Nonsmooth dynamical systems. Numerical Time–integration schemes and the Siconos platform

Vincent Acary

vincent.acary@inrialpes.fr
INRIA Rhône–Alpes

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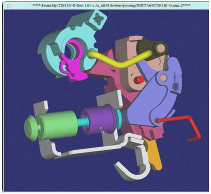
Applications

Mechanical systems with contact, impact and friction

Simulation of Circuit breakers (INRIA/Schneider Electric)



Vincent Acary

vincent.acary@inrialpes.fr
INRIA Rhône–Alpes

Applications

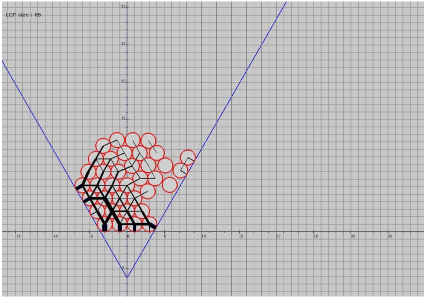
Mechanical systems with contact, impact and friction

Bipedal Robot INRIA BIPOP

Applications

Mechanical systems with contact, impact and friction

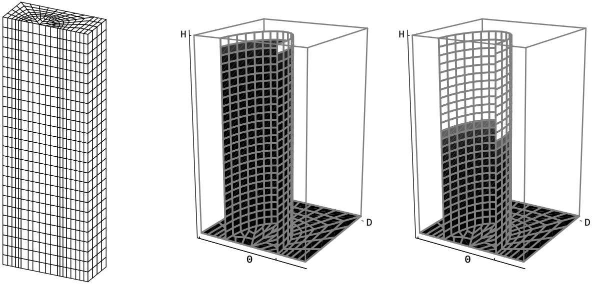
Stack of beads with perturbation



Applications

Mechanical systems with contact, impact and friction

FEM models with contact, friction cohesion, etc...

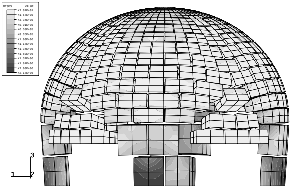


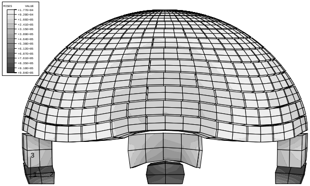
Joint work with Y. Monerie, IRSN.

Applications

Mechanical systems with contact, impact and friction

Divided Materials and Masonry





Vincent Acary

vincent.acary@inrialpes.fr
INRIA Rhône–Alpes

Introduction & Motivations

Lagrangian dynamical systems with unilateral constraints and friction

Numerical time–integration schemes

Adaptive time-step size time–stepping scheme

Time–stepping schemes of any order

Splitting based Time–stepping schemes

Conclusions & Perspectives

The Siconos Platform

The smooth multibody dynamics

Definition (Smooth multibody dynamics)

$$\begin{cases} M(q)\dfrac{dv}{dt} + F(t,q,v) = 0, \\ v = \dot{q} \end{cases} \tag{1}$$

where

- $F(t,q,v) = N(q,v) + F_{int}(t,q,v) - F_{ext}(t)$

Definition (Boundary conditions)

- Initial Value Problem (IVP):

$$t_0 \in \mathbb{R}, \quad q(t_0) = q_0 \in \mathbb{R}^n, \quad v(t_0) = v_0 \in \mathbb{R}^n, \tag{2}$$

- Boundary Value Problem (BVP):

$$(t_0, T) \in \mathbb{R} \times \mathbb{R}, \quad \Gamma(q(t_0), v(t_0), q(T), v(T)) = 0 \tag{3}$$

Perfect unilateral constraints

Unilateral constraints

- Finite set of ν unilateral constraints on the generalized coordinates :

$$g(q,t) = [g_\alpha(q,t) \geqslant 0, \quad \alpha \in \{1 \ldots \nu\}]^T . \tag{4}$$

- Admissible set $\mathcal{C}(t)$

$$\mathcal{C}(t) = \{q \in \mathcal{M}(t), g_\alpha(q,t) \geqslant 0, \alpha \in \{1 \ldots \nu\}\} . \tag{5}$$

Normal cone to $\mathcal{C}(t)$

$$N_{\mathcal{C}(t)}(q(t)) = \left\{ y \in \mathbb{R}^n, y = -\sum_\alpha \lambda_\alpha \nabla g_\alpha(q,t),\ \lambda_\alpha \geqslant 0,\ \lambda_\alpha g_\alpha(q,t) = 0 \right\} \tag{6}$$

Unilateral constraints as an inclusion

Definition (Perfect unilateral constraints on the smooth dynamics)

Introduction of the multipliers $\mu \in \mathbb{R}^m$

$$\begin{cases} M(q)\dfrac{dv}{dt} + F(t,q,v) = r = \nabla_q^T h(q,t)\,\lambda \\ \\ -r \in N_{\mathcal{C}(t)}(q(t)) \end{cases} \tag{7}$$

where $r = \nabla_q^T g(q,t)\,\lambda$ generalized forces or generalized reactions due to the constraints.

Remark

- The unilateral constraints are said to be perfect due to the normality condition.
- Notion of normal cones can be extended to more general sets. see (Clarke, 1975, 1983 ; **?**)
- The right hand side is neither bounded (and then nor compact).
- The inclusion and the constraints concern the second order time derivative of q.

➜ Standard Analysis of DI does no longer apply.

Non Smooth Lagrangian Dynamics

Fundamental assumptions.

- The velocity $v = \dot{q}$ is of Bounded Variations (B.V)
  ➜ The equation are written in terms of a right continuous B.V. (R.C.B.V.) function, v^+ such that

$$v^+ = \dot{q}^+ \tag{8}$$

- q is related to this velocity by

$$q(t) = q(t_0) + \int_{t_0}^{t} v^+(t)\,dt \tag{9}$$

- The acceleration, ($\ddot{q}$ in the usual sense) is hence a differential measure dv associated with v such that

$$dv(]a,b]) = \int_{]a,b]} dv = v^+(b) - v^+(a) \tag{10}$$

Non Smooth Lagrangian Dynamics

Definition (Non Smooth Lagrangian Dynamics)

$$\begin{cases} M(q)dv + F(t,q,v^+)dt = dr \\ \\ v^+ = \dot{q}^+ \end{cases} \tag{11}$$

where dr is the reaction measure and dt is the Lebesgue measure.

Remarks

- The non smooth Dynamics contains the impact equations and the smooth evolution in a single equation.
- The formulation allows one to take into account very complex behaviors, especially, finite accumulation (Zeno-state).
- This formulation is sound from a mathematical Analysis point of view.

References

(Schatzman, 1973, 1978 ; Moreau, 1983, 1988)

Decomposition of measure

$$\left\{ \begin{array}{llll} dv = & \gamma \, dt + & (v^+ - v^-) \, d\nu + & dv_s \\ dr = & f \, dt + & p \, d\nu + & dr_s \end{array} \right. \tag{12}$$

where

- $\gamma = \ddot{q}$ is the acceleration defined in the usual sense.
- f is the Lebesgue measurable force,
- $v^+ - v^-$ is the difference between the right continuous and the left continuous functions associated with the B.V. function $v = \dot{q}$,
- $d\nu$ is a purely atomic measure concentrated at the time t_i of discontinuities of v, i.e. where $(v^+ - v^-) \neq 0$,i.e. $d\nu = \sum_i \delta_{t_i}$
- p is the purely atomic impact percussions such that $pd\nu = \sum_i p_i \delta_{t_i}$
- dv_S and dr_S are singular measures with the respect to $dt + d\eta$.

vincent.acary@inrialpes.fr
INRIA Rhône–Alpes

Impact equations and Smooth Lagrangian dynamics

Substituting the decomposition of measures into the non smooth Lagrangian Dynamics, one obtains

Definition (Impact equations)

$$M(q)(v^+ - v^-)d\nu = pd\nu, \tag{13}$$

or

$$M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \tag{14}$$

Definition (Smooth Dynamics between impacts)

$$M(q)\gamma dt + F(t, q, v)dt = fdt \tag{15}$$

or

$$M(q)\gamma^+ + F(t, q, v^+) \quad = \quad f^+ \quad [dt - a.e.] \tag{16}$$

The Moreau's sweeping process of second order

Definition (Moreau (1983, 1988))

A key stone of this formulation is the inclusion in terms of velocity. Indeed, the inclusion (7) is "replaced" by the following inclusion

$$\begin{cases} M(q)dv + F(t,q,v^+)dt = dr \\ \\ v^+ = \dot{q}^+ \\ \\ -dr \in N_{T_C(q)}(v^+) \end{cases} \tag{17}$$

Comments

This formulation provides a common framework for the non smooth dynamics containing inelastic impacts without decomposition.
➜ Foundation for the time–stepping approaches.

