Periodic motions of coupled impact oscillators

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Introduction and Motivations

General Motivations

- Understand the dynamics of nonlinear lattices (i.e. large networks of coupled nonlinear oscillators) subjected to unilateral contact and impacts.
- Computation of spatially periodic waves (standing waves or periodic traveling waves) and spatially localized waves (breathers) with impacts.
- Develop theoretical and numerical tools for the analysis of nonlinear waves in nonsmooth mechanical systems.
- Develop a notion of nonsmooth modes in granular media or discrete models of continuum systems.

Introduction and Motivations

Few items in the literature

Literature on vibrating strings

- Haraux, A. and Cabannes, H. Almost periodic motion of a string vibrating against a straight fixed obstacle. *Nonlinear Anal. 7* (1983) 129–141
  H. Cabannes. Presentation of software for movies of vibrating strings with obstacles, *Appl. Math. Lett. 10* (1997), 79-84.
- V.K. Astashev and V.L. Krupenin. Experimental investigation of vibrations of strings interacting with point obstacles, *Doklady Physics 46* (2001), 522-525.
  V.K. Astashev and V.L. Krupenin. Standing waves with line inflection in distributed objects colliding with extended and combined limiters *Journal of Machinery Manufacture and Reliability 41* (2012), 1-6

Literature on discrete mechanical systems

- M. Homer and S. Hogan. Impact dynamics of large dimensional systems *International Journal of Bifurcation and Chaos. 17* (2007), p. 561 - 573
- O.V. Gendelman and L.I. Manevitch, Discrete breathers in vibroimpact chains: analytic solutions, *Phys. Rev. E 78* (2008), 026609.
  O.V. Gendelman, Exact solutions for discrete breathers in a forced-damped chain, *Phys. Rev. E 87* (2013), 062911.

Problem Setting

Multi-supported string.

An infinite chain of impact ocillators is considered with positions described by an infinite vector $y(t) \in l_\infty(\mathbb{Z})$ (the space of bounded sequences on $\mathbb{Z}$).
Mechanical motivations : suspension or cable-stayed bridges.

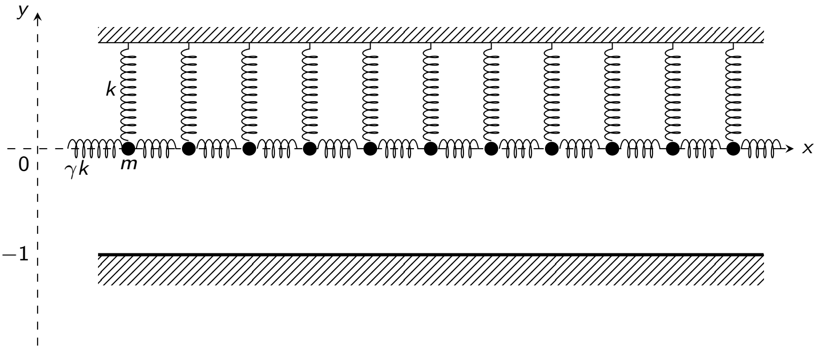


Figure : A chain of linearly coupled impact oscillators

Problem Setting

Equation of motion (undimensional system)

The dynamics is described by the following complementarity system

$$\ddot{y}_n + y_n - \gamma\,(\Delta y)_n = \lambda_n, \quad n \in \mathbb{Z}, \tag{1}$$

$$0 \leqslant \lambda \perp (y + \mathbb{1}) \geqslant 0, \tag{2}$$

$$\text{if } \dot{y}_n(t^-) < 0 \text{ and } y_n(t) = -1 \text{ then } \dot{y}_n(t^+) = -\dot{y}_n(t^-), \tag{3}$$

where

- $(\Delta y)_n = y_{n+1} - 2\,y_n + y_{n-1}$ defines a discrete Laplacian operator,
- $\mathbb{1}$ denotes the constant sequence with all terms equal to unity and
- $\gamma \geqslant 0$ is a parameter that couples the adjacent oscillators.

