The nonsmooth generalized-α scheme with a simultaneous enforcement of constraints at position and velocity levels.

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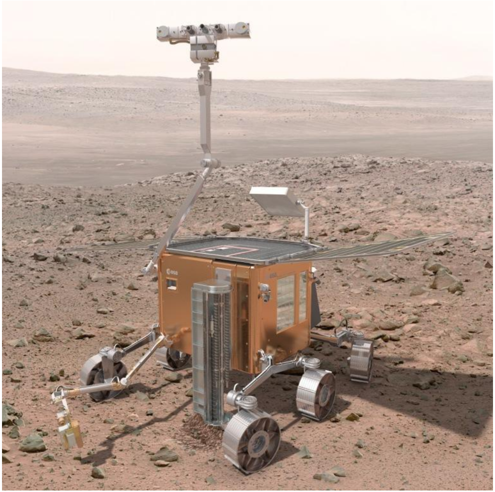
Mechanical systems with contact, impact and friction

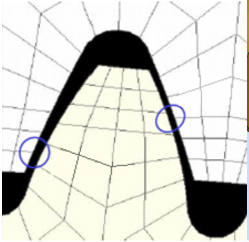
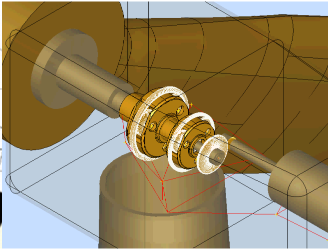
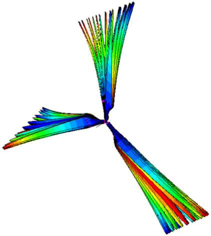
Simulation of flexible multibody systems.

Simulation of the ExoMars Rover (INRIA/Trasys Space/ESA)



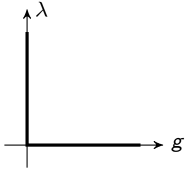
Mechanical systems with contact, impact and friction

Simulation of flexible multibody systems.
Simulation of wind turbines (DYNAWIND project)
Joint work with O. Brüls, Q.Z. Chen and G. Virlez (Université de Liège)

NonSmooth Multibody Systems

Scleronomous holonomic perfect unilateral constraints and joints



- Nonsmooth equations of motion

$$\begin{aligned} \dot{\mathbf{q}}^+ &= \mathbf{v}^+ && \text{(1a)} \\ \mathbf{M}(\mathbf{q})\,\mathrm{d}\mathbf{v} - \mathbf{g}_\mathbf{q}^T\,\mathrm{d}\mathbf{i} &= \mathbf{f}(\mathbf{q},\mathbf{v},t)\,\mathrm{d}t && \text{(1b)} \\ \mathbf{g}^{\overline{\mathcal{U}}}(\mathbf{q}) &= \mathbf{0} && \text{(1c)} \\ \mathbf{0} \leqslant \mathbf{g}^{\mathcal{U}}(\mathbf{q}) &\perp \mathrm{d}\mathbf{i}^{\mathcal{U}} \geqslant \mathbf{0} && \text{(1d)} \end{aligned}$$

where

- $\mathbf{g}_\mathbf{q} = \nabla g(q)$.
- $\mathcal{U}$ index set of indices of the unilateral constraints,
- $\overline{\mathcal{U}}$ the set of bilateral constraints,
- $\mathcal{C} = \mathcal{U} \cup \overline{\mathcal{U}}$

- Newton Impact law $\mathbf{g}_\mathbf{q}^{\mathcal{U}}\mathbf{v}^+(t) = -e\,\mathbf{g}_\mathbf{q}^{\mathcal{U}}\mathbf{v}^-(t)$
  e is the coefficient of restitution.

The Moreau's sweeping process of second order

Definition (Moreau [1983, 1988])

A key stone of this formulation is the inclusion in terms of velocity.

$$
\begin{aligned}
\dot{\mathbf{q}} &= \mathbf{v} && (2a)\\
\mathbf{M}(\mathbf{q})\,\mathrm{d}\mathbf{v} - \mathbf{g}_{\mathbf{q}}^{T}\,\mathrm{d}\mathbf{i} &= \mathbf{f}(\mathbf{q},\mathbf{v},t)\,\mathrm{d}t && (2b)\\
\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}\,\mathbf{v} &= \mathbf{0} && (2c)\\
\text{if } g^{j}(\mathbf{q}) \leqslant 0 \text{ then } 0 \leqslant g_{\mathbf{q}}^{j}\,\mathbf{v} + e\,g_{\mathbf{q}}^{j}\,\mathbf{v}^{-} &\perp \mathrm{d}i^{j} \geqslant 0, \quad \forall j \in \mathcal{U} && (2d)
\end{aligned}
$$

Comments

This formulation provides a common framework for the nonsmooth dynamics containing inelastic impacts without decomposition.
➜ Foundation for the Moreau–Jean time–stepping approach.