The Moreau's sweeping process of second order

Comments

- *The inclusion concerns measures.* Therefore, it is necessary to define what is the inclusion of a measure into a cone.
- *The inclusion in terms of velocity* v^+ rather than of the coordinates q.

Interpretation

- Inclusion of measure, $-dr \in K$
  - Case $dr = r'dt = fdt$.
$$-f \in K \tag{18}$$
  - Case $dr = p_i \delta_i$.
$$-p_i \in K \tag{19}$$
- Inclusion in terms of the velocity. Viability Lemma
If $q(t_0) \in C(t_0)$, then
$$v^+ \in T_C(q), t \geqslant t_0 \Rightarrow q(t) \in C(t), t \geqslant t_0$$
➜ The unilateral constraints on q are satisfied. The equivalence needs at least an impact inelastic rule.

The Moreau's sweeping process of second order

The Newton-Moreau impact rule

$$-\,dr \in N_{T_C(q(t))}(v^+(t) + ev^-(t)) \tag{20}$$

where e is a coefficient of restitution.

Vincent Acary

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INRIA Rhône–Alpes

The Moreau's sweeping process of second order

The case of C is finitely represented

$$\mathcal{C} = \{q \in \mathcal{M}(t), g_\alpha(q) \geqslant 0, \alpha \in \{1 \ldots \nu\}\}. \tag{21}$$

Decomposition of dr and v^+ onto the tangent and the normal cone.

$$dr = \sum_\alpha \nabla_q^T g_\alpha(q)\, d\lambda_\alpha \tag{22}$$

$$U_\alpha^+ = \nabla_q g_\alpha(q)\, v^+, \alpha \in \{1 \ldots \nu\} \tag{23}$$

Complementarity formulation (under constraints qualification condition)

$$-d\lambda_\alpha \in N_{T_{\mathbb{R}_+}(g_\alpha)}(U_\alpha^+) \Leftrightarrow \text{ if } g_\alpha(q) \leqslant 0, \text{ then } 0 \leqslant U_\alpha^+ \perp d\lambda_\alpha \geqslant 0 \tag{24}$$

The case of C is $\mathbb{R}_+$

$$-dr \in N_C(q) \Leftrightarrow 0 \leqslant q \perp dr \geqslant 0 \tag{25}$$

is replaced by

$$-dr \in N_{T_C(q)}(v^+) \Leftrightarrow \text{ if } q \leqslant 0, \text{ then } 0 \leqslant v^+ \perp dr \geqslant 0 \tag{26}$$

Coulomb's friction

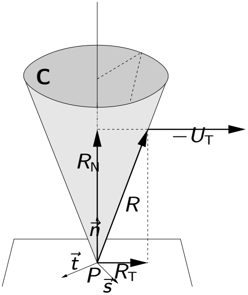


Figure: Coulomb's friction. The sliding case.

Definition (Coulomb's friction)

Coulomb's friction says the following. If $g(q) = 0$ then:

$$\begin{cases} \text{If } U_{\mathrm{T}}(t) = 0 & \text{then } R \in \ \mathbf{C} \\ \text{If } U_{\mathrm{T}}(t) \neq 0 & \text{then } ||R_{\mathrm{T}}(t)|| = \mu |R_{\mathrm{N}}| \text{ and there exists a scalar } a \geqslant 0 \\ & \text{such that } R_{\mathrm{T}}(t) = -a U_{\mathrm{T}}(t) \end{cases} \tag{27}$$

where $C = \{R, ||R_{\mathrm{T}}(t)|| \leqslant \mu |R_{\mathrm{N}}| \ \}$ is the Coulomb friction cone

Coulomb's friction

Definition (Coulomb's friction as an inclusion into a disk)

Let us introduce the following inclusion (Moreau, 1988), using the indicator function $\psi_{\mathbf{D}}(\cdot)$:

$$-U_{\mathrm{T}} \in \partial\psi_{\mathbf{D}}(R_{\mathrm{T}}) \tag{28}$$

where $D = \{R_{\mathrm{T}}, ||R_{\mathrm{T}}(t)|| \leqslant \mu|R_{\mathrm{N}}|\ \}$ is the Coulomb friction disk

Definition (Coulomb's friction as a variational inequality (VI))

Then (28) appears to be equivalent to

$$\begin{cases} R_{\mathrm{T}} \in \mathbf{D} \\ \langle U_{\mathrm{T}}, z - R_{\mathrm{T}} \rangle \geqslant 0 \text{ for all } z \in \mathbf{D} \end{cases} \tag{29}$$

and to

$$R_{\mathrm{T}} = \mathrm{proj}_{\mathbf{D}}[R_{\mathrm{T}} - \rho U_{\mathrm{T}}], \text{ for all } \rho > 0 \tag{30}$$

Definition (Coulomb's Friction as a Second–Order Cone Complementarity Problem)

Let us introduce the modified velocity $\widehat{U}$ defined by

$$\widehat{U} = [U_{\mathrm{N}} + \mu \,||U_{\mathrm{T}}||, U_{\mathrm{T}}]^T. \tag{31}$$

This notation provides us with a synthetic form of the Coulomb friction as

$$-\widehat{U} \in \partial\psi_{\mathbf{C}}(R), \tag{32}$$

or

$$\mathbf{C}^* \ni \widehat{U} \perp R \in \mathbf{C}. \tag{33}$$

where $\mathbf{C}^* = \{v \in \mathbb{R}^n \mid r^T v \geqslant 0, \forall r \in \mathbf{C}\}$ is the dual cone.

Coulomb's friction

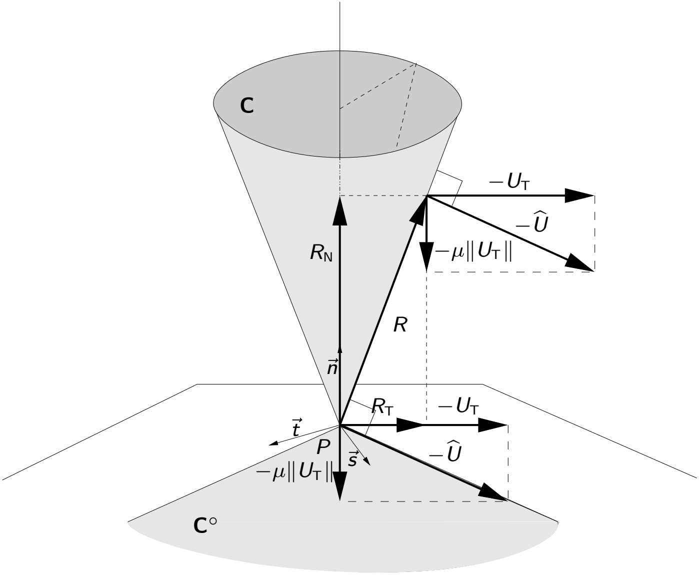


Figure: Coulomb's friction and the modified velocity $\widehat{U}$. The sliding case.

State–of–the–art

Numerical time–integration methods for Nonsmooth Multibody systems (NSMBS):

Nonsmooth event capturing methods (Time–stepping methods)

- $\oplus$ robust, stable and proof of convergence
- $\oplus$ low kinematic level for the constraints
- $\oplus$ able to deal with finite accumulation
- $\ominus$ very low order of accuracy even in free flight motions

Nonsmooth event tracking methods (Event–driven methods)

- $\oplus$ high level integration of free flight motions
- $\ominus$ no proof of convergence
- $\ominus$ sensibility to numerical thresholds
- $\ominus$ reformulation of constraints at higher kinematic levels.
- $\ominus$ unable to deal with finite accumulation

Objectives & means

Objectives

Design nonsmooth event capturing methods with

- same properties as standard methods (robustness, accumulation, ...)
- Higher resolution (ratio error/computational cost)
- Higher order of accuracy

Means

1. Adaptive time–step size and order strategies for standard methods
2. Mixed integrators with rough pre-detection of events
3. Splitting strategies
4. Ad hoc detection of discontinuity and order of discontinuity methods.

