dissipation inequality is automatically satisfied

\[ d = \sigma : \dot{\varepsilon}^p + a \cdot \dot{\alpha} \geq 0 \quad \text{if} \quad 0 \in C. \]
Plasticity in the generalized standard materials framework

\[
\begin{align*}
\begin{pmatrix} \dot{\varepsilon}^p \\ \dot{\alpha} \end{pmatrix} \in N_c \begin{pmatrix} \sigma \\ a \end{pmatrix} & \quad \iff \quad C = \left\{ (\sigma, a), f(\sigma, a) \geq 0 \right\} \\
\begin{pmatrix} \sigma \\ a \end{pmatrix} = \text{proj}_c \left( \begin{pmatrix} \sigma \\ a \end{pmatrix} - \rho \begin{pmatrix} \dot{\varepsilon}^p \\ \dot{\alpha} \end{pmatrix} \right), \rho > 0
\end{align*}
\]

\[\dot{\varepsilon}^p = -\nabla_{\sigma}^T f(\sigma, a) \lambda \]
\[\dot{\alpha} = -\nabla_{a}^T f(\sigma, a) \lambda \]
\[0 \leq \lambda \perp f(\sigma, a) \geq 0,\]
Unilateral contact

$g_N(x)$: gap function

$v_N(x) = \frac{d}{dt} g_N(x)$: relative velocity

Signorini contact law:

$$0 \leq g_N \perp r_N \geq 0 \iff -r_N \in N_{\mathbb{R}^+}(g_N).$$  \hspace{1cm} (6)

Signorini contact law at the velocity level:

$$0 \leq r_N \perp v_N \geq 0 \text{ if } g_N = 0, \text{ else } r_N = 0$$

$$\uparrow$$

$$-r_N \in N_{T_{\mathbb{R}^+}(g_N)}(v_N),$$  \hspace{1cm} (7)

where $T_{\mathbb{R}^+}(g_N)$ is the tangent cone of $\mathbb{R}^+$ at $g_N$. 

Space discretization

Standard FEM discretization to make is as simple as possible.

Finite dimensional smooth linear dynamics:

\[ M \ddot{v}(t) + B^T \sigma(t) = f_{\text{ext}}(t) + r(t). \]  

\( \triangleright \) \( u, v \) nodal displacement and velocity vector,

\( \triangleright \) \( \sigma \) stress at Gauss points,

\( \triangleright \) \( M \) constant mass matrix,

\( \triangleright \) \( B^T \) discrete divergence operator, \( B \) is the discrete gradient.

Elasto-plastic relation at Gauss points and contact kinematics

\[
\left\{
\begin{array}{l}
\sigma = E \varepsilon^e = E (\varepsilon - \varepsilon^p) \\
a = -D \dot{\alpha} \\
(\dot{\varepsilon}^p, \dot{\alpha}) \in N_C \begin{pmatrix} \sigma \\ a \end{pmatrix},
\end{array}
\right. \tag{9}
\]

\[
\nu_N = H^T(u) v \\
r = H(u) r_N. \tag{10}
\]
Measure differential equation

Ability to deal with velocity jumps and impulsive forces in discrete systems

\[ M \, dv + B^\top \sigma(t) \, dt = f_{\text{ext}}(t) \, dt + H(u(t)) \, di_N. \]  
\[ (11) \]

- \( dv \) differential measure (”acceleration as measure”)
- \( di_N \) contact reaction measure

Unilateral contact and Newton impact law

\[ - \, di_N \in N_{\mathbb{IR}^m} (g_N(t)) \left( v_N(t) + e v_N^-(t) \right). \]  
\[ (12) \]

where \( e \) is the coefficient of restitution
After the differentiation of the constitutive law with \( S = E^{-1} \) (similar to incremental formulation) and the introduction of slack variable

\[
y = -\dot{\alpha} \quad \text{and} \quad z = -\dot{\epsilon}^p
\]

