Concurrent multiple impact in rigid bodies: Formulation and simulation

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# Motivations

1. What is a concurrent multiple impact?
2. Industrial motivations
3. Main objectives
4. Previous results
5. Proposed approach

# Chain of balls

# Lagrangian systems

# Conclusion and perspectives
A multiple impact in a multibody system may be defined as:
The occurrence of several impacts at the same instant on a rigid body.

Academical examples:

- Newton's cradle
- The rocking block

Major difficulties:
- The standard laws (Newton, Poisson, ... ) do no longer apply correctly
- The continuity with the respect to initial conditions is usually lost.
### Motivations

**Chain of balls**

**Lagrangian systems**

**Conclusion**

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**What is a concurrent multiple impact?** Continued...

<table>
<thead>
<tr>
<th>Initial conditions</th>
<th>Newton</th>
<th>Poisson</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Initial conditions" /></td>
<td>$v_1 = 1$</td>
<td>$v_1^+ = -1/3$</td>
</tr>
<tr>
<td><img src="image2" alt="Newton" /></td>
<td>$e_A = e_B = 1$</td>
<td>$v_2^+ = v_3^+ = 2/3$</td>
</tr>
<tr>
<td><img src="image3" alt="Poisson" /></td>
<td>$e_A = 0, e_B = 1$</td>
<td>$v_1^+ = v_2^+ = 0$</td>
</tr>
</tbody>
</table>

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Industrial motivations

Long term collaboration with Schneider Electric (M. Abadie)
Study of the stability of circuits breakers w.r.t. external impact excitations
Main objectives

Find a multiple impact law, i.e.

$$\dot{q}^+ = F(\dot{q}^-, q, t)$$

with the following properties:

- Single-valued mapping
- Ensure the fundamental principles of the mechanics of multibody system
  - Equations of motion
  - Unilateral constraints
  - Energetic balance
- Fit qualitatively and quantitatively the experiments
  - Measurable parameters with a physical meaning
- Efficient numerical treatment
  - Formulation in terms of a mathematical program with constraints
Previous works

- Sequential approach [Han & Gilmore, 1993]
  - Heuristic choice of solutions
  - Existence and uniqueness problems (closed-loop systems)

- Generalization to multi-constrained systems
  (Newton [Moreau, 1988], Poisson [Glocker & Pfeiffer, 1995]).
  - Efficient numerical algorithm
  - Do not fit the experiments.

- Introduction of an impulse ratio [Hurmuzlu, 2001]
  - Lack of efficient numerical algorithm
  - Do not ensure the fundamental principles of mechanics

- Non local formulation of multi-body systems with impacts [Frémond, 1995].
  - Ensure the fundamental principles of mechanics
  - Lack of physical interpretation of the parameter

- Geometric framework for impacting systems [Aeberhard & Glocker]
Proposed approach

Fundamental problem:
- Taking into account the impact propagation by wave effects.
- Incorporating a continuum mechanics model for the deformation behaviour.

Quasi-rigid body model: general assumption of the Hertz’s contact
- Local deformation in the area of contact
- Quasi-static behavior laws (no inertia effects in the contact region)
- Negligible wave effects inside the bodies (focusing on bodies without special aspect ratios)

⇒ Multibody systems are modeled by a discrete dynamical systems of rigid bodies and unilateral contact force model (non-linear stiffness and damping)

A typical example of such systems: a chain of balls [Falcon & Laroche, 1998]
- Balls are modeled by discrete masses
- Contact interaction: Non linear Hertzian model
Outline

1 – Motivations

2 – Chain of balls
   2.1 – The unidimensional chain of 3 balls
   2.2 – The unidimensional chain of n balls
   2.3 – The pool: A bidimensional chain of balls