NonSmooth Multibody Systems (NSMBS)

General definition

$$\begin{cases} M(q)\dot{v} = F(t,q,v) + G(t,q)\lambda & (34a)\\ \dot{q} = v & (34b)\\ w = g(t,q,v) & (34c)\\ 0 \in S(w,\lambda,t) + T(w,\lambda,t) & (34d)\\ v^+ = \mathcal{F}(v^-,q,t) & (34e)\end{cases}$$

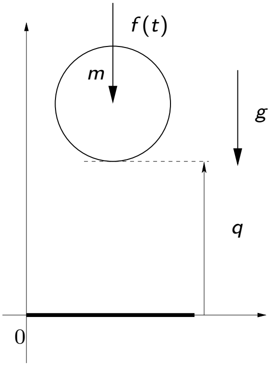
- $S : \mathbb{R}^{m\times m} \times \mathbb{R} \mapsto \mathbb{R}^{m\times m}$ continuously differentiable mapping
- $T : \mathbb{R}^{m\times m} \times \mathbb{R} \rightsquigarrow \mathbb{R}^{m\times m}$ multivalued mapping with a closed graph.

With scleronomous holonomic perfect unilateral constraints

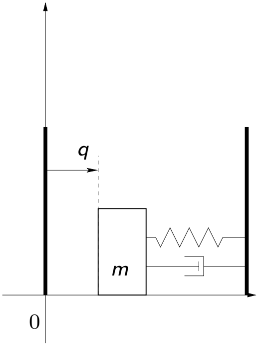
$$\begin{cases} M(q)\dot{v} = F(t,q,v) + G(q)\,\lambda\\ \dot{q} = v\\ 0 \leqslant y = g(q) \perp \lambda \geqslant 0\\ v^+ = \mathcal{F}(v^-,q,t)\end{cases} \tag{35}$$

where $G(q) = \nabla g(q)$

Academic examples I



(a) Bouncing ball example



(b) Linear Oscillator example

Figure: Academic test examples with analytical solutions

NonSmooth Multibody Systems (NSMBS)

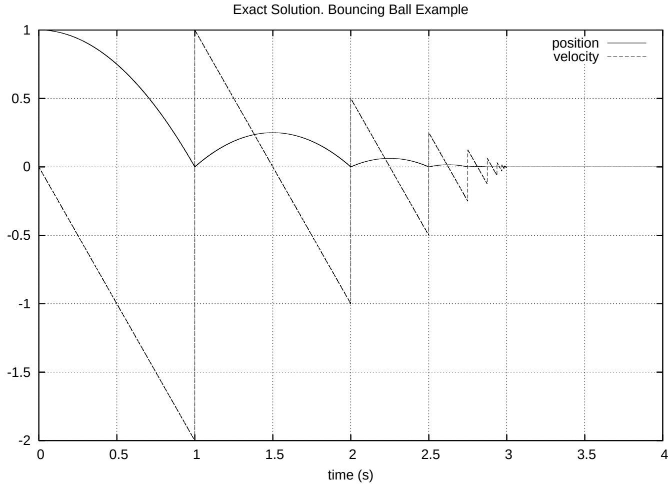


Figure: Analytical solutions. Bouncing ball example]

NonSmooth Multibody Systems (NSMBS)

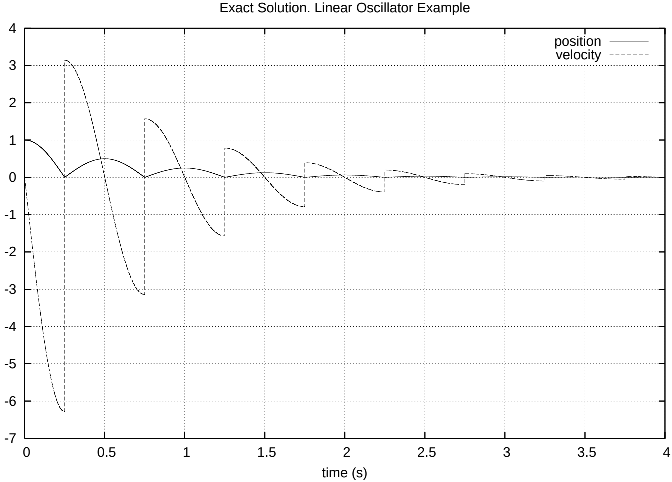
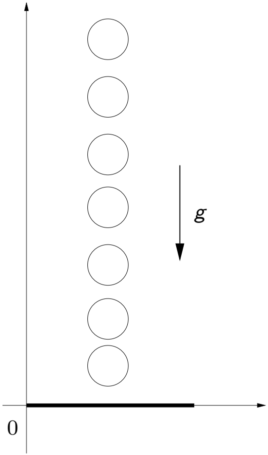


Figure: Analytical solutions. Linear Oscillator

NonSmooth Multibody Systems (NSMBS)

Academic examples II



(a) N Bouncing balls example

Figure: Academic test examples

Moreau's Time stepping scheme

Principle

$$\begin{cases} M(q_{k+\theta})(v_{k+1} - v_k) - h\tilde{F}_{k+\theta} = G(q_{k+\theta})P_{k+1}, & (36a) \\ q_{k+1} = q_k + hv_{k+\theta}, & (36b) \\ U_{k+1} = G^T(q_{k+\theta})\, v_{k+1} & (36c) \\ -P_{k+1} \in \partial\psi_{T_{\mathbb{R}^m_+}(\tilde{y}_{k+\gamma})}(U_{k+1} + eU_k), & (36d) \\ \tilde{y}_{k+\gamma} = y_k + h\gamma U_k, \quad \gamma \in [0,1]. & (36e) \end{cases}$$

with $\theta \in [0,1], \gamma \geqslant 0$ and $x_{k+\alpha} = (1-\alpha)x_{k+1} + \alpha x_k$ and $\tilde{y}_{k+\gamma}$ is a prediction of the constraints.

Properties

- Convergence results for one constraints
- Convergence results for multiple constraints problems with acute kinetic angles
- No theoretical proofs of order

Schatzman–Paoli's Time stepping scheme

Principle

$$\begin{cases} M(q_k+1)(q_{k+1}-2q_k+q_{k-1}) - h^2 F(t_{k+\theta}, q_{k+\theta}, v_{k+\theta}) = p_{k+1}, & (37a) \\ v_{k+1} = \dfrac{q_{k+1}-q_{k-1}}{2h}, & (37b) \\ -p_{k+1} \in N_K\left(\dfrac{q_{k+1}+eq_{k-1}}{1+e}\right), & (37c) \end{cases}$$

where N_K defined the normal cone to K.
For $K = \{q \in \mathbb{R}^n, y = g(q) \geqslant 0\}$

$$0 \leqslant g\left(\frac{q_{k+1}+eq_{k-1}}{1+e}\right) \perp \nabla g\left(\frac{q_{k+1}+eq_{k-1}}{1+e}\right) P_{k+1} \geqslant 0 \qquad (38)$$

Properties

- Convergence results for one constraints
- Convergence results for multiple constraints problems with acute kinetic angles
- No theoretical proof of order

Measuring error and convergence

Convergence in the sense of filled-in graph (Moreau (1978))

$$gr^\star(f) = \{(t,x) \in [0,T] \times \mathbb{R}^n, 0 \leqslant t \leqslant T \text{ and } x \in [f(t^-), f(t^+)]\}. \quad (39)$$

Such graphs are closed bounded subsets of $[0,T] \times \mathbb{R}^n$, hence, we can use the Hausdorff distance between two such sets with a suitable metric:

$$d((t,x),(s,y)) = \max\{|t-s|, \|x-y\|\}. \quad (40)$$

Defining the excess of separation between two graphs by

$$e(gr^\star(f), gr^\star(g)) = \sup_{(t,x)\in gr^\star(f)} \inf_{(s,y)\in gr^\star(g)} d((t,x),(s,y)), \quad (41)$$

the Hausdorff distance between two filled-in graphs $h^\star$ is defined by

$$h^\star(gr^\star(f), gr^\star(g)) = \max\{e(gr^\star(f), gr^\star(g)), e(gr^\star(g), gr^\star(f))\}. \quad (42)$$

Measuring error and convergence

An equivalent grid-function norm to the function norm in $\mathcal{L}_1$

$$\|e\|_1 = h \sum_{i=0}^{N} |f_i - f(t_i)| \tag{43}$$

In the same way, the $p - norm$ can be defined by

$$\|e\|_p = \left(h \sum_{i=0}^{N} |f_i - f(t_i)|^p \right)^{1/p} \tag{44}$$

The computation of this two last norm is easier to implement for piecewise continuous analytical function than the Hausdorff distance.

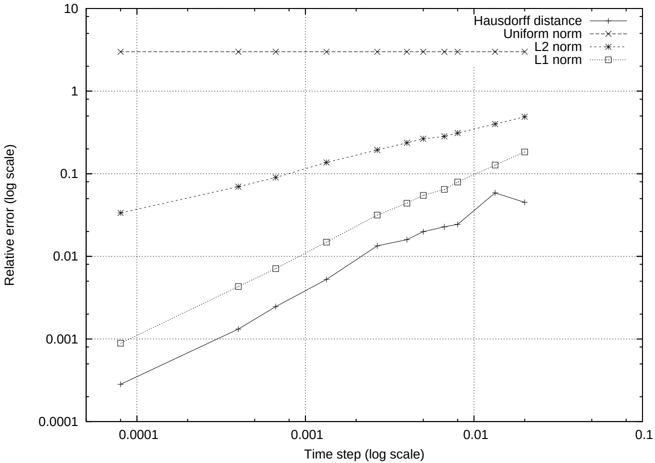
Global order of convergence.

