Formulations and extensive comparisons of 3D frictional contact solvers based on performance profiles

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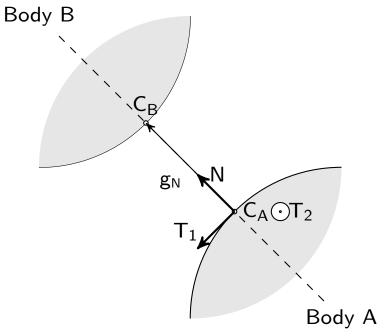
Signorini's condition and Coulomb's friction



- gap function $g_N = (C_B - C_A)N$.
- reaction forces velocities

$$r = r_N N + r_T, \quad \text{with } r_N \in \mathbf{R} \text{ and } r_T \in \mathbf{R}^2.$$

$$u = u_N N + u_T, \quad \text{with } u_N \in \mathbf{R} \text{ and } u_T \in \mathbf{R}^2.$$

- Signorini conditions

$$\text{position level} : 0 \leqslant g_N \perp r_N \geqslant 0.$$

$$\text{velocity level} : \begin{cases} 0 \leqslant u_N \perp r_N \geqslant 0 & \text{if } g_N \leqslant 0 \\ r_N = 0 & \text{otherwise.} \end{cases}$$

Signorini's condition and Coulomb's friction

Modeling assumption

Let μ be the coefficient of friction. Let us define the Coulomb friction cone K which is chosen as the isotropic second order cone

$$K = \{r \in \mathbb{R}^3 \mid \|r_{\mathrm{T}}\| \leqslant \mu r_n\}. \tag{1}$$

The Coulomb friction states

- for the sticking case that

$$u_{\mathrm{T}} = 0, \quad r \in K \tag{2}$$

- and for the sliding case that

$$u_{\mathrm{T}} \neq 0, \quad r \in \partial K, \exists\, \alpha > 0, r_{\mathrm{T}} = -\alpha u_{\mathrm{T}}. \tag{3}$$

Disjunctive formulation of the frictional contact behavior

$$\begin{cases} r = 0 & \text{if } g_{\mathrm{N}} > 0 \quad \text{(no contact)} \\ r = 0, u_{\mathrm{N}} \geqslant 0 & \text{if } g_{\mathrm{N}} \leqslant 0 \quad \text{(take–off)} \\ r \in K, u = 0 & \text{if } g_{\mathrm{N}} \leqslant 0 \quad \text{(sticking)} \\ r \in \partial K, u_{\mathrm{N}} = 0, \exists\, \alpha > 0, u_{\mathrm{T}} = -\alpha r_{\mathrm{T}} & \text{if } g_{\mathrm{N}} \leqslant 0 \quad \text{(sliding)} \end{cases} \tag{4}$$

Signorini's condition and Coulomb's friction

Second Order Cone Complementarity (SOCCP) formulation [De Saxcé(1992)]

- Modified relative velocity $\hat{u} \in \boldsymbol{R}^3$ defined by

$$\hat{u} = u + \mu \|u_\mathrm{T}\| \mathrm{N}. \tag{5}$$

- Second-Order Cone Complementarity Problem (SOCCP)

$$K^\star \ni \hat{u} \perp r \in K \tag{6}$$

if $g_\mathrm{N} \leqslant 0$ and $r = 0$ otherwise. The set $K^\star$ is the dual convex cone to K defined by

$$K^\star = \{u \in \boldsymbol{R}^3 \mid r^\top u \geqslant 0, \quad \text{for all } r \in K\}. \tag{7}$$

Signorini's condition and Coulomb's friction

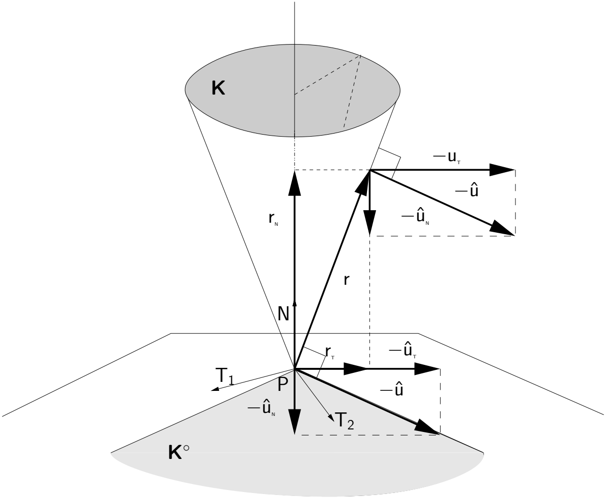


Figure: Coulomb's friction and the modified velocity $\hat{u}$. The sliding case.