we get a differential measure variational inequality

\[
\begin{aligned}
M \dot{\mathbf{v}} + B^\top \sigma(t) dt &= f_{\text{ext}}(t) dt + H(u(t)) d\mathbf{i}_N \\
\dot{\mathbf{u}}(t) &= \mathbf{v}(t) \\
S \dot{\sigma}(t) &= B \mathbf{v}(t) + z(t) \\
D^{-1} \dot{a}(t) &= y(t) \\
\nu_N(t) &= H^\top(u(t)) \mathbf{v}(t) \\
- \begin{pmatrix}
\mathbf{z}(t) \\
\mathbf{y}(t) \\
d\mathbf{i}_N
\end{pmatrix} &\in N_C \times \tau_{IR^+}(g_N(t)) \begin{pmatrix}
\sigma(t) \\
a(t) \\
(\nu_N(t) + e\nu_N^-(t))
\end{pmatrix}.
\end{aligned}
\]

A single variational inequality.
Time stepping scheme

Extension of the Moreau–Jean scheme with $\theta \in (0, 1]$

$$\begin{align*}
M(v_{k+1} - v_k) + hB^\top \sigma_{k+\theta} &= hf_{\text{ext}, k+\theta} + Hp_{N, k+1} \\
S(\sigma_{k+1} - \sigma_k) - hBv_{k+\theta} &= hz_{k+\theta}, \\
D^{-1}(a_{k+1} - a_k) &= hy_{k+\theta}, \\
v_{N, k+1} &= H^\top v_{k+1} \\
\end{align*}$$

(14)

$$\begin{pmatrix}
\mathbf{z}_{k+\theta} \\
\mathbf{y}_{k+\theta} \\
p_{N, k+1}
\end{pmatrix} \in \mathbb{N}_C \times \mathbb{R}^m_+ \begin{pmatrix}
\sigma_{k+\theta} \\
a_{k+\theta} \\
v_{N, k+1} + ev_{N, k}
\end{pmatrix}$$

where

- notation: $h$ time step, $x_{k+\theta} = \theta x_{k+1} + (1 - \theta)x_k$
- the Signorini condition only on the active contact ($g_{N, k} \leq 0$)
- displacements are updated afterwards $u_{k+1} = u_k + hv_{k+\theta}$
- impulses $p_{N, k+1}$ as primary variable
- elasto-plastic law with hardening is solved at $k + \theta$. 
Variational approach: saddle point problem

Proposition (Saddle point problem)

The solutions of \((v, \dot{\varepsilon}, \sigma, a, v_N)\) of the first order optimality conditions of

\[
\begin{align*}
\min_{v, \dot{\varepsilon}} \max_{\sigma, a} & \quad \frac{1}{2} (v - v_k)^\top M (v - v_k) \\
& - \frac{1}{2} (\sigma - \sigma_k)^\top S (\sigma - \sigma_k) - \frac{1}{2} (a - a_k)^\top D^{-1} (a - a_k) \\
& + h\theta \sigma^\top \dot{\varepsilon} - h\theta f_{\text{ext},k+1}^\top v
\end{align*}
\]

s.t.

\[
Bv = \dot{\varepsilon} \\
\theta v_N = H^\top v - (1 - \theta)v_{N,k} \\
\begin{pmatrix}
\sigma \\
a \\
v_N + ev_{N,k}
\end{pmatrix} \in C \times \mathbb{R}^m_+.
\]

are solutions of (14) for \((v_{k+\theta}, z_{k+\theta}, \sigma_{k+\theta}, a_{k+\theta}, v_{N,k+1})\).

\[\blacktriangleright\] A kind of discrete \(\partial'\)Alembert principle for elastoplasticity with contact.
Variational approach: saddle point problem

Assumption (1)

The matrices $M$, $S$ and $D$ are symmetric definite positive matrices.