Definition

A one-step time–integration scheme is of order q for a given norm $\|\cdot\|$ if there exists a constant C such that

$$\|e\| = Ch^q + \mathcal{O}(h^{q+1}) \tag{45}$$

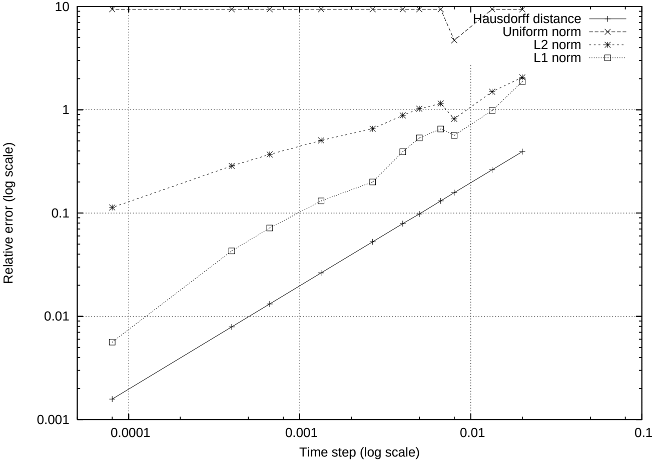
Empirical order of convergence. Moreau's time–stepping scheme



(a) The bouncing ball example

Figure: Empirical order of convergence of the Moreau's time-stepping scheme.

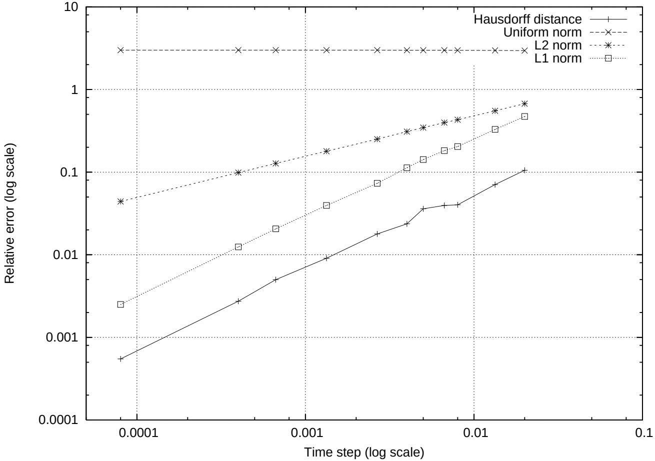
Empirical order of convergence. Moreau's time–stepping scheme



(a) The linear oscillator example

Figure: Empirical order of convergence of the Moreau's time-stepping scheme.

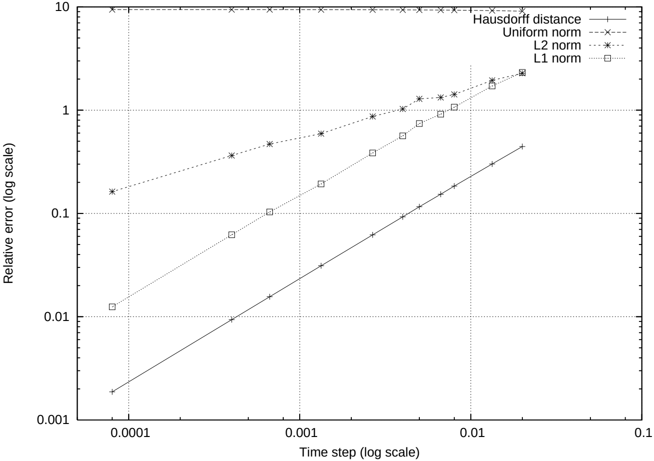
Empirical order of convergence. Schatzman–Paoli's time–stepping scheme



(a) The bouncing ball example

Figure: Empirical order of convergence of the Schatzman-Paoli's time-stepping scheme.

Empirical order of convergence. Schatzman–Paoli's time–stepping scheme



(a) The linear oscillator example

Figure: Empirical order of convergence of the Schatzman-Paoli's time-stepping scheme.

Local error estimates for the Moreau's time–stepping

Notation

$$e = x(t_k + h) - x_{k+1} = \left[\begin{array}{c} e_v \\ e_q \end{array} \right] = \left[\begin{array}{c} v^+(t_k + h) - v_{k+1} \\ q(t_k + h) - q_{k+1} \end{array} \right] \quad (46)$$

One impact at time $t_* \in (t_k, t_{k+1}]$

Assumption

$$di = p\delta_{t_*}, \quad \text{or equivalently} \quad dI = P\delta_{t_*}, \text{with } P = G(t_*)p. \tag{47}$$

Notation

$$\mathcal{I} = \{\alpha, \alpha \in \{1..m\}\} \tag{48}$$

$$\mathcal{I}_* = \{\alpha \in \mathcal{I}, P^\alpha \geqslant 0, U^{\alpha,+}(t_*) - U^{\alpha,-}(t_*) = -(1+e)U^{\alpha,-}(t_*)\} \tag{49}$$

$$\mathcal{I}_p = \{\alpha \in \mathcal{I}, P^\alpha_{k+1} \geqslant 0, U^\alpha_{k+1} - U^\alpha_k = -(1+e)U^\alpha_k\} \tag{50}$$

Lemma

Let us assume that we have only one elastic impact at time $t_* \in (t_k, t_{k+1}]$ without persistent contact, i.e. , $di = p\delta_{t_*}$.

1. *If $\mathcal{I}_* = \mathcal{I}_p$, then the local order of consistency of the scheme is given by*

$$\begin{aligned} e_v &= K_v h + \mathcal{O}(h^2) \\ e_q &= K_q h + \mathcal{O}(h^2) \end{aligned} \tag{51}$$

2. *If $\mathcal{I}_* \neq \mathcal{I}_p$, then the local order of consistency of the scheme is given by*

$$\begin{aligned} e_v &= K_v + \mathcal{O}(h) \\ e_q &= K_q h + \mathcal{O}(h^2) \end{aligned} \tag{52}$$

Smooth Lagrange multiplier in persistent contact without impact in $(t_k, t_{k+1}]$

Assumption

$$di = \lambda(t)dt, \tag{53}$$

or equivalently

$$dI = \Lambda(t)dt, \text{with } \Lambda(t) = G(t)\lambda(t). \tag{54}$$

Notation

$$\mathcal{I}_\Lambda(t) = \{\alpha \in \mathcal{I}, \Lambda^\alpha(t) \geqslant 0, U^{\alpha,+}(t) = U^{\alpha,-}(t) = 0\} \tag{55}$$

$$\mathcal{I}_{\Lambda,k+1} = \{\alpha \in \mathcal{I}, \Lambda^\alpha_{k+1} \geqslant 0, U^\alpha_{k+1} = U^\alpha_k = 0\} \tag{56}$$

Lemma

Assuming that $\mathcal{I}_\Lambda(t) = \mathcal{I}_{\Lambda,k+1}$ for all $t \in (t_k, t_{k+1}]$. The local order of consistency of the scheme is one that is

$$\begin{array}{rcl} e_v & = & Kh^2 + \mathcal{O}(h^3) \\ e_q & = & K_q h^2 + \mathcal{O}(h^3) \end{array} \tag{57}$$

Local error estimates for the Moreau's time–stepping

Other cases

- *One impact and smooth Lagrange multiplier* The same result holds as in the first Lemma.
- *losing contact event (take–off) without impact* The order of the time–integration scheme depends on the regularity of the contact forces (at least continuous).
- *Finite accumulation* The order of the time–integration should be at least 0. Idea of the proof : use the fact that the velocity vanishes and is of bounded variations

Practical error estimates for the Moreau's time–stepping

Order "0" case

Standard error estimates do not apply for Order 0.
We propose to extend it to the order 0 of consistency by assuming that the constant can be evaluated by

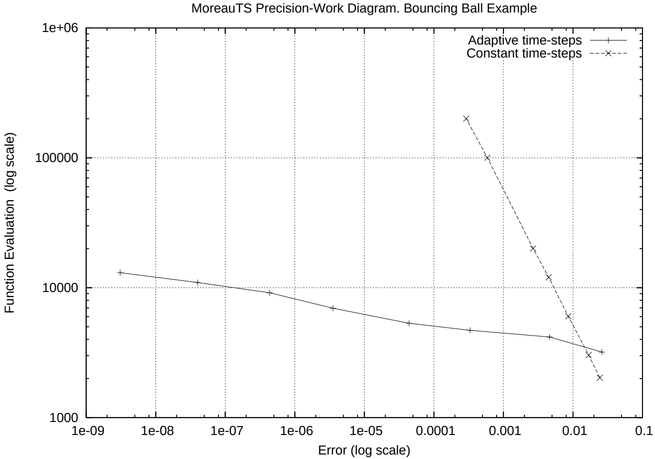
$$C = \frac{2(e_1 - e_{1/2})}{h} \tag{58}$$

and the local error estimate by

$$e_{1/2} = 2(x_{1/2} - x_1) + \mathcal{O}(h^2) \tag{59}$$

The adaptive time–step control exposed for smooth ODE is then apply directly.