Anticontinuum limit for $\gamma \to 0$

- R. S. MacKay and S. Aubry (1994) Proof of existence of breathers for time-reversible or Hamiltonian networks of weakly coupled oscillators. Nonlinearity 7 1623
- J. L. Marin, S. Aubry (1996) Breathers in nonlinear lattices: numerical calculation from the anticontinuous limit. Nonlinearity 9 1501

Problem Setting

Main assumptions for periodic solutions

Given a period $T \in (0, 2\pi)$, we seek for T-periodic solutions with the following assumptions:

- each particle impacts the obstacle at most one time per period of oscillations
- an impact pattern is defined for a family of solutions with the following index sets:
  - Impacting particles at the end of the period

$$I_2 = \{k \mid y_k(pT) = -1, p \in \mathbb{Z}\} \tag{4}$$

  - Impacting particles at the half of the period

$$I_1 = \{k \mid y_k(pT + T/2) = -1, p \in \mathbb{Z}\} \tag{5}$$

  - Non impacting particles.

$$I_0 = \mathbb{Z} \setminus (I_1 \cup I_2) \tag{6}$$

Problem Setting

This symmetry assumption allows us to define a reduced problem on $[0, T/2]$.

Reduced boundary value problem on $[0, T/2]$

Reformulation of the problem on $[0, T/2]$

$$\ddot{y}_n + y_n - \gamma\,(\Delta y)_n = 0, \quad n \in \mathbb{Z}, \quad t \in (0, T/2), \tag{7}$$

with time boundary conditions

$$\begin{array}{llll} \dot{y}^{(i)}(0) = 0, & \dot{y}^{(i)}(T/2) = 0 & \text{for} & i \in I_0 \\ \dot{y}^{(i)}(0) = 0, & y^{(i)}(T/2) = -1 & \text{for} & i \in I_1 \\ \dot{y}^{(i)}(T/2) = 0, & y^{(i)}(0) = -1 & \text{for} & i \in I_2 \end{array} \tag{8}$$

Remarks

- The problem (7–8) is a linear BVP without complementarity condition.
- The following constraint has to be satisfied by the solution *a posteriori*

$$y(t) + \mathbb{1} > 0, \quad t \in (0, T/2). \tag{9}$$

Problem Setting

Solution procedure for the reduced boundary value problem.

- First order linear system with time boundary conditions and space periodic boundary conditions (finite chain of N particles)

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} \dot{y}(t) \\ \ddot{y}(t) \end{bmatrix} = A(\gamma)x(t) \\ M\,x(0) + N\,x(T/2) = B \end{cases} \tag{10}$$

- the initial conditions $x(0)$ is given by the following linear system

$$(N \exp(A(\gamma)\,T/2) + M)x(0) = B \tag{11}$$

There is *a priori* no “actual” continuation procedure with respect to γ.

Some properties

- For $\gamma = 0$, an explicit solution is known as the solution of decoupled oscillators.
- The question of the invertibility of $(N \exp(A(\gamma)\,T/2) + M)$ for general value of γ is a question that remains open.

Problem Setting

Particular values of γ

Minimal value of γ such there exists a linear normal mode of period T

- Classical dispersion formula between the frequency ω and the wave number q

$$\omega^2 = 1 + 4\gamma \sin^2(q/2), q \in [-\pi, \pi] \tag{12}$$

- For a given period T, the lowest value of γ for the out of phase mode with $q = \pi$ and we get

$$\gamma_c = \frac{1}{4}\left[(\frac{2\pi}{T})^2 - 1\right] \tag{13}$$

  Example : $T = 3\pi/2$, $\gamma_c = 0.194\ldots$

- For $\gamma = \gamma_c$, there exists a linear normal mode with grazing contacts.

Numerical Methods & Software

A classical shooting method with continuation

Rather than solving the linear system $(N \exp(A(\gamma)\,T/2) + M)x(0) = B$, we use a standard shooting method with continuation on the parameter γ.

- It allows to check the constraint $y(t) + \mathbb{1} > 0, \quad t \in (0, T/2)$.
- Efficient way to compute the exponential matrix
- Convergence of the Newton method in one iteration for non degenerate linear systems.
- It enables the introduction of nonlinear local and interaction potentials.

Siconos Software

The open source Siconos software is used to integrate the general mechanical system with unilateral constraints and Newton impact law.