Moreau–Jean time stepping scheme [Moreau, 1988, Jean, 1999]

Principle

$$P_{n+1} \approx di((t_n, t_{n+1}]) = \int_{(t_n, t_{n+1}]} \mathrm{d}i \tag{3}$$

$$\begin{aligned}
\mathbf{q}_{n+1} &= \mathbf{q}_n + h\mathbf{v}_{n+\theta}, && \text{(4a)}\\
M(\mathbf{q}_{n+\theta})(\mathbf{v}_{n+1} - \mathbf{v}_n) - hf_{n+\theta} &= g_q(\mathbf{q}_{n+\theta})P_{n+1}, && \text{(4b)}\\
\text{if} \quad \bar{g}^j_n \leqslant 0, 0 \leqslant g^j_{\mathbf{q},n+1}\,\mathbf{v}_{n+1} + e\,g^j_{\mathbf{q},n}\,\mathbf{v}_n &\perp P^j_{n+1} \geqslant 0 && \text{(4c)}\\
& && \text{(4d)}
\end{aligned}$$

with

- $\theta \in [0, 1]$
- $x_{n+\theta} = (1-\theta)x_{n+1} + \theta x_n$
- $f_{n+\theta} = f(t_{n+\theta}, \mathbf{q}_{n+\theta}, \mathbf{v}_{n+\theta})$
- $\bar{g}_n$ is a prediction of the constraints, e.g. $\bar{g}_n = g_n + h/2 g^j_{\mathbf{q},n}\,\mathbf{v}_n$

Objectives & Motivations

Limitations of the Moreau–Jean scheme

- Moreau–Jean time–stepping : strong numerical damping for $\theta \gg 1/2$.
  ➜ Improve numerical damping with a controlled damping of high frequencies.
- Constraint treated at the velocity level : penetration at the position level.
  ➜ solve the constraints at position level.
- Rough activation of constraints at the velocity level

Means

- Splitting between impulsive and non impulsive terms and use of α–scheme. [Chen et al., 2013]
- Gear–Gupta–Leimkuhler (GGL) enforcement of the unilateral constraint at the position level. [Acary, 2013]
- Nonsmooth Newton method viewed as an active set method.

The nonsmooth generalized α scheme

GGL approach to stabilize the constraints at the position level

$$\mathrm{d}\mathbf{w} = \mathrm{d}\mathbf{v} - \dot{\tilde{\mathbf{v}}}\,\mathrm{d}t \tag{5}$$

Smooth (non-impulsive) part

Solutions of the following DAE

$$\begin{aligned}
\dot{\tilde{\mathbf{q}}} &= \tilde{\mathbf{v}} && \text{(6a)}\\
\mathbf{M}(\mathbf{q})\,\dot{\tilde{\mathbf{v}}} - \mathbf{g}_{\mathbf{q}}^{T}(\mathbf{q})\,\tilde{\boldsymbol{\lambda}} &= \mathbf{f}(\mathbf{q},\mathbf{v},t) && \text{(6b)}\\
\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}(\mathbf{q})\,\tilde{\mathbf{v}} &= \mathbf{0} && \text{(6c)}\\
\tilde{\boldsymbol{\lambda}}^{\mathcal{U}} &= \mathbf{0} && \text{(6d)}
\end{aligned}$$

with the initial value $\tilde{\mathbf{v}}(t_n) = \mathbf{v}(t_n)$, $\tilde{\mathbf{q}}(t_n) = \mathbf{q}(t_n)$.

