Numerical solutions of the Coulomb friction contact problem from the perspective of optimisation and mathematical programming

ICCCM, Munich, July 2025

Vincent Acary

Inria - Centre de l'Université Grenoble Alpes - Laboratoire Jean Kuntzmann

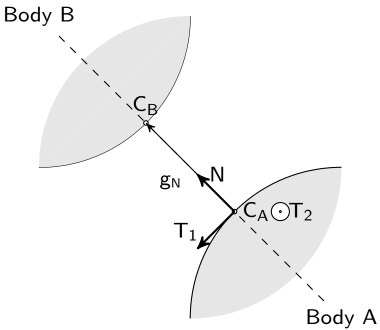
Thanks to the collaborators for stimulating discussions and developments:

Pierre Alart, Paul Armand, Florent Cadoux, Frédéric Dubois,
Claude Lémaréchal, Jérôme Malick and Mathieu Renouf

Motivations & contents

1. Introduce a sufficiently generic and representative discrete 3D frictional contact problem.
2. Interpret this problem in the context of numerical optimisation and mathematical programming.
3. Provide an existence result, whose assumption can be verified numerically.
4. Compare the main existing numerical methods based on a large collection of problems (FCLIB) and a common implementation (SICONOS/Numerics).
5. Propose a new solution method based on the interior point method.

Signorini's condition and Coulomb's friction



- gap function $g_N = (C_B - C_A)N$.
- reaction forces and velocities

$$r = r_N N + r_T, \quad \text{with } r_N \in \mathbb{R}, \quad r_T \in \mathbb{R}^2.$$

$$u = u_N N + u_T, \quad \text{with } u_N \in \mathbb{R} \quad u_T \in \mathbb{R}^2.$$

- Signorini conditions

$$\text{position level : } 0 \leqslant g_N \perp r_N \geqslant 0.$$

$$\text{velocity level : } \begin{cases} 0 \leqslant u_N \perp r_N \geqslant 0 & \text{if } g_N \leqslant 0 \\ r_N = 0 & \text{otherwise.} \end{cases}$$

Signorini's condition and Coulomb's friction

Coulomb friction modeling assumption

Let μ be the coefficient of friction. Let us define the Coulomb friction cone K which is chosen as the isotropic second order cone

$$K = \{r \in \mathbb{R}^3 \mid \|r_{\mathsf{T}}\| \leqslant \mu r_{\mathsf{N}}\}. \tag{1}$$

Coulomb friction postulates

- for the sticking case that

$$u_{\mathsf{T}} = 0, \quad r \in K, \tag{2}$$

- and for the sliding case that

$$u_{\mathsf{T}} \neq 0, \quad \|r_{\mathsf{T}}\| = \mu r_{\mathsf{N}}, \quad r_{\mathsf{T}} = -\frac{u_{\mathsf{T}}}{\|u_{\mathsf{T}}\|}\|r_{\mathsf{T}}\|. \tag{3}$$

Disjunctive formulation of the frictional contact behavior

$$\begin{cases} r = 0 & \text{if } g_{\mathsf{N}} > 0 \quad \text{(no contact)} \\ r = 0, u_{\mathsf{N}} \geqslant 0 & \text{if } g_{\mathsf{N}} \leqslant 0 \quad \text{(take–off)} \\ r \in K, u = 0 & \text{if } g_{\mathsf{N}} \leqslant 0 \quad \text{(sticking)} \\ r \in \partial K, u_{\mathsf{N}} = 0, r_{\mathsf{T}} = -\frac{u_{\mathsf{T}}}{\|u_{\mathsf{T}}\|}\|r_{\mathsf{T}}\| & \text{if } g_{\mathsf{N}} \leqslant 0 \quad \text{(sliding)} \end{cases} \tag{4}$$

Signorini's condition and Coulomb's friction

Second Order Cone Complementarity (SOCCP) formulation

- Modified relative velocity $\tilde{u} \in \mathbb{R}^3$ (De Saxcé, 1992) defined by

$$\tilde{u} = u + \mu \|u_{\mathrm{T}}\| \mathrm{N}. \tag{5}$$

- Second-Order Cone Complementarity Problem (SOCCP)

$$K^\star \ni \tilde{u} \perp r \in K \tag{6}$$

if $g_{\mathrm{N}} \leqslant 0$ and $r = 0$ otherwise.
The set $K^\star$ is the dual convex cone to K defined by

$$K^\star = \{u \in \mathbb{R}^3 \mid r^\top u \geqslant 0, \quad \text{for all } r \in K\}. \tag{7}$$

(Acary and Brogliato, 2008; Acary et al., 2011)