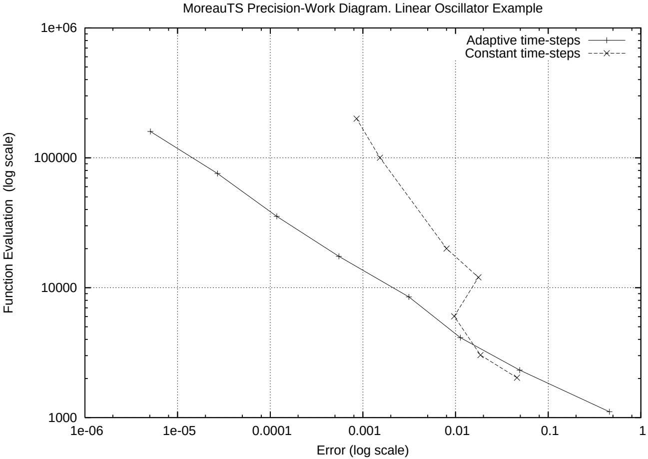
Order "0" time–step adjustment for the Moreau's time–stepping



(a) The bouncing ball example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 0

Order “0” time–step adjustment for the Moreau’s time–stepping



(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau’s time-stepping scheme. Order 0

Sizing the error in the violation of constraints

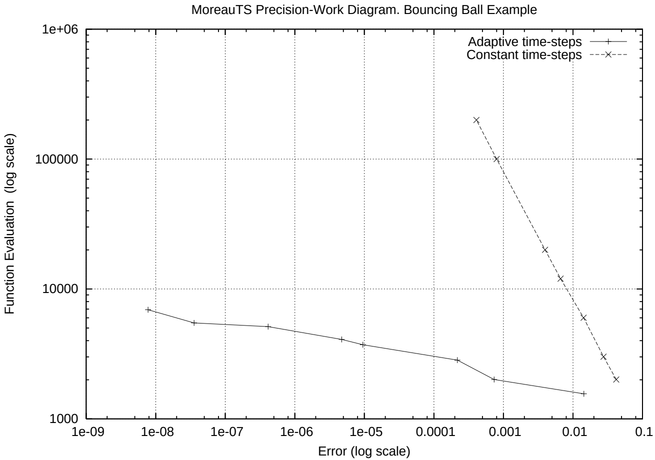
The violation of constraints is sized by the following rule:

$$e_{\mathrm{violation}} = \| \min(0, g(q)) \circ invtol \|_{\infty} \tag{60}$$

Assuming that the scheme is of order 1 almost everywhere in smooth phase and may be controlled by $e_{\mathrm{violation}}$ when an nonsmooth vent occurs, the step size adjustment is implemented by the means of the following error estimation

$$\mathrm{error} = \max(e_{\mathrm{violation}}, \|e_k \circ invtol\|_{\infty}) \tag{61}$$

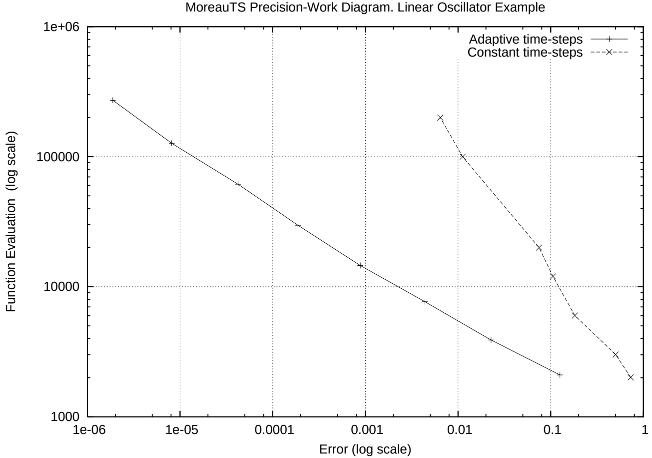
Results on two academic test examples



(a) The bouncing ball example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 0 + violation error

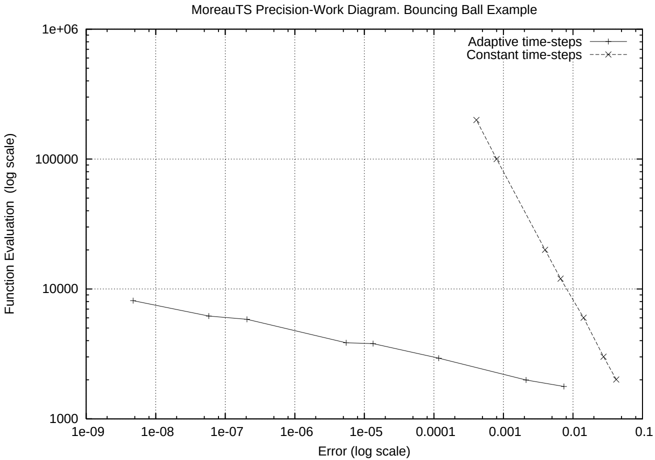
Results on two academic test examples



(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 0 + violation error

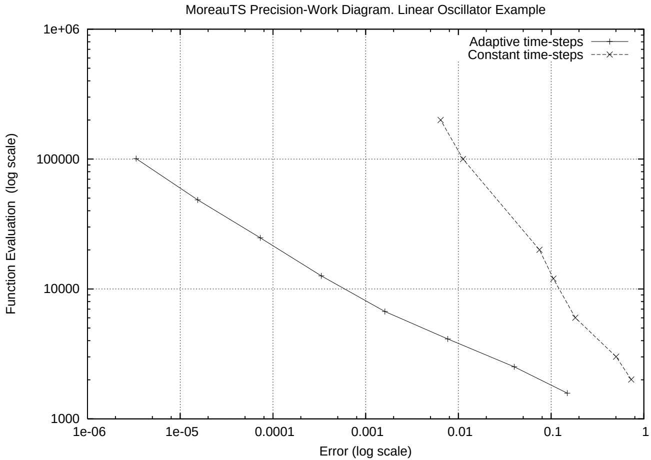
Results on two academic test examples



(a) The bouncing ball example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 1 + violation error

Results on two academic test examples



(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau's time-stepping scheme. Order 1 + violation error

Higher Order Time–stepping schemes

Background

Work of Mannshardt (1978) on time–integration schemes of any order for ODEs with discontinuities (with tranversality assumption)

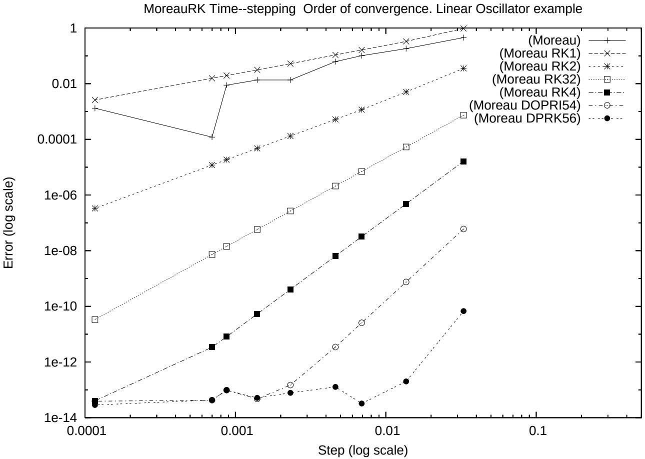
Principle

- Let us assume only one event per time–step at instants t_*.
- Choose any ODE solvers of order p
- Perform a rough location of the event inside the time step of length h
  Find an interval $[t_a, t_b]$ such that

  $$t_* \in [t_a, t_b] \text{ and } |t_b - t_a| = Ch^{p+1} + \mathcal{O}(h^{p+2}) \tag{62}$$

  Dichotomy, Newton, Local Interpolants, Dense output,...
- Perform an integration on $[t_k, t_a]$ with the ODE solver of order p
- Perform an integration on $[t_a, t_b]$ with Moreau's time–stepping scheme
- Perform an integration on $[t_b, t_{k+1}]$ with the ODE solver of order p

Results on the linear oscillator



(a) The linear oscillator example

Figure: Precision Work diagram for the Moreau's time-stepping scheme.