The nonsmooth generalized α scheme

Splitting the dynamics between smooth and nonsmooth part

$$
\begin{aligned}
\dot{\mathbf{q}} &= \mathbf{v} && \text{(7a)}\\
\mathrm{d}\mathbf{v} &= \mathrm{d}\mathbf{w} + \dot{\tilde{\mathbf{v}}}\,\mathrm{d}t && \text{(7b)}\\
\mathbf{M}(\mathbf{q})\,\dot{\tilde{\mathbf{v}}} - \mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}},T}\,\tilde{\boldsymbol{\lambda}}^{\overline{\mathcal{U}}} &= \mathbf{f}(\mathbf{q},\mathbf{v},t) && \text{(7c)}\\
\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}\,\tilde{\mathbf{v}} &= \mathbf{0} && \text{(7d)}\\
\tilde{\boldsymbol{\lambda}}^{\mathcal{U}} &= \mathbf{0} && \text{(7e)}\\
\mathbf{M}(\mathbf{q})\,\mathrm{d}\mathbf{w} - \mathbf{g}_{\mathbf{q}}^{T}\,(\mathrm{d}\mathbf{i} - \tilde{\boldsymbol{\lambda}}\,\mathrm{d}t) &= \mathbf{0} && \text{(7f)}\\
\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}\mathbf{v} &= \mathbf{0} && \text{(7g)}\\
\text{if } g^{j}(\mathbf{q}) \leqslant 0 \text{ then } 0 \leqslant g_{\mathbf{q}}^{j}\,\mathbf{v} + e\,g_{\mathbf{q}}^{j}\,\mathbf{v}^{-} &\perp \mathrm{d}i^{j} \geqslant 0, \quad \forall j \in \mathcal{U} && \text{(7h)}
\end{aligned}
$$

The nonsmooth generalized α scheme

Splitting the dynamics between smooth and nonsmooth part

The equations of motion become

$$\begin{aligned}
\mathbf{M}(\mathbf{q})\,\dot{\mathbf{q}} - \mathbf{g}_{\mathbf{q}}^{T}\,\boldsymbol{\mu} &= \mathbf{M}(\mathbf{q})\,\mathbf{v} && \text{(8a)}\\
\cancel{\dot{\mathbf{q}} = \mathbf{v}} \rightarrow \qquad \mathbf{g}^{\overline{\mathcal{U}}}(\mathbf{q}) &= \mathbf{0} && \text{(8b)}\\
\mathbf{0} \leqslant \mathbf{g}^{\mathcal{U}}(\mathbf{q}) &\perp \boldsymbol{\mu}^{\mathcal{U}} \geqslant \mathbf{0} && \text{(8c)}\\
\mathrm{d}\mathbf{v} &= \mathrm{d}\mathbf{w} + \dot{\tilde{\mathbf{v}}}\,\mathrm{d}t && \text{(8d)}\\
\mathbf{M}(\mathbf{q})\,\dot{\tilde{\mathbf{v}}} - \mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}},T}\,\tilde{\boldsymbol{\lambda}}^{\overline{\mathcal{U}}} &= \mathbf{f}(\mathbf{q},\mathbf{v},t) && \text{(8e)}\\
\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}\,\tilde{\mathbf{v}} &= \mathbf{0} && \text{(8f)}\\
\tilde{\boldsymbol{\lambda}}^{\mathcal{U}} &= \mathbf{0} && \text{(8g)}\\
\mathbf{M}(\mathbf{q})\,\mathrm{d}\mathbf{w} - \mathbf{g}_{\mathbf{q}}^{T}\,(\mathrm{d}\mathbf{i} - \tilde{\boldsymbol{\lambda}}\,\mathrm{d}t) &= \mathbf{0} && \text{(8h)}\\
\mathbf{g}_{\mathbf{q}}^{\overline{\mathcal{U}}}\mathbf{v} &= \mathbf{0} && \text{(8i)}\\
\text{if } g^{j}(\mathbf{q}) \leqslant 0 \text{ then } 0 \leqslant g_{\mathbf{q}}^{j}\,\mathbf{v} + e\,g_{\mathbf{q}}^{j}\,\mathbf{v}^{-} &\perp \mathrm{d}i^{j} \geqslant 0, \quad \forall j \in \mathcal{U} && \text{(8j)}
\end{aligned}$$