Signorini's condition and Coulomb's friction

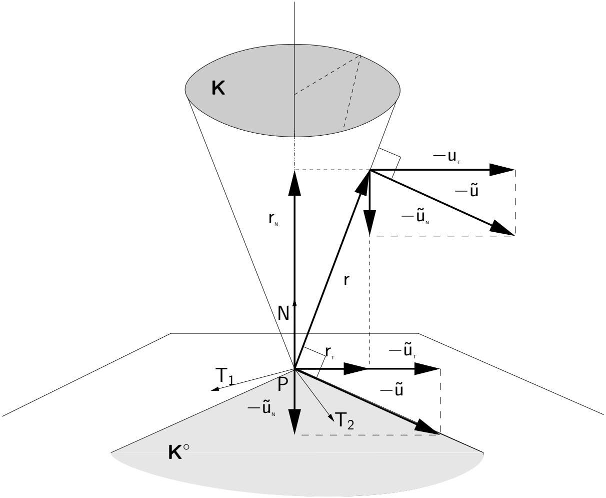


Figure: Coulomb's friction and the modified velocity $\tilde{u}$. The sliding case.

Discrete frictional contact problems

Problem 1 (General discrete frictional contact problem)

Given

- *a symmetric positive definite matrix* $M \in \mathbb{R}^{n \times n}$,
- *a vector* $f \in \mathbb{R}^n$,
- *a matrix* $H \in \mathbb{R}^{n \times m}$,
- *a vector* $w \in \mathbb{R}^m$,
- *a vector of coefficients of friction* $\mu \in \mathbb{R}^{n_c}$,

find three vectors $v \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ *and* $r \in \mathbb{R}^m$, *denoted by* $\mathrm{FC/I}(M, H, f, w, \mu)$ *such that*

$$\begin{cases} Mv = Hr + f \\ u = H^\top v + w \\ \tilde{u} = u + g(u) \\ K^\star \ni \tilde{u} \perp r \in K \end{cases} \tag{8}$$

with $g(u) = [[\mu^\alpha \| u_T^\alpha \| \mathrm{N}^\alpha]^\top, \alpha = 1 \ldots n_c]^\top$. □

Discrete frictional contact problems

Wide range of applications

The problem is:

- is generic enough to include a large number of cases in practice,
- is really representative in the linear, or the linearized, case (Newton procedure),
- can be generalised to non-linear cases.

See for instance (Acary and Cadoux, 2013)

Origin of the linear relation $u = H^\top v + w$

- H is the contact configuration matrix (similar to the Jacobians of the constraints)
- w can contain
  - impact laws terms or prescribed velocity in velocity level formulations
  - displacements, or increments of displacements, in position level formulations

Discrete frictional contact problems

Origin of the linear relation $Mv = Hr + f$

- Time–discretization of the discrete dynamical mechanical system.
  Event–capturing or event–detecting time–stepping schemes
- Space discretization of the quasi–static problem of solids (FEM)
  (M is the tangent stiffness matrix !).
- Time–discretization and space discretization of the dynamic problem of solids.
  (FEM, MPM, PFEM, . . .)
- Flexible or rigid multi-body Systems,
- Spectral methods, harmonic balance method, . . .

Discrete frictional contact problems

Problem 2 (Reduced discrete frictional contact problem)

Given

- *a symmetric positive semi–definite matrix* $W \in \mathbb{R}^{m\times m}$,
- *a vector* $q \in \mathbb{R}^{m}$,
- *a vector* $\mu \in \mathbb{R}^{n_c}$ *of coefficients of friction,*

find two vectors $u \in \mathbb{R}^{m}$ *and* $r \in \mathbb{R}^{m}$, *denoted by* $\mathrm{FC/II}(W, q, \mu)$ *such that*

$$\begin{cases} u = Wr + q \\ \tilde{u} = u + g(u) \\ K^{\star} \ni \tilde{u} \perp r \in K \end{cases} \tag{9}$$

with $g(u) = [[\mu^{\alpha}\|u_T^{\alpha}\|\mathrm{N}^{\alpha}]^{\top}, \alpha = 1 \dots n_c]^{\top}$. □

Relation with the general problem

$W = H^{\top}M^{-1}H$ and $q = H^{\top}M^{-1}f + w$.

From the optimization point of view

Discrete frictional contact are complementarity problems / variational inequalities.

Finite dimensional Second-Order Cone Complementarity Problems (SOCCP)

$$K^\star \ni Wr + q + g(Wr + q) \perp r \in K \tag{10}$$

of more generally,

Variational Inequality (VI) (normal cone inclusion)

$$-(Wr + q + g(Wr + q)) \stackrel{\Delta}{=} -F(r) \in N_K(r). \tag{11}$$

Properties

- nonsmooth since $g()$ is nonsmooth
- nonmonotone since the mapping F is not monotone for large μ
- many possible reformulations such as nonsmooth equations $G(r) = 0$

From the optimization point of view

Important Remarks

- The variational inequality is NOT the optimality condition of a (convex) optimization problem.
- The problem is hard to solve efficiently and robustly at tight accuracy.
- Even harder if H is not full rank (constraints redundancy)
- Generic numerical methods for VI/CP exist and can be applied
- Numerous of existing methods for FC3D problems are adaptations of mathematical programming methods.