Higher Order Time–stepping schemes

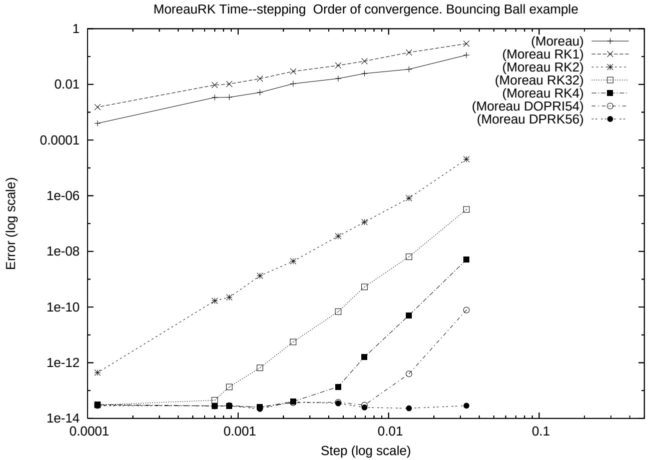
Finite accumulation

- Repeat the whole process on the remaining part of the interval $[t_b, t_k]$
- By induction, repeat this process up to the end of the original time step.

Vincent Acary

vincent.acary@inrialpes.fr
INRIA Rhône–Alpes

Results on the Bouncing Ball



(a) The Bouncing Ball example

Figure: Precision Work diagram for the Moreau's time-stepping scheme.

Splitting–based methods.

Principle for smooth ODEs

Let us consider a smooth ODE which can be written as

$$\dot{x}(t) = f(x,t) + g(x,t) \tag{63}$$

A example of splitting–based method is given by the following procedure

1. Perform the integration of f on $[t_k, t_{k+1}]$ to obtain $\tilde{x}(t_{k+1})$ that is

$$\tilde{x}(t_{k+1}) = x(t_k) + \int_{t_k}^{t_{k+1}} f(x,t)\,dt \tag{64}$$

2. Perform the integration of g on $[t_k, t_{k+1}]$ with initial value $\tilde{x}(t_{k+1})$ to obtain $\hat{x}(t_{k+1})$ that is

$$\hat{x}(t_{k+1}) \quad = \quad \tilde{x}(t_{k+1}) + \int_{t_k}^{t_{k+1}} g(x,t)\,dt \tag{65}$$

Properties

- $x(t_k+1) \neq \hat{x}(t_{k+1})$ is the general case. (except special linear case, constant dynamics, ...)
- $\hat{x}_(t_{k+1}) \rightarrow x(t_{k+1})$ when $t_{k+1} \rightarrow t_k$

Splitting–based methods.

Splitting–based for Moreau scheme without continuous contact forces

- The first part is

$$\begin{cases} M(q)\dot{v} = F(t,q,v), \\ \dot{q} = v, \\ q(t_k) = q_k, \qquad v(t_k) = v_k \end{cases} \tag{66}$$

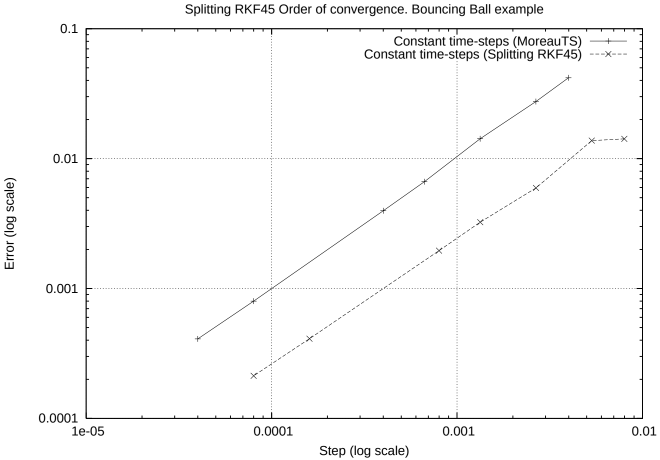
yielding to the approximations $q_1 = q(t_{k+1})$ and $v_1 = v(t_{k+1})$ which can integrated by any smooth ODE solvers.

- The second one is given by

$$\begin{cases} M(q)\dot{v} = G(q)\lambda, \\ \dot{q} = 0, \\ y = g(q) \\ -\lambda \in \partial\psi_{T_{\mathbb{R}_+}(y)}(\dot{y}(t^+) + e\dot{y}(t^-)) \\ q(t_k) = q_1; v(t_k) = v_1; \end{cases} \tag{67}$$

and leads to the approximation $q_{k+1} = q(t_{k+1})$ and$q_{k+1} = q(t_{k+1})$.

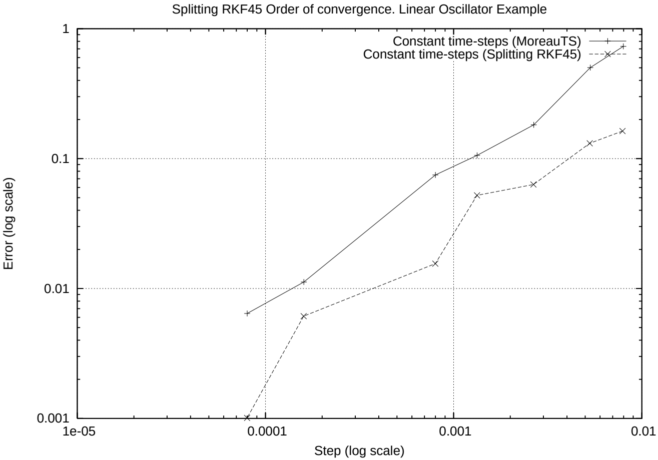
Splitting–based methods with constants time–step.



(a) The bouncing ball example

Figure: Empirical order of convergence of the Splitting RKF45 time-stepping scheme

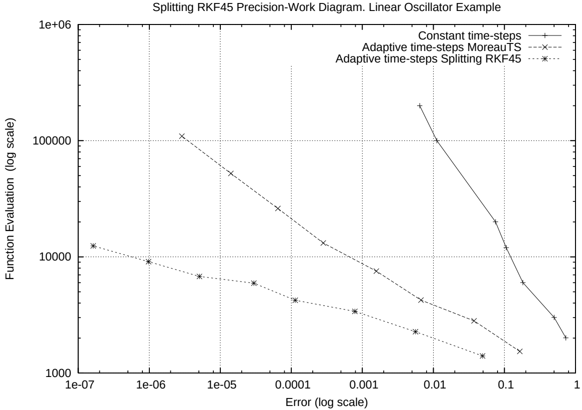
Splitting–based methods with constants time–step.



(a) The linear oscillator example

Figure: Empirical order of convergence of the Splitting RKF45 time-stepping scheme

Splitting–based methods with adaptive time–step.



(a) The linear oscillator example

Figure: Empirical order of convergence of the Splitting RKF45 time-stepping scheme

Vincent Acary

vincent.acary@inrialpes.fr
INRIA Rhône–Alpes

Splitting–based methods.

Splitting–based for Moreau scheme with continuous contact forces

- The first part is

$$\begin{cases} M(q)\dot v = F(t,q,v) + r(t), \\ \dot q = v, \\ y = g(q) \\ -r(t) \in \partial\psi_{T_{\mathbb{R}_+}(y)}(\dot y(t)) \\ q(t_k) = q_k, \qquad v(t_k) = v_k \end{cases} \tag{68}$$

  yielding to the approximations $q_1 = q(t_{k+1})$ and $v_1 = v(t_{k+1})$ which can integrated by any smooth ODE solvers.

- The second one is given by

$$\begin{cases} M(q)\dot v = G(q)\lambda, \\ \dot q = 0, \\ y = g(q) \\ -\lambda \in \partial\psi_{T_{\mathbb{R}_+}(y)}(\dot y(t^+) + e\dot y(t^-)) \\ q(t_k) = q_1; v(t_k) = v_1; \end{cases} \tag{69}$$

  and leads to the approximation $q_{k+1} = q(t_{k+1})$ and$q_{k+1} = q(t_{k+1})$.