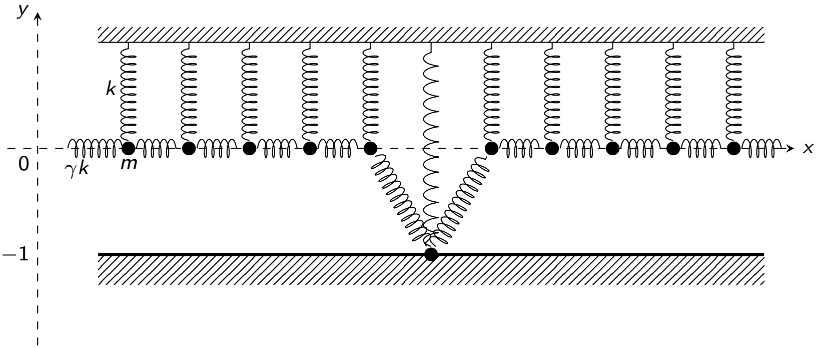
- Event-driven strategy
- LSODAR solver for ordinary differential equations.

http://siconos.gforge.inria.fr

A first illustration.

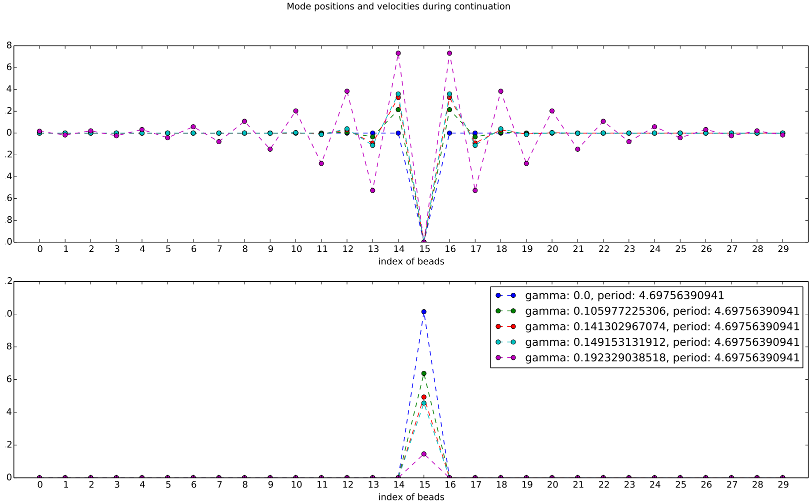
Example 1 (particle centered localized standing wave with one impacting particle)

- 30 *particles.* $I_2 = \{15\}$, $I_1 = \emptyset$, $I_0 = \{1, \dots 30\} \setminus \{15\}$
- $T = 3\pi/2, \gamma_c \approx 0.197$



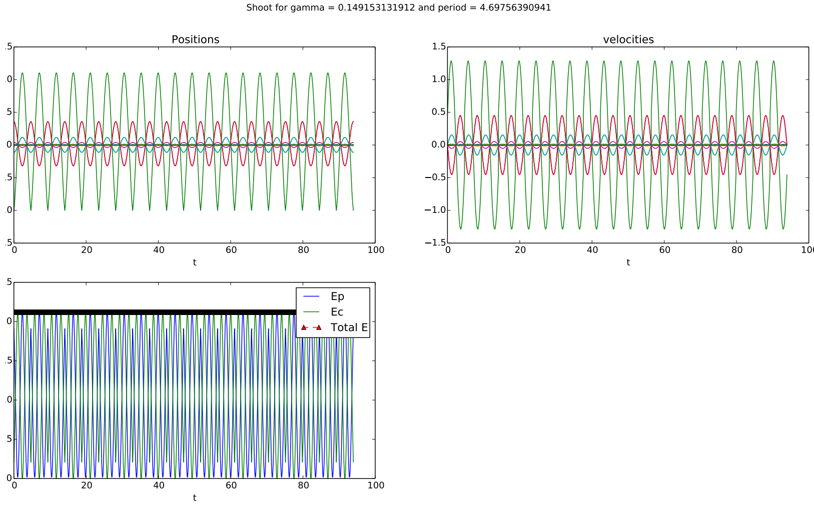
A first illustration.

q_0, v_0 computed with the nonsmooth model.



A first illustration.

Integration over $[0, 20\,T]$ with the nonsmooth model



Comparison with Hertz contact law.