3 – Lagrangian systems

4 – Conclusion and perspectives
The unidimensional chain of 3 balls

Equation of motion at the instant of impact \((v_2 = 0)\) :

\[
\begin{align*}
\begin{cases}
m(v_1^+ - v_1) &= -p_1 \\
m(v_2^+) &= p_1 - p_2 \\
m(v_3^+ - v_3) &= p_2
\end{cases}
\end{align*}
\] (1)

Unilateral constraints

\[
\begin{align*}
\begin{cases}
v_1^+ &\leq v_2^+ \leq v_3^+ \\
p_1 &\geq 0, \quad p_2 \geq 0
\end{cases}
\end{align*}
\] (2)
**Choice of a parametrization**

- One global energetic coefficient \( e \):
  \[
  (v_1^+)^2 + (v_2^+)^2 + (v_3^+)^2 = e (v_1^2 + v_3^2)
  \]  

- One impulse ratio
  \[
  \alpha = \frac{p_2}{p_1}
  \]  

→ Parametrization of the solutions:

\[
\begin{align*}
p_1 &= \frac{(1 + \sqrt{\Delta})m}{2(1 - \alpha + \alpha^2)} \\
p_2 &= \frac{(1 + \sqrt{\Delta})m\alpha}{2(1 - \alpha + \alpha^2)} \\
\dot{q}_1^+ &= -\frac{1 + \sqrt{\Delta}}{2(1 - \alpha + \alpha^2)} + \dot{q}_1^- \\
\dot{q}_2^+ &= -\frac{(1 + \sqrt{\Delta})(\alpha - 1)}{2(1 - \alpha + \alpha^2)} \\
\dot{q}_3^+ &= -\frac{(1 + \sqrt{\Delta})\alpha}{2(1 - \alpha + \alpha^2)}
\end{align*}
\]

with \( \Delta = -1 + 2e - 2\alpha e + 2\alpha + 2\alpha^2 e - 2\alpha^2 \).
Evaluation of $e$ and $\alpha$

- Bounds imposed by $\Delta > 0$ and unilateral constraints

\[
\frac{1}{2} \leq \alpha \leq \frac{2e - 2 - \sqrt{6e - 2}}{e - 3}, \quad \frac{1}{3} \leq e \leq 1
\]  

- Evaluation by a regularized system with Hertzian springs.

\[
\begin{aligned}
\begin{cases}
m\ddot{\delta}_1 = -2f_1(\delta_1) + f_2(\delta_2) \\
m\ddot{\delta}_2 = -2f_2(\delta_2) + f_1(\delta_1) \\
0 \leq f \perp f - K(\delta) \cdot \delta \geq 0
\end{cases}
\end{aligned}
\]  

\[
\delta_i = q_{i+1} - q_i \quad \text{Indentation,} \quad K(\delta) = \begin{bmatrix} k_1\delta_1^{1/2} & 0 \\ 0 & k_2\delta_2^{1/2} \end{bmatrix} \quad \kappa = \frac{k_1}{k_2}
\]  

- Evaluation of the impulse ratio

\[
\alpha = \frac{\int_0^{t_f} f_1(t) \, dt}{\int_0^{t_f} f_2(t) \, dt}
\]
Numerical integration of 3 balls chain. Hertzian spring contact.

Forces between balls versus time.

- No sequential or simultaneous process
Rigid body model of a unidimensional chain of n balls:
- Unknowns: \( n \) velocities \( \dot{q}_i \), \( n - 1 \) impulses \( p_i \)
- Equations:
  - \( n \) equations of motion
  - 1 Energetic balance \( e \)
  - \( n - 2 \) impulse ratios, \( \alpha_{ij} = p_i / p_j \)
  ➜ Unique determination of post-impact velocities and impulses

Regularised quasi-rigid model:
- Visco-elastic contact model
  \[
  f = K \delta^\nu + C \delta^\nu \dot{\delta}
  \]  
  \[(11)\]
- Evaluation of the impulse ratios
  \[
  \alpha_{\gamma/\beta} = \frac{\int_0^{t_f} f_{\gamma}(t) \, dt}{\int_0^{t_f} f_{\beta}(t) \, dt}
  \]  
  \[(12)\]
The unidimensional chain of n balls (Continued ...)