The nonsmooth generalized α scheme

Velocity jumps and position correction

The multipliers $\boldsymbol{\Lambda}(t_n;t)$ and $\boldsymbol{\nu}(t_n;t)$ are defined as

$$\boldsymbol{\Lambda}(t_n;t) = \int_{(t_n,t]} (\mathrm{d}\mathbf{i} - \tilde{\boldsymbol{\lambda}}(\tau)\,\mathrm{d}\tau) \tag{9a}$$

$$\boldsymbol{\nu}(t_n;t) = \int_{t_n}^{t} (\boldsymbol{\mu}(\tau) + \boldsymbol{\Lambda}(t_n;\tau))\,\mathrm{d}\tau \tag{9b}$$

with $\boldsymbol{\Lambda}(t_n;t_n) = \boldsymbol{\nu}(t_n;t_n) = \mathbf{0}$.
The velocity jump and position correction variables

$$\mathbf{W}(t_n;t) = \int_{(t_n,t]} \mathrm{d}\mathbf{w} = \mathbf{v}(t) - \tilde{\mathbf{v}}(t) \tag{10a}$$

$$\mathbf{U}(t_n;t) = \int_{t_n}^{t} (\dot{\mathbf{q}} - \tilde{\mathbf{v}})\mathrm{d}t = \mathbf{q}(t) - \tilde{\mathbf{q}}(t) \tag{10b}$$

➜ Low-order approximation of impulsive terms.
➜ Higher–order approximation of non impulsive terms.

The nonsmooth generalized α scheme

$$
\begin{aligned}
\mathbf{M}(\mathbf{q}_{n+1})\mathbf{U}_{n+1} - \mathbf{g}^{T}_{\mathbf{q},n+1}\,\boldsymbol{\nu}_{n+1} &= \mathbf{0} && \text{(11a)}\\
\mathbf{g}^{\overline{\mathcal{U}}}(\mathbf{q}_{n+1}) &= \mathbf{0} && \text{(11b)}\\
\mathbf{0} \leqslant \mathbf{g}^{\mathcal{U}}(\mathbf{q}_{n+1}) \perp \boldsymbol{\nu}^{\mathcal{U}}_{n+1} &\geqslant \mathbf{0} && \text{(11c)}\\
\mathbf{M}(\mathbf{q}_{n+1})\dot{\tilde{\mathbf{v}}}_{n+1} - \mathbf{f}(\mathbf{q}_{n+1},\mathbf{v}_{n+1},t_{n+1}) - \mathbf{g}^{\overline{\mathcal{U}},T}_{\mathbf{q},n+1}\,\tilde{\boldsymbol{\lambda}}^{\overline{\mathcal{U}}}_{n+1} &= \mathbf{0} && \text{(11d)}\\
\mathbf{g}^{\overline{\mathcal{U}}}_{\mathbf{q},n+1}\,\tilde{\mathbf{v}}_{n+1} &= \mathbf{0} && \text{(11e)}\\
\mathbf{M}(\mathbf{q}_{n+1})\mathbf{W}_{n+1} - \mathbf{g}^{T}_{\mathbf{q},n+1}\boldsymbol{\Lambda}_{n+1} &= \mathbf{0} && \text{(11f)}\\
\mathbf{g}^{\overline{\mathcal{U}}}_{\mathbf{q},n+1}\mathbf{v}_{n+1} &= \mathbf{0} && \text{(11g)}\\
\text{if } g^{j}(\mathbf{q}^{*}_{n+1}) \leqslant 0 \text{ then } 0 \leqslant g^{j}_{\mathbf{q},n+1}\,\mathbf{v}_{n+1} + e\,g^{j}_{\mathbf{q},n}\,\mathbf{v}_{n} \perp \Lambda^{j}_{n+1} &\geqslant 0, \forall j \in \mathcal{U}
\end{aligned}
$$

The nonsmooth generalized α scheme

Nonsmooth generalized α-scheme

$$
\begin{aligned}
\tilde{\mathbf{q}}_{n+1} &= \mathbf{q}_n + h\mathbf{v}_n + h^2(0.5-\beta)\mathbf{a}_n + h^2\beta\mathbf{a}_{n+1} && (12a)\\
\mathbf{q}_{n+1} &= \tilde{\mathbf{q}}_{n+1} + \mathbf{U}_{n+1} && (12b)\\
\tilde{\mathbf{v}}_{n+1} &= \mathbf{v}_n + h(1-\gamma)\mathbf{a}_n + h\gamma\mathbf{a}_{n+1} && (12c)\\
\mathbf{v}_{n+1} &= \tilde{\mathbf{v}}_{n+1} + \mathbf{W}_{n+1} && (12d)\\
(1-\alpha_m)\mathbf{a}_{n+1} + \alpha_m\mathbf{a}_n &= (1-\alpha_f)\dot{\tilde{\mathbf{v}}}_{n+1} + \alpha_f\dot{\tilde{\mathbf{v}}}_n && (12e)
\end{aligned}
$$