From the optimization point of view

Semismooth Newton methods for nonsmooth equations $G(r) = 0$.

Not just adaptations, but sometimes pioneering methods.

- The natural map F^{nat} associated with the VI (11)

$$F^{\text{nat}}(r) = r - P_K(r - F(r))$$

- Pioneering work of Alart and Curnier, 1991

$$\begin{cases} r_{\text{N}} - P_{\mathbb{R}_+^{n_c}}(r_{\text{N}} - \rho_{\text{N}} u_{\text{N}}) = 0, \\ r_{\text{T}} - P_{D(\mu, r_{\text{N},+} + \rho u_{\text{N}})}(r_{\text{T}} - \rho_{\text{T}} u_{\text{T}}) = 0, \end{cases}$$

- other SOCCP functions (Fisher-Bursmeister function)

From the optimization point of view

An optimization problem

$$\begin{array}{ll} \min\limits_{v,u,r} & \tilde{u}^\top r = u^\top r + \mu r_{\mathrm{N}} \|u_{\mathrm{T}}\| \stackrel{\Delta}{=} b(u,r) \\ \text{s.t.} & Mv = Hr + f \\ & \tilde{u} = H^\top v + w + g(u) \in K^\star \\ & r \in K \end{array} \tag{12}$$

$b(u,r)$ is the de Saxcé bi-potential.

- A solution of the discrete frictional contact problem is a solution of the optimization problem (12) with $b(u,r) = 0$
- A solution of the optimization problem (12) is a solution of the discrete frictional contact problem if $b(u,r) = 0$
- With constraints qualification, the problem has a solution.
- The problem is not convex and non smooth, may have a lot of local minima.

➜ In practice, finding a minimum is difficult, and a global minimum is not ensured.

An existence result via convex optimization

PhD of F. Cadoux with C. Lemaréchal and J. Malick (Acary et al., 2011; Cadoux, 2009)

Let us introduce a slack variable

$$s^{\alpha} := \|u^{\alpha}_{\mathrm{T}}\|$$

New formulation of the modified velocity with $A \in \mathrm{IR}^{m \times n_c}$

$$\tilde{u} := u + As \qquad (g(u) = As)$$

The problem $\mathrm{FC/I}(M, H, f, w, \mu)$ can be reformulated as

$$\begin{cases} Mv = Hr + f \\ \widetilde{u} = H^{\top} v + w + As \\ K^{\star} \ni \tilde{u} \perp r \in K \end{cases} \tag{13}$$

with

$$s^{\alpha} := \|u^{\alpha}_{\mathrm{T}}\| \tag{14}$$

An existence result via convex optimization

The problem (13) appears to be the KKT condition of

Primal problem

$$\begin{cases} \min \quad J(v) := \frac{1}{2} v^\top M v + f^\top v \\ H^\top v + w + As \in K^\star \end{cases} \qquad (D_s)$$

Dual problem

$$\begin{cases} \min \quad J_s(r) := \frac{1}{2} r^\top W r - q_s^\top r \\ r \in K \end{cases} \qquad (P_s)$$

with $q_s = q + As$

Interest

Two convex programs ➜ existence of solutions under feasibility conditions.

An existence result via convex optimization

Fixed point problem

Introducing

$$u(s) := \operatorname{argmin}_u(P_s) = \operatorname{argmin}_u(D_s)$$

practically computable by optimization software, and

$$F^\alpha(s) := \|u_T^\alpha(s)\|,$$

the incremental problem becomes a fixed point problem

$$F(s) = s$$

An existence result via convex optimization

Key assumption

$$\exists v \in \mathbb{R}^m \ : \ Hv + w \in \operatorname{int} K^\star \tag{15}$$

Using Assumption (15),

- the application $F : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is well-defined, continuous and bounded
- apply Brouwer's theorem

Theorem 3

A fixed point exists

An existence result via convex optimization

Numerical validation of the key assumption

Solving a SOC linear program: find $x^\star \in \boldsymbol{R}$

$$\begin{array}{ll} \max\limits_x & x \\ \text{s.t.} & Hv + w - ax \in K^\star \end{array}$$

where $a = \mathrm{col}(\mathrm{N}^\alpha, \alpha \in [\![1, \mathrm{m}]\!]) \in \mathbb{R}^{\mathrm{m}}$.

If $x^\star > 0$, then the assumption is satisfied.

Numerical interest

The fixed point equation $F(s) = s$ can be tackled by

- fixed-point iterations
$$s \leftarrow F(s)$$
- Newton iterations
$$s \leftarrow \mathrm{Jac}[F](s) \backslash F(s)$$

The inner problem can be solved by QP solvers with SOC constraints (ADMM, IPM, AL, . . .)