3D frictional contact problem

Multiple contact notation

For each contact $\alpha \in \{1, \dots n_c\}$, we have

- the local velocity : $u^\alpha \in \boldsymbol{R}^3$, and

$$u = [[u^\alpha]^\top, \alpha = 1 \dots n_c]^\top$$

- the local reaction vector $r^\alpha \in \boldsymbol{R}^3$

$$r = [[r^\alpha]^\top, \alpha = 1 \dots n_c]^\top$$

- the local Coulomb cone

$$K^\alpha = \{r^\alpha, \|r^\alpha_{\mathrm{T}}\| \leqslant \mu^\alpha |r^\alpha_{\mathrm{N}}|\} \subset \boldsymbol{R}^3$$

and the set K is the cartesian product of Coulomb's friction cone at each contact, that

$$K = \prod_{\alpha=1\dots n_c} K^\alpha \tag{8}$$

and $K^\star$ is dual.

3D frictional contact problems

Problem 1 (General discrete frictional contact problem)

Given

- *a symmetric positive definite matrix* $M \in \mathbb{R}^{n \times n}$,
- *a vector* $f \in \mathbb{R}^n$,
- *a matrix* $H \in \mathbb{R}^{n \times m}$,
- *a vector* $w \in \boldsymbol{R}^m$,
- *a vector of coefficients of friction* $\mu \in \boldsymbol{R}^{n_c}$,

find three vectors $v \in \mathbb{R}^n$, $u \in \boldsymbol{R}^m$ *and* $r \in \boldsymbol{R}^m$, *denoted by* $\mathrm{FC/I}(M, H, f, w, \mu)$ *such that*

$$\begin{cases} Mv = Hr + f \\ u = H^\top v + w \\ \hat{u} = u + g(u) \\ K^\star \ni \hat{u} \perp r \in K \end{cases} \tag{9}$$

with $g(u) = [[\mu^\alpha \| u_T^\alpha \| \mathrm{N}^\alpha]^\top, \alpha = 1 \dots n_c]^\top$. □

3D frictional contact problems

Problem 2 (Reduced discrete frictional contact problem)

Given

- *a symmetric positive semi–definite matrix* $W \in \mathbb{R}^{m\times m}$,
- *a vector* $q \in \mathbb{R}^m$,
- *a vector* $\mu \in \boldsymbol{R}^{n_c}$ *of coefficients of friction,*

find two vectors $u \in \boldsymbol{R}^m$ *and* $r \in \boldsymbol{R}^m$, *denoted by* $\mathrm{FC/II}(W, q, \mu)$ *such that*

$$\begin{cases} u = Wr + q \\ \hat{u} = u + g(u) \\ K^\star \ni \hat{u} \perp r \in K \end{cases} \tag{10}$$

with $g(u) = [[\mu^\alpha \| u_T^\alpha \| \mathrm{N}^\alpha]^\top, \alpha = 1 \dots n_c]^\top$. □

Relation with the general problem

$W = H^\top M^{-1} H$ and $q = H^\top M^{-1} f + w$.

From the mathematical programming point of view

Nonmonotone and nonsmooth problem

$$K^\star \ni Wr + q + g(Wr + q) \perp r \in K \tag{11}$$

Possible reformulation

- Variational inequality or normal cone inclusion

$$-(Wr + q + g(Wr + q)) \triangleq -F(r) \in N_K(r). \tag{12}$$

- Nonsmooth equations $G(r) = 0$
  - The natural map F^{nat} associated with the VI (12) $F^{\mathrm{nat}}(z) = z - P_X(z - F(z))$.
  - Variants of this map (Alart-Curnier formulation, ...)
  - one of the SOCCP-functions. (Fisher-Bursmeister function)
- and many other ...