Conclusions

Adaptive time–step strategies

- Higher resolution schemes
- Work with finite accumulation of events

Higher order schemes

- Schemes of any orders
- Work with finite accumulation of events

Splitting based methods

- Higher resolution schemes
- Work with finite accumulation of events

Vincent Acary

vincent.acary@inrialpes.fr
INRIA Rhône–Alpes

Perspectives

- Theoretical works on orders and practical error estimations
- Adaptive time–step strategies on the higher order time–stepping schemes.
- Improve the pre–detection process of the event and the order of discontinuity
- Test on nonsmooth and nonlinear mechanical systems.
- Adapt the schemes with a step without external forces when the Moreau's scheme is used
- Other types of time–stepping schemes . . .

Vincent Acary

vincent.acary@inrialpes.fr
INRIA Rhône–Alpes

Overview of the Siconos Platform

Context

The Siconos Platform is one of the main outcome of the Siconos EU project.

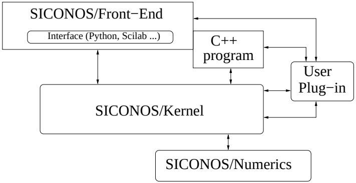
Functionalities

Modeling, simulation, (analysis and control) of Non Smooth Dynamical Systems.

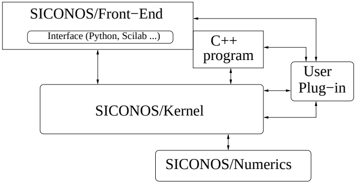
Constraints and Requirements

- various applications fields (Mechanics, Electronics . . .) and corresponding modeling habits and formulations
- various mathematical and numerical tools
- various skills in computer science (from the high perfomance computing to the Matlab users)
- links and interfaces with existing softwares:
  - low-level numerical libraries (BLAS, LAPACK, ODEPACK, . . .)
  - Matlab or Scilab dedicated user toolbox
  - simulation tools for an application field: Scicos, Simulink, FEM and DEM Sofware (LMGC90, . . .), Hybrid Modeling Language (Modelica, . . .)

Siconos components diagram



Siconos components diagram



- *SICONOS/Numerics API C:*
  shared dynamic library that provides low-level solvers and algorithms in C and fortran.
  Sources: NSSpack (LCP, Friction ...), odepack (Lsodar ...).
- *SICONOS/Kernel:* API C++: compiled command files with high level methods (C++ Constructors and/or XML file data loading.)
  $\Rightarrow$ from $simulation \rightarrow run()$ to $DynamicalSystem \rightarrow computeFext(t)$
- *SICONOS/Frond-End*: "user-friendly" interface providing a more interactive way of using the platform.
  - API C++ with interactive environment Python scripting (Swig wrapper).
  - API C: Scilab and Matlab interfaces.

Modeling Principle:

Non Smooth Dynamical system

Dynamical system

$$\dot{x} = f(x, t) + r$$

Modeling Principle:

Non Smooth Dynamical system

Dynamical system

$$\dot{x} = f(x,t) + r$$

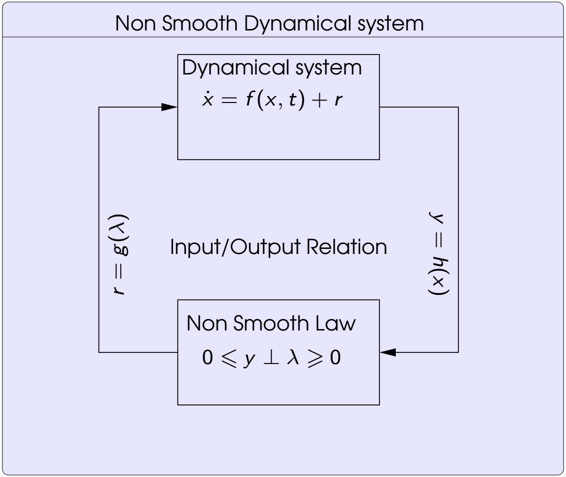
Non Smooth Law

$$0 \leqslant y \perp \lambda \geqslant 0$$

Vincent Acary

vincent.acary@inrialpes.fr
INRIA Rhône–Alpes

Modeling Principle:



Vincent Acary

vincent.acary@inrialpes.fr
INRIA Rhône–Alpes

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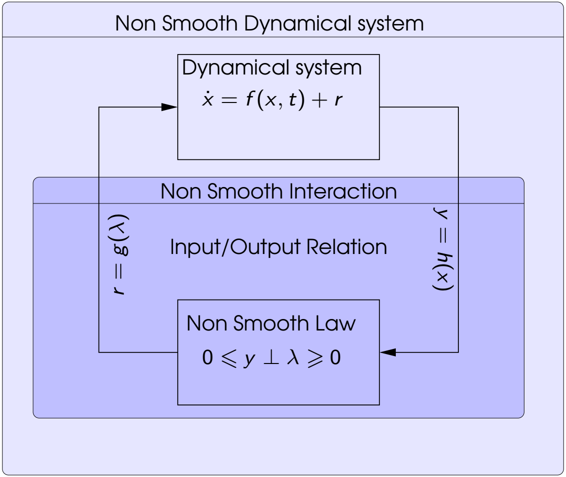
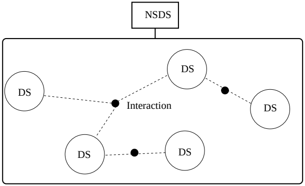


Kernel Modeling Part

Siconos Non Smooth Dynamical System:



- *Dynamical System*: a set of ODEs
- *Interaction*: a set of relations (ie constraints) and a non-smooth law

Kernel Modeling Part

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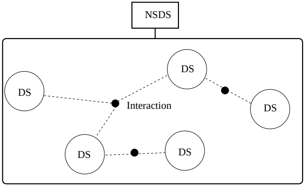
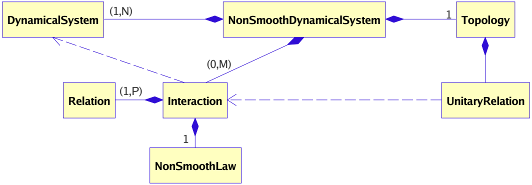
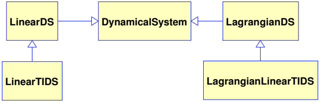


- *Dynamical System*: a set of ODEs
- *Interaction*: a set of relations (ie constraints) and a non-smooth law
- *Topology*: link with the simulation, handles relative degrees, index sets ...

Simplified Modeling Tools class diagram:



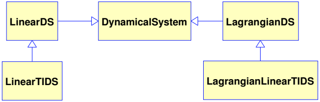
Dynamical Systems in Siconos/Kernel



- Parent Class **DynamicalSystem**

$$\dot{x} = f(x, \dot{x}, t) + T(x)u(x, t) + r$$

Dynamical Systems in Siconos/Kernel



- Parent Class **DynamicalSystem**

$$\dot{x} = f(x, \dot{x}, t) + T(x)u(x, t) + r$$

- Derived Classes
  - **LinearDS** Linear Dynamical Systems

  $$\dot{x} = A(t)x + Tu(t) + b(t) + r$$

  - **LagrangianDS** Lagrangian Dynamical Systems

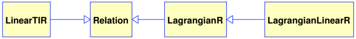
  $$M(q)\ddot{q} + NNL(q, \dot{q}) + F_{int}(\dot{q}, q, t) = F_{ext}(t) + T(q)u(q, t) + p$$

  - **LagrangianLinearTIDS** Lagrangian Linear Time Invariant Systems

  $$M\ddot{q} + C\dot{q} + Kq = F_{ext}(t) + Tu(t) + p$$

Note: all operators ($f(x, t)$, $M(q)$, ...) can be set either as matrices (when constant) or with a user-defined external function (plug-in).

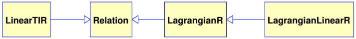
Relations



- Parent Class **Relation**

$$y = h(x, t, \ldots) \quad , \quad r = g(\lambda, t, \ldots)$$

Relations



- Parent Class **Relation**

$$y = h(x, t, ...) \quad , \quad r = g(\lambda, t, ...)$$

- Derived Classes:
  - **LinearTIR** Linear Time Invariant Relation

$$y = Cx + Fu + D\lambda + e, \quad r = B\lambda$$

  - **LagrangianR** Lagrangian Relation

$$\dot{y} = H(q, t, \ldots)\dot{q}, \quad p = H^t(q, t, \ldots)\lambda$$

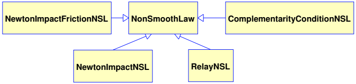
  - **LagrangianLinearR** Lagrangian Linear Relation

$$\dot{y} = H\dot{q} + b, \quad p = H^t\lambda$$

Vincent Acary

vincent.acary@inrialpes.fr
INRIA Rhône–Alpes

Non Smooth laws



- Parent Class **NonSmoothLaw**
- Derived Classes
  - **ComplementarityConditionNSL** Complementarity condition or unilateral contact

$$0 \leqslant y \perp \lambda \geqslant 0$$

  - **Relay** condition.

$$\begin{cases} \dot{y} = 0, |\lambda| \leqslant 1 \\ \dot{y} \neq 0, \lambda = \mathrm{sign}(y) \end{cases}$$

  - **NewtonImpactLawNSL** Newton impact Law.

$$\text{if } y(t) = 0, \quad 0 \leqslant \dot{y}(t^+) + e\dot{y}(t^-) \perp \lambda \geqslant 0$$

  - **NewtonImpactFrictionNSL** Newton impact and Friction (Coulomb) Law.