Special cases

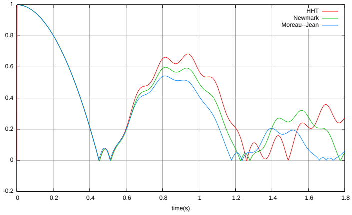
- $\alpha_m = \alpha_f = 0$ ➜ Nonsmooth Newmark
- $\alpha_m = 0, \alpha_f \in [0, 1/3]$ ➜ Nonsmooth Hilber-Hughes–Taylor (HHT)

Spectral radius at infinity $\rho_\infty \in [0, 1]$

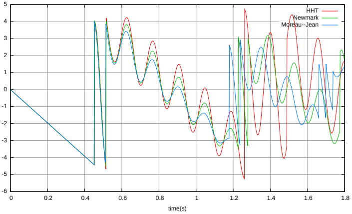
$$
\alpha_m = \frac{2\rho_\infty - 1}{\rho_\infty + 1}, \quad \alpha_f = \frac{\rho_\infty}{\rho_\infty + 1}, \quad \beta = \frac{1}{4}(\gamma + \frac{1}{2})^2. \qquad (13)
$$

Numerical Illustrations

Two ball oscillator with impact.

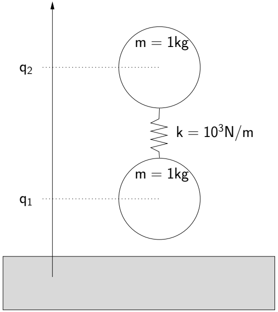


Position of the first ball



Velocity of the first ball

Time–step : $h = 2e-3$.
Moreau ($\theta = 1.0$).
Newmark ($\gamma = 1.0, \beta = 0.5, \alpha_m = \alpha_f = 0$).
HHT ($\gamma = 1.0, \beta = 0.5, \alpha_f = 0.1, \alpha_m = 0$)



Numerical Illustrations

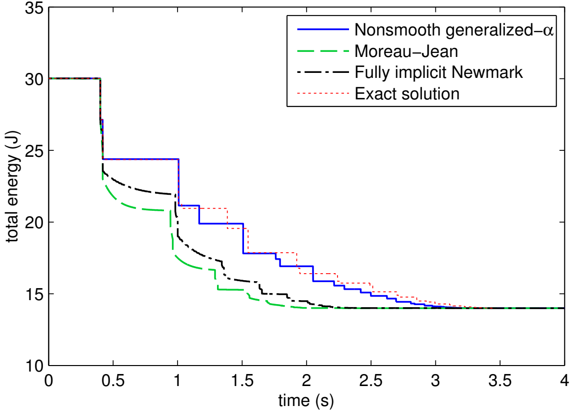
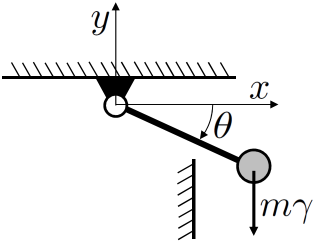


Figure 7. Numerical results for the total energy of the bouncing oscillator.

Numerical Illustrations

Bouncing Pendulum

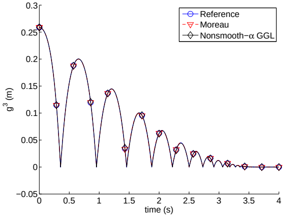
$$\mathbf{q} = [x, y, \theta]^T$$
$$g_1(\mathbf{q}) = x - l\cos\theta = 0$$
$$g_2(\mathbf{q}) = y - l\sin\theta = 0$$
$$g_3(\mathbf{q}) = x - \sqrt{2}/2 \geqslant 0$$

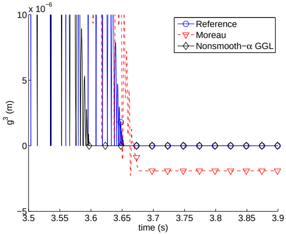


Time–step : $h = 2e - 3$.
Moreau ($\theta = 1/1.8$).
α-schemes ($\rho_\infty = 0.8$)