VI based methods

Standard methods

- Basic fixed point iterations with projection [FP-VI]

$$z_{k+1} \leftarrow P_X(z_k - \rho_k F(z_k))$$

- Extragradient method [EG-VI]

$$z_{k+1} \leftarrow P_X(z_k - \rho_k F(P_X(z_k - \rho_k F(z_k))))$$

With fixed ρ, we get the Uzawa Algorithm of De Sacxé-Feng [FP-DS]

Self-adaptive procedure for ρ_k

[UPK]

$$\text{Armijo-like}: \; m_k \in \mathbb{N} \quad \text{such that} \begin{cases} \rho_k = \rho 2^{m_k}, \\ \rho_k \|F(z_k) - F(\bar{z}_k)\| \leqslant \|z_k - \bar{z}_k\| \end{cases}$$

Nonsmooth Equations based methods

Nonsmooth Newton on $G(z) = 0$

$$z_{k+1} = z_k - \Phi^{-1}(z_k)(G(z_k)), \qquad \Phi(z_k) \in \partial G(z_k)$$

- Alart–Curnier Formulation [Alart and Curnier(1991)] [NSN-AC]

$$\begin{cases} r_{\mathrm{N}} - P_{\mathbb{R}^{n_c}_+}(r_{\mathrm{N}} - \rho_{\mathrm{N}} u_{\mathrm{N}}) = 0, \\ r_{\mathrm{T}} - P_{D(\mu, r_{\mathrm{N},+} + \rho u_{\mathrm{N}})}(r_{\mathrm{T}} - \rho_{\mathrm{T}} u_{\mathrm{T}}) = 0, \end{cases}$$

- Jean–Moreau Formulation [NSN-MJ]

$$\begin{cases} r_{\mathrm{N}} - P_{\mathbb{R}^{n_c}_+}(r_{\mathrm{N}} - \rho_{\mathrm{N}} u_{\mathrm{N}}) = 0, \\ r_{\mathrm{T}} - P_{D(\mu, r_{\mathrm{N},+})}(r_{\mathrm{T}} - \rho_{\mathrm{T}} u_{\mathrm{T}}) = 0, \end{cases}$$

- Direct normal map reformulation [NSN-NM]

$$r - P_K\,(r - \rho(u + g(u))) = 0$$

- Extension of Fischer-Burmeister function to SOCCP [NSN-FB]

$$\phi_{\mathrm{FB}}(x, y) = x + y - (x^2 + y^2)^{1/2}$$

Matrix block-splitting and projection based algorithms [Moreau(1994), Jean and Touzot(1988)]

Block splitting algorithm with $W^{\alpha\alpha} \in \mathbb{R}^3$ [NSGS-*]

$$\begin{cases} u^{\alpha}_{i+1} - W^{\alpha\alpha} P^{\alpha}_{i+1} = q^{\alpha} + \sum_{\beta<\alpha} W^{\alpha\beta} r^{\beta}_{i+1} + \sum_{\beta>\alpha} W^{\alpha\beta} r^{\beta}_{i} \\ \widehat{u}^{\alpha}_{i+1} = \left[u^{\alpha}_{\mathrm{N},i+1} + \mu^{\alpha} \, ||u^{\alpha}_{\mathrm{T},i+1}||, u^{\alpha}_{\mathrm{T},i+1} \right]^T \\ \mathbf{K}^{\alpha,*} \ni \widehat{u}^{\alpha}_{i+1} \perp r^{\alpha}_{i+1} \in \mathbf{K}^{\alpha} \end{cases} \tag{13}$$

for all $\alpha \in \{1 \ldots m\}$.

Over-Relaxation [PSOR-*]

One contact point problem

- closed form solutions
- Any solver listed before.

Proximal point technique [Moreau(1962), Moreau(1965), Rockafellar(1976)]

Principle

We want to solve

$$\min_x f(x) \tag{14}$$

We define the approximation problem for a given x_k

$$\min_x f(x) + \rho\|x - x_k\|^2 \tag{15}$$

with the optimal point $x^\star$.

$$x^\star \triangleq \text{prox}_{f,\rho}(x_k) \tag{16}$$

Proximal point algorithm [PPA-*]

$$x_{k+1} = \text{prox}_{f,\rho_k}(x_k)$$

Special case for solving $G(x) = 0$

$$f(x) = \frac{1}{2} G^\top(x) G(x)$$

Optimization based methods

- Alternating optimization problems (Panagiotopoulos et al.) [PANA-*]
- Successive approximation with Tresca friction (Haslinger et al.) [TRESCA-*]

$$\begin{cases} \theta = h(r_{\mathrm{N}}) \\ \min \ \dfrac{1}{2} r^\top W r + r^\top q \\ \quad \text{s.t.} \qquad r \in C(\mu, \theta) \end{cases} \tag{17}$$

where $C(\mu,\theta)$ is the cylinder of radius $\mu\theta$.