Unilateral Hertz model

$$f = k\delta_+^{3/2}, \text{ with } \delta_+ = \max(0, -1 - y_i)$$

Computational performance

15 values of $\gamma_k \in [0, \gamma_{max}]$, $\gamma_{max} < \gamma_c$. $\gamma_{max} = 0.187$ $\gamma_c = 0.194$

	Nonsmooth law	Hertz law ($k = 100$)
# LSODAR time steps	46194	1 788 158
# LSODAR r.h.s evaluation	83435	10 915 173
# Newton iteration	15	53
CPU time(s)	2.5	401.4

- Hertz model :
  - the LSODAR solver fails to integrate the system near γ_c.
  - numerical issues arise sooner with increasing values of k
- With the nonsmooth model
  - one newton iterations is needed in the generic case.
  - no need of initial solution for starting the continuation process.

Linear Stability of periodic solutions

Computation of the Floquet multipliers the monodromy matrix

- Let us consider the interval of study $[T/4, T + T/4]$ ($t_0 = T/4$).
- Under the assumption that a single particle k hits the obstacle at time $T/2$ and a single particle l hits the obstacle at time T, we get for the Jacobian

$$J(5T/4, t_0, x_0) = \exp(A(5T/4-T))\, S_{T,l} \exp(A(T-T/2))\, S_{T/2,k}\, exp(A(T/2-T/4)) \tag{14}$$

where $S_{t,k}$ is the saltation matrix associated with the jump in the velocity at time t for the particle k.

Computation of the saltation matrix

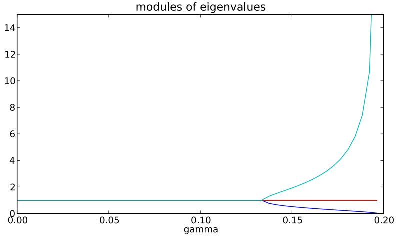
Impact map and reset matrix

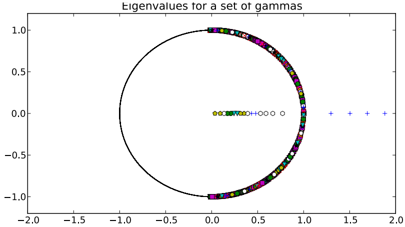
$$x(t^+) = R_k x(t^-). \tag{15}$$

$$S_{t,k} = R_k + \frac{1}{v_k(t^{*,-})} \begin{bmatrix} 0 & x(t^+) - R_k\, x(t^-) & 0 \end{bmatrix} \tag{16}$$

Linear Stability of periodic solutions

Example 1. (particle centered localized standing wave with one impacting particle)

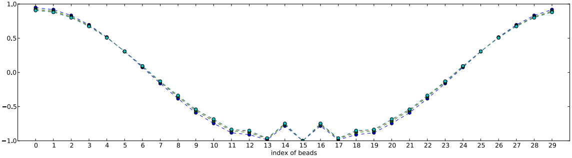




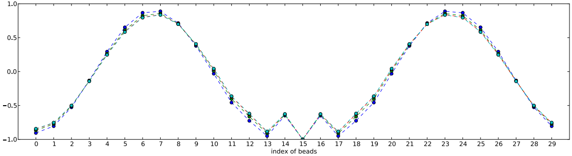
Nonsmooth modes

Nonsmooth modes for $\gamma \approx 1$.

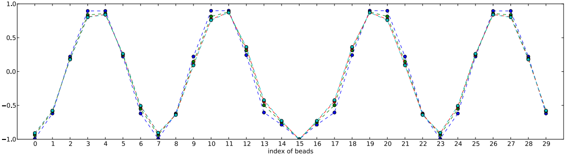
mode 1.
$q = 2\pi/30$,
$T \approx 6.150$



mode 2.
$q = 2\pi/15$,
$T \approx 5.801$



mode 3.
$q = 2\pi/7$,
$T \approx 4.745$



Conclusions & perspectives

Conclusions

- Numerical framework for computing and analyzing the stability of the periodic solutions in large 1D lattice with impacts.
- The nonsmooth framework enables computational efficiency and the exploration of solutions for any value of γ without continuation.

Perspectives

- Theoretical investigations on the existence of periodic solutions for $\gamma \neq 0$
- Stability : differentiability of the flow with multiple impacts ?
- Existence of travelling waves ?
- Exploration of other contact profiles (point obstacles, asymmetric obstacle, ...).
- Nonsmooth normal modes for vibrating strings ($\gamma \approx 1/N^2$) or other continuum mechanical systems

Thank you for your attention.