C++ description of a Model

- Dynamical Systems definition:

```
DynamicalSystem * DS1 = new
LagrangianLinearTIDS(nDof,q0,v0,Mass);
DS1→setComputeFExtFunction("BallPlugin.so", "ballFExt");
```


vincent.acary@inrialpes.fr
INRIA Rhône–Alpes

C++ description of a Model

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```
DynamicalSystem * DS1 = new
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DS1→setComputeFExtFunction("BallPlugin.so", "ballFExt");
```

- Interactions definition: non smooth law and relation:

```
NonSmoothLaw * nslaw = new NewtonImpactNSL(e);
Relation * relation = new LagrangianLinearR(H,b);
Interaction * inter = new Interaction(name, listOfDS,dim, nslaw,
relation);
```


vincent.acary@inrialpes.fr
INRIA Rhône–Alpes

C++ description of a Model

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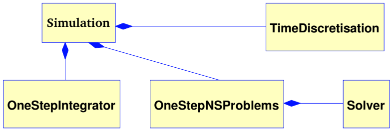
relation);
```

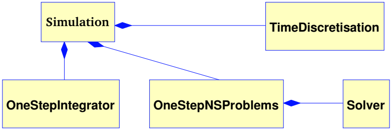
- Non Smooth Dynamical System and Model

```
NonSmoothDynamicalSystem * nsds = new
NonSmoothDynamicalSystem(allDS, allInteractions);
Model * theModel = new Model(t0,T);
theModel→setNonSmoothDynamicalSystemPtr(nsds);
```

Simulation tools in Siconos/Kernel



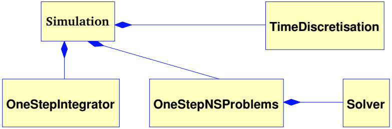
Simulation tools in Siconos/Kernel



Simulation description in C++ input file:

```
Simulation* s = new TimeStepping(theModel);
TimeDiscretisation * t = new TimeDiscretisation(timeStep,s);
OneStepIntegrator * OSI = new Moreau(listOfDS,theta,s);
OneStepNSProblem * osnspb = new LCP(s, "LCP",Lemke,parameters);
```

Simulation tools in Siconos/Kernel



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```
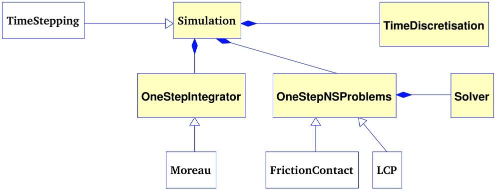

Unitary Relation and Index Sets

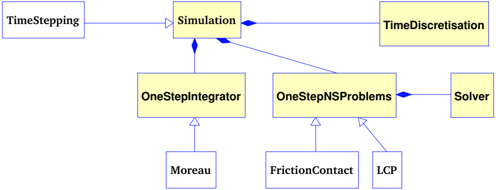
UR: $y^i = h(q, \ldots)$.
Index Sets: set of Unitary Relations (UR).

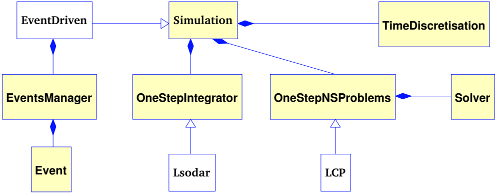
- $I_0 = \{UR_\alpha\}$ all unilateral constraints in the system, ie all the potential interactions/relations of the systems.
- $I_i = \{UR_\alpha, \alpha \in I_{i-1}, y^{(i-1)} = 0\} \subset I_{i-1}$

Simulation tools in Siconos/Kernel



Simulation tools in Siconos/Kernel





OneStepIntegrator:

- **Moreau**: Moreau's Time-stepping integrator
- **Lsodar**: Numerical integration scheme based on the Livermore Solver for Ordinary Differential Equations with root finding.

OnestepNSproblem: Numerical one step non smooth problem formulation and solver.

- **LCP** Linear Complementarity Problem

$$\begin{cases} w = Mz + q \\ 0 \leqslant w \perp z \geqslant 0 \end{cases}$$

- **FrictionContact2D(3D)** Two(three)-dimensional contact friction problem
- **QP** Quadratic programming problem

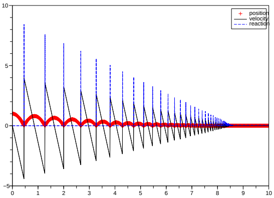
$$\begin{cases} \min \frac{1}{2} z^T Q z + z^T p \\ z \geqslant 0 \end{cases}$$

- **Relay**

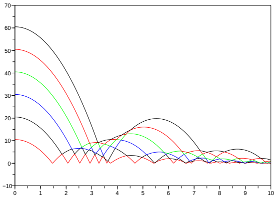
vincent.acary@inrialpes.fr
INRIA Rhône–Alpes

Model: Lagrangian Linear Time Invariant Dynamical Systems with Lagrangian Linear Relations, Newton Impact Law.
Simulation: Moreau's Time Stepping or Event Driven.

Bouncing Ball

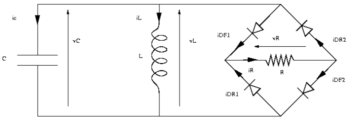


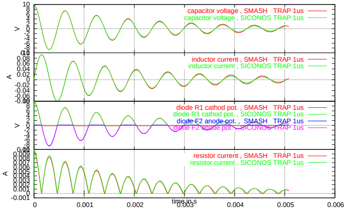
Beads column



A 4 diodes bridge wave rectifier.

Model: Linear Dynamical System with Linear Relations, Complementarity Condition Non Smooth Law.
Simulation: Moreau's Time Stepping

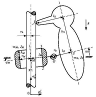


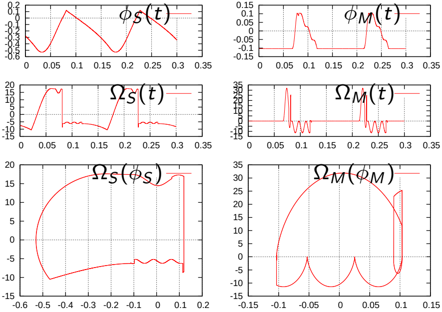


Comparison between the SICONOS Platform (Non Smooth LCS model) and SPICE simulator (Smooth Diode model).

Woodpecker toy (sample from Michael Moeller (CR10))

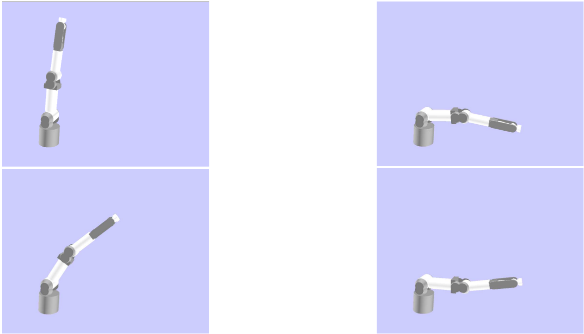
Model: Lagrangian Linear Dynamical System, Lagrangian Linear Relations, Newton impact-friction law.
Simulation: Moreau's Time Stepping





A Robotic Arm (Pa10)

Model: Lagrangian Non Linear Dynamical System with Lagrangian Non Linear Relations, Newton impact.
Simulation: Moreau's Time Stepping



Help and Documentation

- Doxygen tools for automatic documentation in Numerics and Kernel
- Users, developers and theoretical manuals (in progress ...)
- Web pages, Bug tracker, forum ... on Gforge.
- Samples library as templates.

Diffusion

- The SICONOS platform is distributed under GPL licence.
- Visit the Gforge Web site for
  - Documentations
  - Mailing lists
  - Downloads
  - Bug tracker
  - Contributing, ...

  http://gforge.inria.fr/projects/siconos/

vincent.acary@inrialpes.fr
INRIA Rhône–Alpes

Introduction & Motivations

Lagrangian dynamical systems with unilateral constraints and friction

Numerical time–integration schemes

Adaptive time-step size time–stepping scheme

Time–stepping schemes of any order

Splitting based Time–stepping schemes

Conclusions & Perspectives

The Siconos Platform

Thank you for your attention.

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