- Fixed point on the norm of the tangential velocity [A., Cadoux, Lemaréchal, Malick(2011)] [ACLM-*].

$$\begin{cases} s = \|u_{\mathrm{T}}\| \\ \min \ \dfrac{1}{2} r^\top W r + r^\top (q + \alpha s) \\ \quad \text{s.t.} \qquad r \in K \end{cases} \tag{18}$$

Fixed point or Newton Method on $F(s) = s$

Siconos/Numerics

Siconos

Open source software for modelling and simulation of nonsmooth systems

Siconos/Numerics

Collection of C routines to solve FC3D problems in dense, sparse or block sparse versions:

- VI solvers: Fixed point, Extra-Gradient, Uzawa
- VI based projection/splitting algorithm: NSGS, PSOR
- Nonsmooth Newton technique: Alart-Curnier, Jean-Moreau, Natural map, Ficher-Bursmeister
- Proximal point algorithm
- Optimization based solvers. Panagiotopoulos, Tresca, SOCQP
- ...

Collection of routines for optimization and complementarity problems

- LCP solvers (iterative and pivoting (Lemke))
- Standard QP solvers (Projected Gradient (Calamai & Moré), Projected CG (Moré & Toraldo), active set technique)
- linear and nonlinear programming solvers.

Measuring errors

Full error criteria

$$\text{error} = \frac{\|F^{\text{nat}}_{\text{vi-2}}(r)\|}{\|q\|}. \tag{19}$$

Cheap error

$$\text{error}_{\text{cheap}} = \frac{\|r_{k+1} - r_k\|}{\|r_k\|}. \tag{20}$$

The tolerance of solver is then self-adapted in the loop to meet the required tolerance based on the error given by (19).

Performance profiles [Dolan and Moré(2002)]

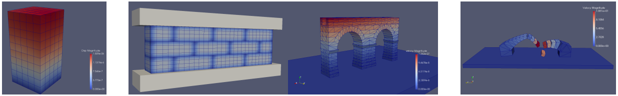
- Given a set of problems $\mathcal{P}$
- Given a set of solvers $\mathcal{S}$
- A performance measure for each problem with a solver $t_{p,s}$ (cpu time, flops, ...)
- Compute the performance ratio

$$\tau_{p,s} = \frac{t_{p,s}}{\min\limits_{s\in\mathcal{S}} t_{p,s}} \geqslant 1 \tag{21}$$

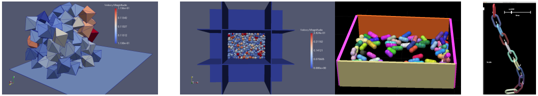
- Compute the performance profile $\rho_s(\tau) : [1, +\infty] \rightarrow [0, 1]$ for each solver $s \in \mathcal{S}$

$$\rho_s(\tau) = \frac{1}{|\mathcal{P}|} \big| \{p \in \mathcal{P} \mid \tau_{p,s} \leqslant \tau\} \big| \tag{22}$$

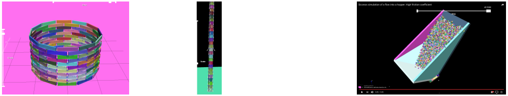
The value of $\rho_s(1)$ is the probability that the solver s will win over the rest of the solvers.