Numerical solution procedure

- VI based methods
- Nonsmooth Equations based methods
- Matrix block–splitting and projection based algorithms
- Proximal point algorithms
- Optimization based approaches

VI based methods

Variational Inequality (VI) reformulation

$$(9) \iff -F(r) := -(Wr + q + g(Wr + q)) \in N_K(r) \qquad (16)$$

Standard methods

- Basic fixed point iterations with projection [FP-VI]

$$r_{k+1} \leftarrow P_K(r_k - \rho_k F(r_k))$$

- with fixed $\rho_k = \rho$, we get the Uzawa Algorithm of Saxcé and Feng, 1998 with similarity with augmented Lagrangian methods(Wriggers, 2006) [FP-DS]
- Extragradient method [EG-VI]

$$r_{k+1} \leftarrow P_K(r_k - \rho_k F(P_K(r_k - \rho_k F(r_k))))$$

Self-adaptive procedure for ρ_k [UPK]

$$\text{Armijo-like}: \; m_k \in \mathbb{N} \quad \text{such that} \begin{cases} \rho_k = \rho 2^{m_k}, \\ \rho_k \|F(r_k) - F(\bar{r}_k)\| \leqslant \|r_k - \bar{r}_k\| \end{cases}$$

Nonsmooth Equations based methods

Nonsmooth Newton on $G(z) = 0$

$$z_{k+1} = z_k - \Phi^{-1}(z_k)(G(z_k)), \qquad \Phi(z_k) \in \partial G(z_k)$$

- Alart–Curnier Formulation (Alart and Curnier, 1991) [NSN-AC]

$$\begin{cases} r_{\mathrm{N}} - P_{\mathbf{R}^{n_c}_{+}}(r_{\mathrm{N}} - \rho_{\mathrm{N}} u_{\mathrm{N}}) = 0, \\ r_{\mathrm{T}} - P_{D(\mu, r_{\mathrm{N},+} + \rho u_{\mathrm{N}})}(r_{\mathrm{T}} - \rho_{\mathrm{T}} u_{\mathrm{T}}) = 0, \end{cases}$$

- Jean–Moreau Formulation [NSN-MJ]

$$\begin{cases} r_{\mathrm{N}} - P_{\mathbf{R}^{n_c}_{+}}(r_{\mathrm{N}} - \rho_{\mathrm{N}} u_{\mathrm{N}}) = 0, \\ r_{\mathrm{T}} - P_{D(\mu, r_{\mathrm{N},+})}(r_{\mathrm{T}} - \rho_{\mathrm{T}} u_{\mathrm{T}}) = 0, \end{cases}$$

- Direct normal map reformulation [NSN-NM]

$$r - P_K(r - \rho(u + g(u))) = 0$$

- Extension of Fischer-Burmeister function to SOCCP [NSN-FB]

$$\phi_{\mathrm{FB}}(x, y) = x + y - (x^2 + y^2)^{1/2}$$

Matrix block-splitting and projection based algorithms (Jean and Touzot, 1988; Moreau, 1994)

Block splitting algorithm with $W^{\alpha\alpha} \in \boldsymbol{R}^3$ [NSGS-*]

$$\begin{cases} u^\alpha_{i+1} - W^{\alpha\alpha} P^\alpha_{i+1} = q^\alpha + \sum_{\beta<\alpha} W^{\alpha\beta} r^\beta_{i+1} + \sum_{\beta>\alpha} W^{\alpha\beta} r^\beta_i \\ \widetilde{u}^\alpha_{i+1} = \left[u^\alpha_{\mathrm{N},i+1} + \mu^\alpha \, ||u^\alpha_{\mathrm{T},i+1}||, u^\alpha_{\mathrm{T},i+1}\right]^T \\ \mathbf{K}^{\alpha,*} \ni \widetilde{u}^\alpha_{i+1} \perp r^\alpha_{i+1} \in \mathbf{K}^\alpha \end{cases} \tag{17}$$

for all $\alpha \in \{1 \ldots m\}$.

Over-Relaxation [PSOR-*]

One contact point problem

- closed form solutions
- Any solver listed before.

Optimization based methods

- Alternating optimization problems (Panagiotopoulos et al.) [PANA-*]
- Successive approximation with Tresca friction (Haslinger et al.) [TRESCA-*]

$$\begin{cases} \theta = h(r_{\mathrm{N}}) \\ \min \ \frac{1}{2} r^\top W r + r^\top q \\ \ \text{s.t.} \qquad r \in C(\mu, \theta) \end{cases} \tag{18}$$

where $C(\mu,\theta)$ is the cylinder of radius $\mu\theta$.