Major results:

- The ratio of impulses is finite and the subspace \( E = \{ \delta \geq 0, \dot{\delta} \geq 0 \} \) is globally attractive.
- The impulse ratios are independent of the absolute value of stiffness and masses but only function of the ratio of stiffness and mass.
- In the linear case \( \nu = 1 \), the impulse ratios are completely determined by the natural modes of the regularized dynamical system and the pre-impact velocities.

Conclusion:

- The definition of the impulse ratio is relevant in the rigid limit.
- The impulse ratio provide us with some informations on the dynamical process.
- The independence of the absolute value of stiffness avoids stiff numerical problem.
The pool: A bidimensional chain of balls

3-balls in the plane
The pool: A bidimensional chain of balls

Solution for $\tau = 0$ and $v_1 = 1$

\[
\begin{align*}
  p_1 &= \frac{1 + \sqrt{\Delta}}{-2\alpha_{12}\cos\theta + 2\alpha_{12}^2 + 2} \\
  p_2 &= \frac{1 + \sqrt{\Delta}}{-2\alpha_{12}\cos\theta + 2\alpha_{12}^2 + 2} \\
  \dot{x}_1^+ &= \cos\theta(1 - \frac{1 + \sqrt{\Delta}}{-2\alpha_{12}\cos\theta + 2\alpha_{12}^2 + 2}) \\
  \dot{y}_1^+ &= \sin\theta(1 - \frac{1 + \sqrt{\Delta}}{-2\alpha_{12}\cos\theta + 2\alpha_{12}^2 + 2}) \\
  \dot{x}_2^+ &= (\cos\theta - \alpha_{12})(\frac{1 + \sqrt{\Delta}}{-2\alpha_{12}\cos\theta + 2\alpha_{12}^2 + 2}) \\
  \dot{y}_2^+ &= \sin\theta(\frac{1 + \sqrt{\Delta}}{-2\alpha_{12}\cos\theta + 2\alpha_{12}^2 + 2}) \\
  \dot{x}_3^+ &= \alpha_{12}(\frac{1 + \sqrt{\Delta}}{-2\alpha_{12}\cos\theta + 2\alpha_{12}^2 + 2}) \\
  \dot{y}_3^+ &= 0 \\
  \text{with} \\
  \Delta &= 2(e - 1)(\alpha_{12}^2 - \alpha_{12}\cos\theta + 1) + 1
\end{align*}
\]
Valid domain for $e$ and $\alpha$

Bounds imposed by $\Delta > 0$ and unilateral constraints

Valid domain of $e$ and $\alpha$ for $\theta = \frac{\pi}{4}$.
Influence of the angle $\tau$.

- The angle $\tau$ determines the occurrence of an impact (single or multiple).

\[-v^-_1 \in T_1 = \{p| - \nabla h_1(q).p \geq 0\}. \quad (13)\]

- The model leads to consistent results whatever the angle $\tau$.

Influence of the angle $\theta$ for $\tau$ such that $-v^-_1 \in T_1$:

1. $\theta \in [0, \frac{\pi}{2})$, one multiple impact
   - We obtain a unique solution. The set of solutions is parametrized by $\epsilon$ and $\alpha$.

2. $\theta \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right)$, two single impacts
   - We must choose a correct value, i.e. $\alpha_{12} = 0$ to have consistent results.
   - or
   - We must choose two standards law for single impact.

3. $\theta = \frac{2\pi}{3}$, The Bernoulli’s problem
   - A impulse ratio $\alpha_{13} = p_3/p_1$ is added.
   - The choice $\alpha_{12} = \alpha_{13}$ leads to consistent results in the symmetric case (same masses and stiffnesses.).
Preliminary conclusion

✦ Validity of the model.
  • In the case of one multiple impact, the set of solutions which respect the equations of motion, the unilateral constraints and the energetic balance is correctly parametrized by the parameter $\epsilon$ and $\alpha_{12}$,
  • In the case of two single impacts, standard single laws must be applied or correct values of the impulse ratios must be chosen.