(a) Cubes_H8 (b) LowWall_FEM (c) Aqueduct_PR (d) Bridge_PR



(e) 100_PR_Periobox (f) 945_SP_Box_PL (g) Capsules (h) Chain



(i) KaplasTower (j) BoxesStack (k) Chute_1000, Chute_4000, Chute_local_problems

Figure: Illustrations of the FClib test problems

Test set	code	friction coefficient μ	# of problems	# of d.o.f.	# of contacts	contact density c	rank ratio(W)
Cubes_H8_2	LMGC90	0.3	15	162	[3 : 5]	[0.02 : 0.09]	1
Cubes_H8_5	LMGC90	0.3	50	1296	[17 : 36]	[0.02 : 0.09]	1
Cubes_H8_20	LMGC90	0.3	50	55566	[361 : 388]	[0.019 : 0.021]	1
LowWall_FEM	LMGC90	0.83	50	{7212}	[624 : 688]	[0.28 : 0.29]	1
Aqueduct_PR	LMGC90	0.8	10	{1932}	[4337 : 4811]	[6.81 : 7.47]	[6.80 : 7.46]
Bridge_PR	LMGC90	0.9	50	{138}	[70 : 108]	[1.5 : 2.3]	[2.27 : 2.45]
100_PR_Periobox	LMGC90	0.8	106	{606}	[14 : 578]	[0.2 : 3]	[1.76 : 3.215]
945_SP_Box_PL	LMGC90	0.8	60	{5700}	[2322 : 5037]	[1.22 : 2.65]	[1.0 : 2.66]
Capsules	Siconos	0.7	249	[96:600]	[17 : 304]	[0.53 : 1.52]	[1.08 : 1.55]
Chain	Siconos	0.3	242	{60}	[8 : 28]	[0.5 : 1.3]	[1.05 : 1.6]
KaplasTower	Siconos	0.7	201	[72 : 792]	[48 : 933]	[3.0 : 3.6]	[2.0 : 3.53]
BoxesStack	Siconos	0.7	255	[6 : 300]	[1 : 200]	[1.86 : 2.00]	[1.875 : 2.0]
Chute_1000	Siconos	1.0	156	[276 : 5508]	[74 : 5056]	[0.69 : 2.95]	[1.0 : 2.95]
Chute_4000	Siconos	1.0	40	[17280 : 20034]	[15965 : 19795]	[2.51 : 3.06]	–
Chute_local_problems	Siconos	1.0	834	3	1	1	1

Table: Description of the test sets of FCLib library (v1.0)

Parameters of the simulation campaign

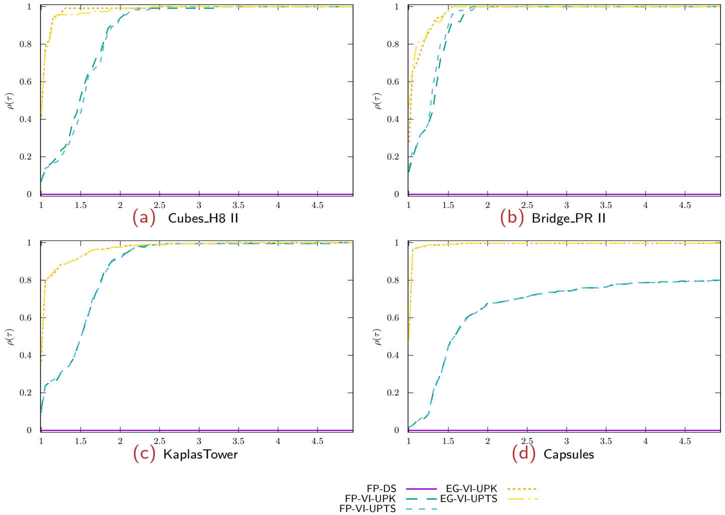
Test set	precision	prescribed time limit (s)	mean performance of the fastest solver $\mu\{\min\{t_{p,s}, s \in S\}\}$	std. deviation performance of the fastest solver $\sigma(\min\{t_{p,s}, s \in S\})$	mean performance of the fastest solver by contact $\mu\{\min\{t_{p,s}/n_{c,p}, s \in S\}\}$	std. deviation performance of the fastest solver by contact $\sigma(\min\{t_{p,s}/n_{c,p}, s \in S\})$	# of unsolved problems
Cubes_H8_⋆	10^{-08}	100	1.73	2.13	4.83^{-03}	5.78^{-03}	0
Cubes_H8_⋆ II	10^{-04}	100	0.92	1.06	2.66^{-03}	2.83^{-03}	0
LowWall_FEM	10^{-08}	400	13.1	3.50	1.91^{-02}	5.09^{-03}	0
LowWall_FEM II	10^{-04}	400	14.8	2.85	2.16^{-02}	4.54^{-03}	0
Aqueduct_PR	10^{-04}	200	5.80	6.36	4.90^{-04}	3.03^{-04}	0
Bridge_PR	10^{-08}	400	10.3	12.9	1.23^{-01}	2.88^{-01}	0
Bridge_PR II	10^{-04}	100	0.048	0.038	1.30^{-03}	1.42^{-03}	0
100_PR_Periobox	10^{-04}	100	0.064	0.062	1.56^{-04}	1.22^{-04}	0
945_SP_Box_PL	10^{-04}	100	3.20	1.71	6.45^{-04}	3.36^{-04}	0
Capsules	10^{-08}	50	$1.46.10^{-02}$	$1.74.10^{-02}$	5.67^{-05}	6.26^{-05}	0
Chain	10^{-08}	50	$6.19.10^{-04}$	$3.68.10^{-04}$	$3.15.10^{-05}$	$1.46.10^{-05}$	0
KaplasTower	10^{-08}	200	$1.27.10^{-01}$	$3.75.10^{-01}$	$1.84.10^{-04}$	$4.57.10^{-04}$	0
KaplasTower II	10^{-04}	100	$2.84.10^{-02}$	$1.51.10^{-01}$	$3.39.10^{-05}$	$1.84.10^{-04}$	0
BoxesStack	10^{-08}	100	$3.42.10^{-02}$	$8.87.10^{-02}$	$3.24.10^{-04}$	$9.77.10^{-04}$	0
Chute_1000	10^{-04}	200	2.62	3.06	6.76^{-04}	6.58^{-04}	0
Chute_4000	10^{-04}	200	10.52	7.88	5.71^{-04}	4.07^{-04}	0
Chute_local_problems	10^{-08}	10	$1.80.10^{-04}$	$1.57.10^{-05}$	$1.80.10^{-04}$	$1.57.10^{-05}$	0