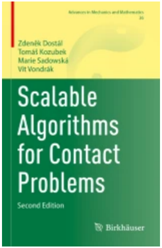
- Fixed point on the norm of the tangential velocity [A., Cadoux, Lemaréchal, Malick(2011)] [ACLM-*].

$$\begin{cases} s = \|u_{\mathrm{T}}\| \\ \min \ \frac{1}{2} r^\top W r + r^\top (q + \alpha s) \\ \ \text{s.t.} \qquad r \in K \end{cases} \tag{19}$$

Fixed point or Newton Method on $F(s) = s$

Optimization based methods

Optimization, contact and huge-scale problems. (Dostál et al., 2023)



siconos/numerics

siconos

Open source software for modelling and simulation of nonsmooth systems

siconos/numerics

Collection of C routines to solve FC3D problems in dense, sparse or block sparse versions:

- VI solvers: Fixed point, Extra-Gradient, Uzawa
- VI based projection/splitting algorithm: NSGS, PSOR
- Semismooth Newton methods
- Optimization based solvers. Panagiotopoulos, Tresca, SOCQP, ADMM
- Interior point methods, ...

Collection of routines for optimization and complementarity problems

- LCP solvers (iterative and pivoting (Lemke))
- Standard QP solvers (Projected Gradient (Calamai & Moré), Projected CG (Moré & Toraldo), active set technique)
- linear and nonlinear programming solvers.

FCLIB : a collection of discrete 3D Frictional Contact (FC) problems

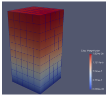
- Few mathematical results: existence, uniqueness, convergence, rate of convergence.
- Our inspiration: MCPLIB or CUTEst in Optimization.
- Without convergence proof, test your method on a large set of benchmarks shared by the community.

What is FCLIB ?

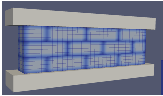
- A open source collection of Frictional Contact (FC) problems stored in a specific HDF5 format
- A open source light implementation of Input/Output functions in C Language to read and write problems (Python and Matlab coming soon)

Goals of the project

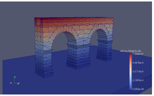
- Provide a standard framework for testing available and new algorithms for solving discrete frictional contact problems
- Share common formulations of problems in order to exchange data in a reproducible manner.



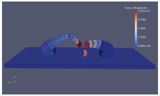
(a) Cubes_H8



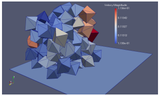
(b) LowWall_FEM



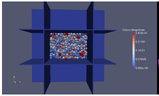
(c) Aqueduct_PR



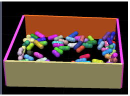
(d) Bridge_PR



(e) 100_PR_Periobox

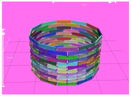


(f) 945_SP_Box_PL

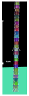


(g) Capsules

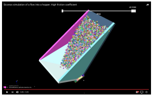
(h) Chain



(i) KaplasTower



(j) BoxesStack



(k) Chute_1000, Chute_4000, Chute_local_problems

Figure: Illustrations of the FClib test problems from Siconos and LMGC90

Parameters of the simulation campaign

- More than 2500 problems
- Around 30 solvers with their variants
- More than 27000 runs between few seconds up to 400s.

Full error criteria

$$\text{error} = \frac{\|F^{\text{nat}}_{\text{vi-2}}(r)\|}{\|q\|}. \tag{20}$$

Performance profiles Dolan and Moré, 2002

- Given a set of problems $\mathcal{P}$
- Given a set of solvers $\mathcal{S}$
- A performance measure for each problem with a solver $t_{p,s}$ (cpu time, flops, ...)
- Compute the performance ratio

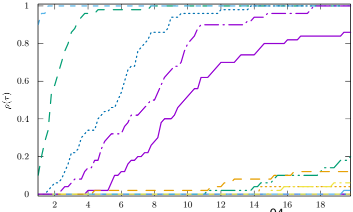
$$\tau_{p,s} = \frac{t_{p,s}}{\min\limits_{s \in \mathcal{S}} t_{p,s}} \geqslant 1 \tag{21}$$

- Compute the performance profile $\rho_s(\tau) : [1, +\infty] \rightarrow [0, 1]$ for each solver $s \in \mathcal{S}$

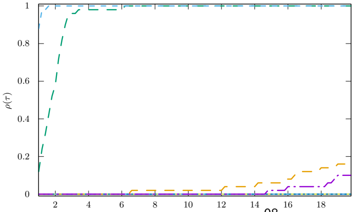
$$\rho_s(\tau) = \frac{1}{|\mathcal{P}|} \Big| \{ p \in \mathcal{P} \mid \tau_{p,s} \leqslant \tau \} \Big| \tag{22}$$

The value of $\rho_s(1)$ is the probability that the solver s will win over the rest of the solvers.