Assumption (2)

It exists $v^0$, $\sigma^0$, $a^0$ such that

\[
\begin{cases}
\left( \begin{array}{c}
\sigma^0 \\
a^0
\end{array} \right) \in C \\
\sigma^0 a^0 \notin C \\
H^T v^0 + (\theta(1 + e) - 1)v_{N,k} \geq 0
\end{cases}
\]

standard feasibility condition

Proposition

Under Assumptions 1 and 2, the saddle-point problem (15) has a unique solution $(v, \dot{v}, \sigma, a, v_N)$. 
Variational approach: convex quadratic problem

Substitution of $v_{k+1}$ in the linear equations in (14)

Reduction to local variables.

$$- \begin{pmatrix} Q \left( \begin{array}{c} \sigma_{k+\theta} \\ a_{k+\theta} \\ p_{N,k+1} \end{array} \right) + p \end{pmatrix} \in \mathcal{N}_C \times \mathbb{R}_+^m \begin{pmatrix} \sigma_{k+\theta} \\ a_{k+\theta} \\ p_{N,k+1} \end{pmatrix},$$

(17)

with

$$Q = \begin{pmatrix} U & 0 & -V \\ 0 & D^{-1} & 0 \\ -V^T & 0 & W \end{pmatrix} \quad \text{and} \quad p = \begin{pmatrix} s \\ D^{-1} a_k \\ r \end{pmatrix}.$$  

(18)

and

$$W = \theta^2 H^T M^{-1} H \quad \text{Delassus matrix}$$

$$U = S + h^2 \theta^2 BM^{-1} B^T$$

$$V^T = h \theta^2 H^T M^{-1} B^T$$

$$s = -S \sigma - h \theta B \left( v_k + \theta h M^{-1} f_{\text{ext},k+\theta} \right)$$

$$r = \theta^2 \left( e v_{N,k} + H \left( v_k + \theta M^{-1} \left( h f_{\text{ext},k+\theta} \right) \right) \right)$$

(19)
Variational approach: convex quadratic problem

Equivalent convex minimization problem:

\[
\begin{align*}
\min_{\sigma, a, p_N} & \quad \frac{1}{2} \begin{pmatrix} \sigma \\ a \\ p_N \end{pmatrix}^\top Q \begin{pmatrix} \sigma \\ a \\ p_N \end{pmatrix} + p^\top \begin{pmatrix} \sigma \\ a \\ p_N \end{pmatrix} \\
\text{s.t.} & \quad \begin{pmatrix} \sigma \\ a \\ p_N \end{pmatrix} \in C \times \mathbb{R}_+^m.
\end{align*}
\]

(20)

Assumption (3)

*The matrix $H$ has full rank.*

Proposition

*Under Assumptions 1 and 3 and for a sufficiently small time step, the problem (20) has a unique solution $(\sigma, a, p_N)$ if the set $0 \in C$.***
Convex Quadratic problem

\[ C \text{ finitely represented} \]

\[ C = \{ (\sigma, a) \mid f(\sigma, a) \geq 0 \}, \quad (21) \]

\( f \) is a smooth vector-valued function with non-vanishing gradients

\[
\min_{\sigma, a, p_N} \frac{1}{2} \begin{pmatrix} \sigma \\ p_N \\ a \end{pmatrix}^\top Q \begin{pmatrix} \sigma \\ p_N \\ a \end{pmatrix} + p^\top \begin{pmatrix} \sigma \\ p_N \\ a \end{pmatrix} \\
\text{s.t.} \\
f(\sigma, a) \geq 0 \\
p_N \geq 0
\]  

\( (22) \)
Numerical methods from optimization

- Direct pivoting techniques (Active set, Lemke, PATH)
  Direct solution if $C$ is polyhedral
- First order (Nesterov) accelerated techniques (ADMM, APGD, \ldots)
- Second order methods:
  1. interior point methods
  2. semi-smooth Newton method

$$
0 = \begin{pmatrix}
\sigma \\
a \\
p_N
\end{pmatrix}
- \text{proj}_{C \times \mathbb{R}_+^m}
\begin{pmatrix}
\sigma \\
a \\
p_N
\end{pmatrix}
- \rho
\begin{pmatrix}
\sigma \\
a \\
p_N
\end{pmatrix}
+ p
,$$
\rho > 0

Comments on semi-smooth Newton methods
Similarity with return-mapping algorithm [Hager, Wolhmut (2009), Christensen(2009)]
but with more flexibility:

- optimal choice of $\rho$ with self-adaptation (proximal technique)
- rescaling technique are easier
- no explicit need of a consistent tangent operator
Discrete energy balance