Table: Parameters of the simulation campaign

Parameters of the simulation campaign

- More than 2500 problems
- Around 30 solvers with their variants
- More than 27000 runs between few seconds up to 400s.

Comparison of numerical methods FP-DS, FP-VI-⋆ and FP-EG-⋆



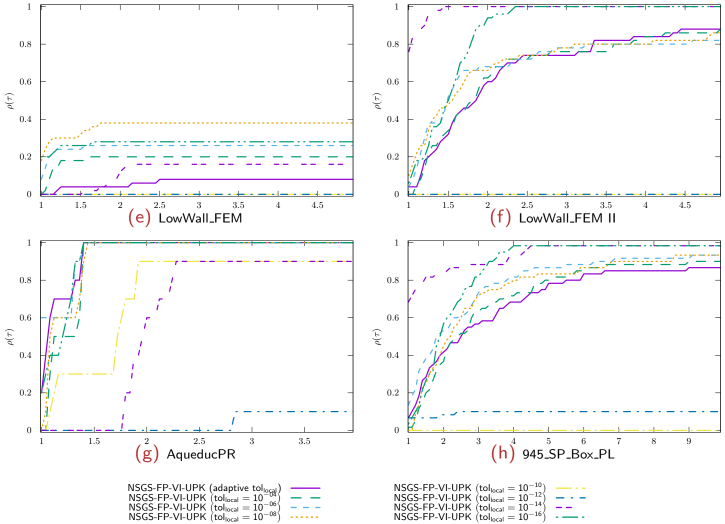
(a) Cubes_H8 II

(b) Bridge_PR II

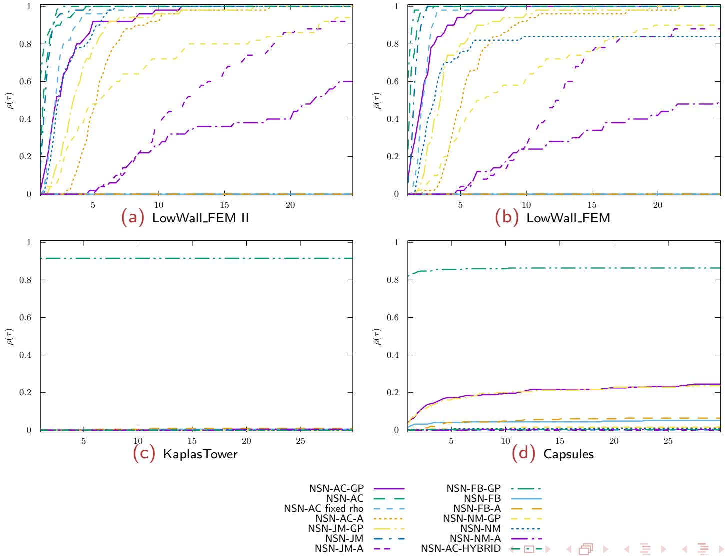
(c) KaplasTower

(d) Capsules

Influence of the local solver in NSGS-$\star$ algorithms.



Comparison of NSN-⋆ algorithms.