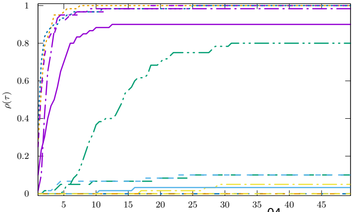
Comparisons by families of solvers



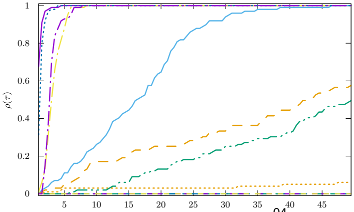
(a) LowWall_FEM II (tol = 1e−04)



(b) LowWall_FEM (tol = 1e−08)



(c) 945_SP_Box_PL (tol = 1e−04)



(d) 100_PR_PerioBox (tol = 1e−04)

Benchmarking : conclusions

Conclusions

1. No "Swiss–knife" solution : choose efficiency OR robustness
2. Newton–based solvers solve efficiently some problems, but robustness issues
3. First order iterative methods (VI, NSGS, PSOR) solves all the problems but very slowly
4. The rank of the H matrix (≈ratio number of contacts unknows/number of d.o.f) plays an important role on the robustness
5. Optimisation-based and proximal-point algorithm solvers are interesting but it is difficult to forecast theirs efficiencies.
6. Need for a second order method when H is rank-deficient (IPM?)

Mode details in Acary et al., 2018

Interior Point Methods

PhD thesis of Hoang Minh Nguyen(2025), with Paul Armand.

- Perturbation of the complementarity condition with a barrier parameter τ

FC/I(M, H, f, w, μ)

$$\begin{aligned} Mv + f &= H^\top r \\ Hv + w + se &= \tilde{u} \\ s &= \|\tilde{u}_\mathrm{T}\| \\ \tilde{u} \circ r &= 0 \\ (\tilde{u}, r) &\in K^2 \end{aligned}$$

Perturbed problem

$$\begin{aligned} Mv + f &= H^\top r \\ Hv + w + se &= \tilde{u} \\ s &= \|\tilde{u}_\mathrm{T}\| \\ \tilde{u} \circ r &= 2\tau e \\ (\tilde{u}, r) &\in \mathrm{int}(K^2) \end{aligned} \tag{23}$$

- Convex problem: IPM is able to solve very accurately and efficiently the problem with a given $s := \|u_\mathrm{T}\|$ even when H is rank-deficient. (see Acary et al., 2023b)
- Extension to general frictional contact problems: nonsmooth interior point method

Nonsmooth Interior-Point Method (NIPM)

Slater's assumption (SA) $\exists v \in \mathbb{R}^m$ such that $Hv + w \in \mathrm{int}(K)$

Propositions

1. Under SA, for each $\tau > 0$, (23) has a solution $(v_\tau, \tilde{u}_\tau, r_\tau, s_\tau)$
2. Under SA, there exists a central path $\{(v_\tau, \tilde{u}_\tau, r_\tau, s_\tau) : \tau > 0\}$, which converges to a solution of $\mathrm{FC/I}(M, H, f, w, \mu)$
3. This central path is not necessarily unique

Main theoretical outcome

Alternative proof of solution existence for $\mathrm{FC/I}(M, H, f, w, \mu)$

Nonsmooth Interior-Point Method (NIPM) - Linearization

System of equations

$$G := \begin{bmatrix} Mv + f - H^\top r \\ Hv + w - \tilde{u} + se \\ s - \|\tilde{u}_\mathrm{T}\| \\ \tilde{u} \circ r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2\tau e \end{bmatrix}$$

Jacobian of G

$$J := \begin{bmatrix} M & -H^\top & 0 & 0 \\ H & 0 & -I & e \\ 0 & 0 & -L & 1 \\ 0 & \tilde{U} & R & 0 \end{bmatrix}$$

where $L = \begin{pmatrix} 0 & \partial\|\tilde{u}_\mathrm{T}\|^\top \end{pmatrix}$, with $\partial\|\tilde{u}_\mathrm{T}\| = \begin{cases} \dfrac{\tilde{u}_\mathrm{T}}{\|\tilde{u}_\mathrm{T}\|} & \text{if } \tilde{u}_\mathrm{T} \neq 0 \\ d \in \mathbb{B} & \text{if } \tilde{u}_\mathrm{T} = 0 \end{cases}$ (unit ball $\mathbb{B}$)

Linear system $J\,d = -G + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2\sigma\tau e \end{bmatrix}$, $\sigma \in (0, 0.5)$: centralization parameter

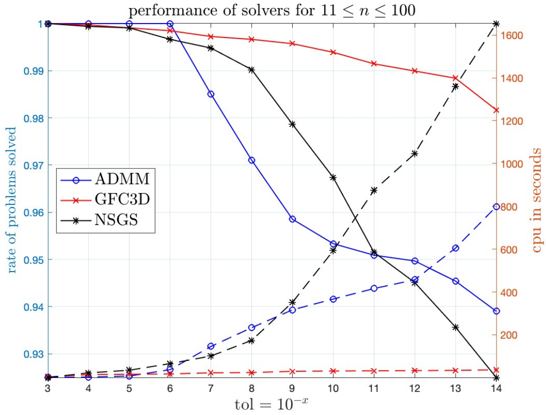
Stopping test

$$\max\left\{ \|Hv + w - \tilde{u}\|_\infty,\ \|Mv + f - H^\top r\|_\infty, |s - \|\tilde{u}_\mathrm{T}\||,\ \|\tilde{u} \circ r\| \right\} \leqslant \text{tol}$$