- Energy balance for discrete systems:

\[
dE(t) = d(T(t) + \Psi(t)) = -D(t)dt + P_{\text{ext}}(t)dt + dP_{\text{impact}},
\]

with

\[
T(t) = \frac{1}{2} v^\top(t) M v(t), \quad \text{and} \quad \Psi(t) = \frac{1}{2} \epsilon^e^\top(t) E \epsilon^e(t) + \frac{1}{2} \alpha^\top(t) D \alpha(t)
\]

- The dissipation and the power of external forces are

\[
D(t) = \sigma^\top(t) \dot{\epsilon}^p(t) + a^\top(t) \dot{\alpha}(t) \quad \text{and} \quad P_{\text{ext}}(t) = f_{\text{ext}}^\top(t)v(t).
\]

- The power of the reaction impulse is given by

\[
dP_{\text{impact}} = \frac{1}{2}(v^+_N + v^-_N)d_i_N.
\]
Discrete energy balance

Approximation of works by the $\theta$-method as

$$W_{\text{ext}}^{k+1} := h\nu_{k+\theta}^T f_{\text{ext}, k+\theta} \approx \int_{t_k}^{t_{k+1}} P_{\text{ext}}(t)dt,$$  \hspace{1cm} (27)

and

$$W_{p}^{k+1} := h\sigma_{k+\theta}^T \dot{\varepsilon}_{k+\theta}^p - h\alpha_{k+\theta}^T y_{k+\theta} \approx \int_{t_k}^{t_{k+1}} D(t)dt,$$  \hspace{1cm} (28)

Approximation of the work dissipated by the percussion

$$W_{c}^{k+1} := v_{N,k+\theta}^T p_{N,k+1} = (1 - \theta (1 + e))v_{N,k}^T p_{N,k+1},$$  \hspace{1cm} (29)

the increment of total energy is then given by

$$\Delta E_{k}^{k+1} = W_{\text{ext}}^{k+1} + W_{p}^{k+1} - W_{c}^{k+1} = \left(\frac{1}{2} - \theta\right) \left(\|v_{k+1} - \nu_k\|_M^2 + \|\varepsilon_{k+1}^e - \varepsilon_k^e\|_E^2 + \|\alpha_{k+1} - \alpha_k\|_D^2\right).$$  \hspace{1cm} (30)
Proposition

Under Assumption 1, energy dissipation of the scheme is as follows

1. When $\theta = \frac{1}{2}$, the time-stepping scheme satisfies the approximation of the discrete energy balance:

$$\Delta E_{k+1}^k - W_{\text{ext}}^{k+1} = -W_p^{k+1} + W_c^{k+1}. \quad (31)$$

2. The dissipated work due to plasticity is always positive

3. When $\theta \leq \frac{1}{1+e}$, the dissipated work due to impact is also positive.

4. When $\frac{1}{2} \leq \theta \leq \frac{1}{1+e} \leq 1$, we have the following dissipation inequality

$$\Delta E_{k+1}^k - W_{\text{ext}}^{k+1} \leq 0. \quad (32)$$
Conclusions & perspectives

Conclusions
A monolithic solver for elastodynamics with contact, impact and plasticity:

▶ A practically stable scheme with a discrete energy balance
▶ A variational formulation (optimization problem) of plasticity with contact
▶ A gateway to a host of multiple optimisation algorithms
▶ Useful also for quasi-static application, even with perfect plasticity

Vincent Acary, Franck Bourrier, Benoit Viano.
Variational approach for nonsmooth elasto-plastic dynamics with contact and impacts. 2023. ⟨hal-03978387v1⟩ to appear in Computer Methods in Applied Mechanics and Engineering

Perspectives

▶ Non associated plasticity and Coulomb friction
  ▶ Implicit standard materials (De Saxcé) ⇒ quasi-variational inequality
▶ Finite strain plasticity and mortar method ([Seitz, Popp, Wall (2015)])
▶ Material point method (PhD Louis Guillet)
  ▶ Application to gravity-driven flows of geomaterials in mountains (mud and debris flows).
A monolithic computational method for elasto-dynamics with plasticity and contact

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