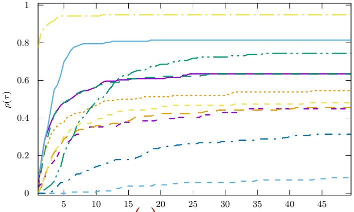
(a) LowWall_FEM II

(b) LowWall_FEM

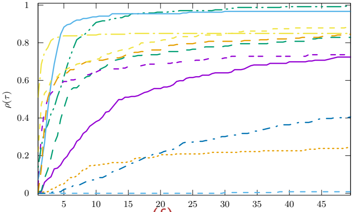
(c) KaplasTower

(d) Capsules

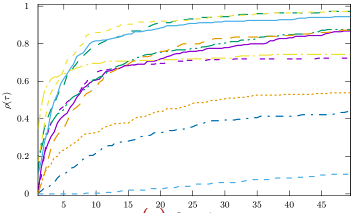
Comparison of the optimization based solvers



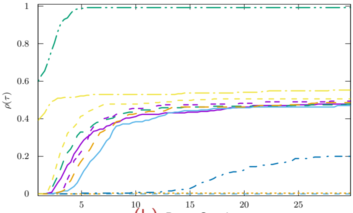
(e) Chute_1000



(f) Chain



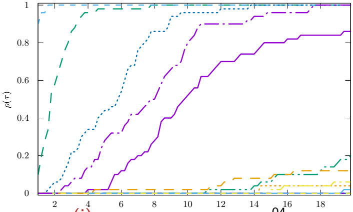
(g) Capsules



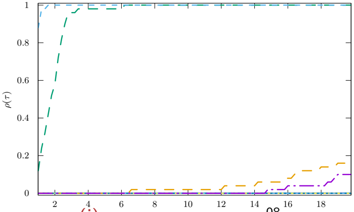
(h) BoxesStack

PANA-PGS-FP-VI-UPK
PANA-PGS-FP-VI-EG-UPK
PANA-CONVEXQP-PG
PANA-PGS-CONVEXQP-PG
TRESCA-NSGS-FP-VI-UPK
TRESCA-CONVEXQP-PG
TRESCA-FP-VI-UPK
SOCLCP-NSGS-PLI
ACLM-NSGS-FP-VI
ACLM-VI-FPP
ACLM-VI-EG

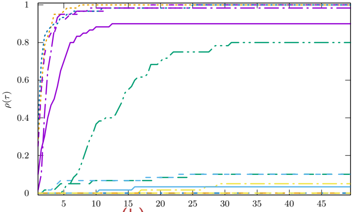
Comparisons by families of solvers



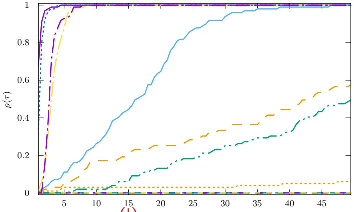
(i) LowWall_FEM II (tol = 1e−04)



(j) LowWall_FEM (tol = 1e−08)



(k) 945_SP_Box_PL



(l) 100_PR_PerioBox

Conclusions & Perspectives

Conclusions

1. A bunch of articles in the literature
2. No "Swiss–knife" solution : choose efficiency OR robustness
3. Newton–based solvers solve efficiently some problems, but robustness issues
4. First order iterative methods (VI, NSGS, PSOR) solves all the problems but very slowly
5. The rank of the H matrix (ratio number of contacts unknows/number of d.o.f) plays an important role on the robustness
6. Optimisation-based and proximal-point algorithm solvers are interesting but it is difficult to forecast their efficiency.

Perspectives

1. Develop new algorithm and compare other algorithm in the literature. (interior point techniques, issues with standard optimization software.)
2. Improve the robustness of Newton solvers and accelerate first-order method
3. Complete the collection of benchmarks ➜ FCLIB

FCLIB : a collection of discrete 3D Frictional Contact (FC) problems

Our inspiration: MCPLIB or CUTEst

What is FCLIB ?

- A open source collection of Frictional Contact (FC) problems stored in a specific HDF5 format
- A open source light implementation of Input/Output functions in C Language to read and write problems (Python and Matlab coming soon)

Goals of the project

Provide a standard framework for testing available and new algorithms for solving discrete frictional contact problems share common formulations of problems in order to exchange data

Call for contribution

http://fclib.gforge.inria.fr

All the results may be found in [Acary et al.(2018)Acary, Brémond, and Huber]

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