Nonsmooth Interior-Point Method (NIPM)

Moderate size problems

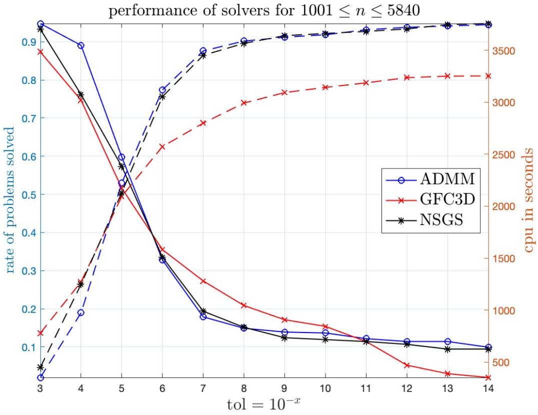
IPM (GFC3D) outperforms NSGS and ADMM



Interior Point Methods

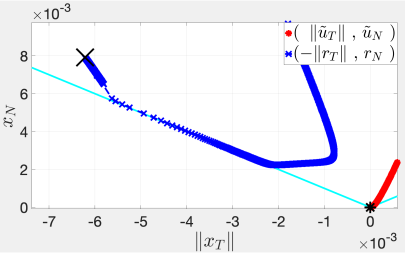
large size problems

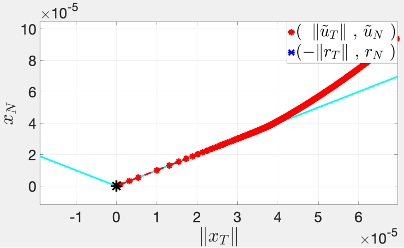
IPM (GFC3D) suffers from robustness



Nonsmooth Interior-Point Method (NIPM) - failures

Failure #1: A special shape of the central path





Solution: $r^{\times} \in \text{int}(K)$, $\tilde{u}^{*} = 0$ (sticking)

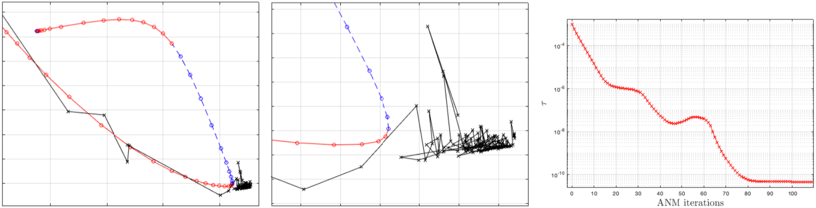
This shape of the central path can cause iterates to get stuck on the boundary, which is not the correct position for the solution.

Nonsmooth Interior-Point Method (NIPM) - failures

Failure #2: Non-monotone parameterization of the central path

- Red-blue curve: Central path $\tau \to r(\tau)$ calculated by Asymptotic Numerical Method (ANM).
Red: τ decreases. Blue: τ increases

- **Black** curve: the path of NIPM iterates



ANM: algorithm based on the computation of series to solve non-linear problems

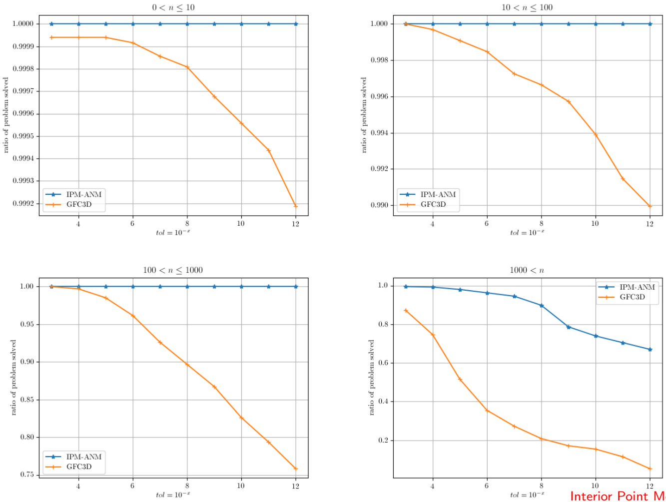
- Capable to calculate the central path with very tight tolerance ($\leqslant 10^{-12}$)

- More robustness = Slower performance

Nonsmooth Interior-Point Method (NIPM)

Moderate size problems

IPM with ANM is robust



Conclusions & Perspectives

Conclusions

- Further research is still needed for an robust AND efficient solver.
- IPM and ANM numerical method provides a robust solver.
- Coupling with other physical phenomena to obtain a monolithic variational inequality :
  - (non associated) plasticity (Acary et al., 2023a; Guillet et al., 2025)
  - fracture with cohesive zone model (Collins-Craft et al., 2022)
  - damage mechanics.

Open software and data collections.