$e = 0.8$

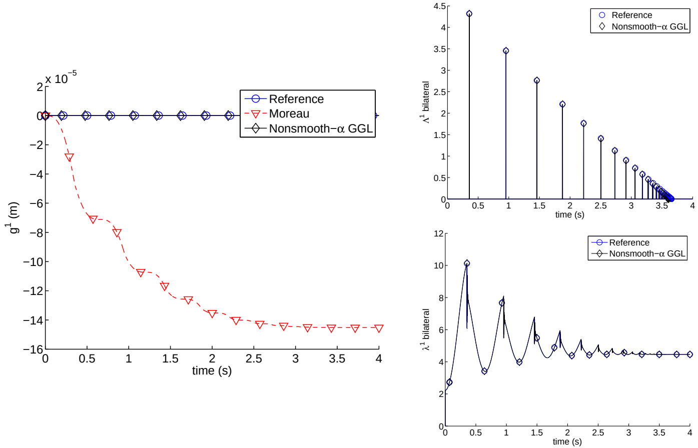
Unilateral constraint





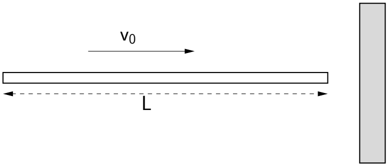
Numerical Illustrations

Bouncing Pendulum



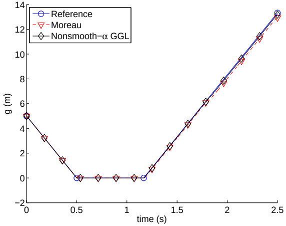
Numerical Illustrations

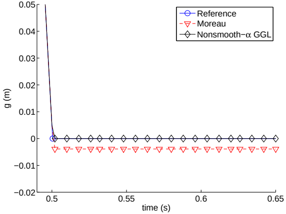
Impacting elastic bar



$g_3(\mathbf{q}) = x_1 \geqslant 0$
$e = 0.0$
200 finite elements
Time–step : $h = 2e - 3$.
Moreau ($\theta = 1/1.8$).
α-schemes ($\rho_\infty = 0.8$)

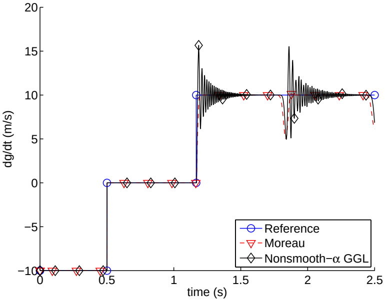
Unilateral constraint

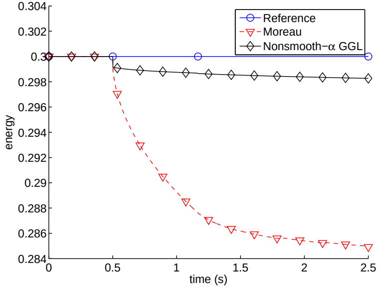




Numerical Illustrations

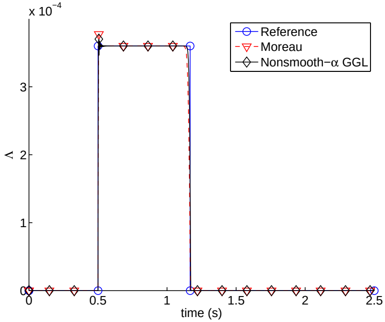
Impacting elastic bar

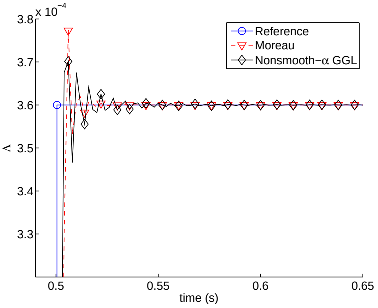




Numerical Illustrations

Impacting elastic bar





The nonsmooth generalized α scheme

Summary

- Improved accuracy and energy behavior when smooth nonlinear contributions are present
- Constraints at position and velocity levels are both satisfied in discrete–time.
- Activation of constraints at velocity levels are solved in a global nonsmooth Newton Method

Thank you for your attention.

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