✦ Open problems
  • How to separate multiple or single impact cases?
  • How to write a single model for all impacting cases?
1 – Motivations
2 – Chain of balls
3 – Lagrangian systems
   3.1 – General model
   3.2 – Angles between constraints
   3.3 – Illustration on an open chain of balls
4 – Conclusion and perspectives
General model

- equations of motion at the instant of impact.

\[ M(q^+ - q^-) = P, \quad q \in \mathbb{R}^n \]  

\[ (14) \]

- unilateral constraints

\[ h_i(q) \geq 0, \quad i = 1, \ldots, m \]  

\[ (15) \]

- kinematic relations:

\[ v = H\dot{q} \]  

\[ P = H^T p \]  

\[ H = \nabla h(q(t_k)) \]  

\[ (16) \]

\[ (17) \]

\[ (18) \]

- unilateral constraints on the relative velocity and the percussions

if \[ h_i(q) = 0, \quad v_i^+ \geq 0, p_i \geq 0 \]  

\[ (19) \]

if \[ h_i(q) > 0, \quad p_i = 0 \]  

\[ (20) \]
Gradient to a constraint in the kinetic metric:

\[ n_i = \frac{M^{-1}(q) \nabla_q h_i(q)}{\sqrt{\nabla_q h_i(q)^TM(q)^{-1}\nabla_q h_i(q)}} \]  

(21)

Angle between two constraints in the kinetic metric:

\[ \cos \theta_{ij} = \langle n_i, n_j \rangle_M = \frac{\nabla_q^T h_i(q)M^{-1}(q)\nabla_q h_j(q)}{\sqrt{\nabla_q h_i(q)^TM(q)^{-1}\nabla_q h_i(q)}} \frac{\sqrt{\nabla_q h_j(q)^TM(q)^{-1}\nabla_q h_j(q)}}{\sqrt{\nabla_q h_j(q)^TM(q)^{-1}\nabla_q h_j(q)}} \]  

(22)

It is noteworthy that \( \nabla_q^T h_i(q)M^{-1}(q)\nabla_q h_j(q) \) is a coefficient of the Delassus matrix \( H^T M^{-1} H \)

Coupling between two adjacent constraints:

\[ \cos \theta_{ij} = \langle n_i, n_j \rangle_M < 0 \implies \text{coupling} \quad \alpha_{ij} = p_i/p_j \]  

(23)

\[ \cos \theta_{ij} = \langle n_i, n_j \rangle_M \geq 0 \implies \text{no coupling due to unilaterality} \]  

(24)
Illustration on an open chain of balls
Illustration on an open chain of balls

First criteria of impact (single or multiple), the velocity $v_1$ must belong to the opposite of the tangent cone $T_1(q)$

$$-v_1^- \in T_1 = \{ p \mid -\nabla h_1(q).p \geq 0 \}. \tag{26}$$

To have a multiple impact of degree $k$, (involving the first $k$-balls) the $k - 1$ first kinetic angle $\theta_{i,i+1}$ must be greater than $\frac{\pi}{2}$:

$$\theta_{i,i+1} > \frac{\pi}{2}, \text{ i.e } \langle n_i, n_{i+1} \rangle_M < 0, i = 1 \ldots k - 1 \tag{25}$$
Outline

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Model with a global energetic coefficient and impulse ratios is a good candidate for propagation of an impact.

Properties of the impulse ratio:
- Well defined in the rigid limit of a regularized model
- Easy to evaluate from the numerical point of view.
- Parametrization of the set of solutions after impact.

A general impact law for multiple and single impacts is a more challenging task:
- How to take into account the kinetic angle between constraints into the formulation?
- How to write a single law valid in all cases?
- How to deal with such law on the numerical point of view?
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