- Siconos/Numerics. A open source collection of solvers. https://github.com/siconos/siconos
- FCLIB: a open collection of discrete 3D Frictional Contact (FC) problems https://github.com/FrictionalContactLibrary contribute ...

Use and contribute ...

Journal of Theoretical Computational and Applied Mechanics

JTCAM

The only overlay Diamond Open Access journal in Mechanics

Right now: an open call for associate editors

- Diamond Open Access: free for readers and authors
- Overlay: based on open archives (arXiv, Hal, ...)
- Publications of the highest scientific calibre with open reviews
- A community–supported journal with ethical and FAIR principles
- Promotion of reproducible and open science

Get involved. Join us now. Contributions are welcome!

Acary, V. and B. Brogliato (2008). *Numerical methods for nonsmooth dynamical systems. Applications in mechanics and electronics.* English. Lecture Notes in Applied and Computational Mechanics 35. Berlin: Springer. xxi, 525 p.

Acary, V. and F. Cadoux (2013). "Recent Advances in Contact Mechanics, Stavroulakis, Georgios E. (Ed.)". In: vol. 56. Lecture Notes in Applied and Computational Mechanics. Springer Verlag. Chap. Applications of an existence result for the Coulomb friction problem.

Acary, V. et al. (2011). "A formulation of the linear discrete Coulomb friction problem via convex optimization". In: *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik* 91.2, pp. 155–175. ISSN: 1521-4001. DOI: 10.1002/zamm.201000073. URL: http://dx.doi.org/10.1002/zamm.201000073.

Acary, Vincent, Franck Bourrier, and Benoit Viano (Feb. 2023a). "Variational approach for nonsmooth elasto-plastic dynamics with contact and impacts". In: *Computer Methods in Applied Mechanics and Engineering* 414, p. 116156. DOI: 10.1016/j.cma.2023.116156. URL: https://inria.hal.science/hal-03978387.

Acary, Vincent, Maurice Brémond, and Olivier Huber (June 2018). "On solving contact problems with Coulomb friction: formulations and numerical comparisons". In: *Advanced Topics in Nonsmooth Dynamics - Transactions of the European Network for Nonsmooth Dynamics*. Ed. by Remco Leine, Vincent Acary, and Olivier Brüls. Springer International Publishing, pp. 375–457. DOI: 10.1007/978-3-319-75972-2_10. URL: https://hal.inria.fr/hal-01878539.

- Acary, Vincent et al. (Apr. 2023b). "Second order cone programming for frictional contact mechanics using interior point algorithm". working paper or preprint. In revision. URL: https://hal.science/hal-03913568.
- Alart, P. and A. Curnier (1991). "A mixed formulation for frictional contact problems prone to Newton like solution method". In: *Computer Methods in Applied Mechanics and Engineering* 92.3, pp. 353–375.
- Cadoux, F. (2009). "Analyse convexe et optimisation pour la dynamique non–regulière". PhD thesis. Université Joseph Fourier, Grenoble I.
- Collins-Craft, Nicholas Anton, Franck Bourrier, and Vincent Acary (Oct. 2022). "On the formulation and implementation of extrinsic cohesive zone models with contact". In: *Computer Methods in Applied Mechanics and Engineering* 400, p. 115545. DOI: 10.1016/j.cma.2022.115545. URL: https://hal.science/hal-03371667.
- De Saxcé, G. (1992). "Une généralisation de l'inégalité de Fenchel et ses applications aux lois constitutives". In: *Comptes Rendus de l'Académie des Sciences* t 314,série II, pp. 125–129.
- Dolan, E.D. and J.J. Moré (2002). "Benchmarking optimization software with performance profiles". In: *Mathematical Programming* 91.2, pp. 201–213.
- Dostál, Zdeněk et al. (2023). *Scalable algorithms for contact problems*. Vol. 36. Springer.

- Guillet, Louis et al. (2025). "Implicit Material Point Method for non-associated plasticity of geomaterials". working paper or preprint. URL: https://hal.science/hal-05070887.
- Jean, M. and G. Touzot (1988). "Implementation of unilateral contact and dry friction in computer codes dealing with large deformations problems". In: *J. Méc. Théor. Appl.* 7.1, pp. 145–160.
- Moreau, J.J. (1994). "Some numerical methods in multibody dynamics: Application to granular materials". In: *European Journal of Mechanics - A/Solids* supp.4, pp. 93–114.
- Saxcé, G. De and Z.-Q. Feng (1998). "The bipotential method: A constructive approach to design the complete contact law with friction and improved numerical algorithms". In: *Mathematical and Computer Modelling* 28.4. Recent Advances in Contact Mechanics, pp. 225–245. ISSN: 0895-7177. DOI: https://doi.org/10.1016/S0895-7177(98)00119-8. URL: http://www.sciencedirect.com/science/article/pii/S0895717798001198.
- Wriggers, P. (2006). *Computational Contact Mechanics*. Second. originally published by John Wiley & Sons Ltd., 2002. Springer Verlag.