An introduction to non-smooth dynamics and its applications in geomechanics.

GEM Seminar

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Vincent Acary, Inria

Laboratoire Jean Kuntzmann - Inria, centre de l'Université Grenoble Alpes

Outline

INRIA Tripop Team

INRIA

French national institute for computer sciences, applied mathematics and automatic control.

TRIPOP team-project

- Research object:
  Modeling, Simulation and Control of Nonsmooth Dynamics.
- Current main application: natural gravitational risks in mountains :
  - rockfall, rock slope stability, rock avalanche, landslides and debris flows
  - design of protection structures

Motivation for nonmooth dynamics

- Many mechanical systems involve abrupt changes:
  - Mechanical systems with impacts (e.g., bouncing ball)
  - Friction and stick-slip motion
  - Plasticity, damage, fracture, ...
- Traditional smooth differential equations are sometimes insufficient for
  - modeling thresholds and inequalities
  - numerical simulation of abrupt changes
- Nonsmooth dynamics provides a framework to
  - model and analyze such systems.
  - efficiently simulate such systems (robust and energy preserving time–integration).

Nonsmooth	Dynamics

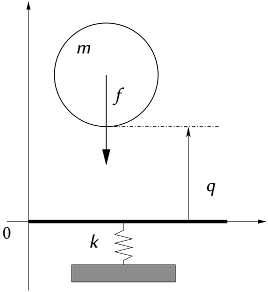
- nonsmooth = lack of differentiability ($\notin \mathcal{C}^1$),
- graphs with peaks, kinks, jumps.

- systems that evolves with time,
- branch of mechanics concerned with the motion of objects.

Where is nonsmoothness?

- nonsmooth solutions in time and space:
  - continuous, functions of bounded variations, measures and distributions.
- nonsmooth modeling of constitutive laws:
  - set–valued mapping, inequality constraints, complementarity, impact laws,
  - ODE with discontinuous r.h.s, differential inclusion, measure equation.

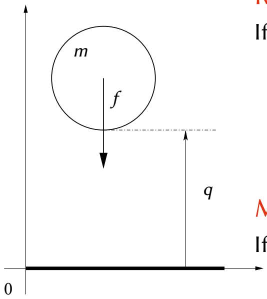
A famous nonsmooth dynamical system: the bouncing ball



A compliant model with piecewise linear stiffness

$$m\ddot{q}(t) = f(t) + \lambda = f(t) + \left\{ \begin{array}{lll} -kq(t) & \text{if} & q(t) < 0 \\ 0 & \text{if} & q(t) \geqslant 0 \end{array} \right. \tag{1}$$

A famous nonsmooth dynamical system: the bouncing ball



Rigid limit

If we let $k \to +\infty$ (rigid contact with restitution) we get

$$\begin{cases} m\ddot{q}(t) = f(t) + \lambda(t) \\ 0 \leqslant q(t) \perp \lambda(t) \geqslant 0 \end{cases} \tag{2}$$

Mandatory impact law (for discrete systems)

If $\dot{q}(t^-) < 0$ and $q(t) = 0$

$$\dot{q}(t^+) = -e\dot{q}(t^-) \tag{3}$$

Therefore we pass from a piecewise linear system to a complementarity system

What do we gain doing so (compliance replaced by rigidity)?

A famous nonsmooth dynamical system: the bouncing ball

Euler discretization of the compliant system (finite k)

$$\left\{\begin{array}{l} \dfrac{v_{i+1}-v_i}{h} = kq_{i+1} \\ \dfrac{q_{i+1}-q_i}{h} = v_i \end{array}\right. \Leftrightarrow \begin{pmatrix} v_{i+1} \\ q_{i+1} \end{pmatrix} = \begin{pmatrix} kh^2+1 & kh \\ h & 1 \end{pmatrix} \begin{pmatrix} v_i \\ q_i \end{pmatrix} \tag{4}$$

This problem is **stiff** because the eigenvalues γ_1 and γ_2 of $\begin{pmatrix} kh^2+1 & kh \\ h & 1 \end{pmatrix}$ satisfy $\frac{\gamma_1}{\gamma_2} \rightarrow +\infty$ when $k \rightarrow +\infty$.

$\boxed{\textit{stiff integrators}}$

A famous nonsmooth dynamical system: the bouncing ball

Euler discretization (Moreau–Jean's scheme) of the complementarity system (infinite k)

$$\left\{\begin{array}{l} v_{i+1} - v_i = hf_{i+1} + \lambda_{i+1} \\ 0 \leqslant v_{i+1} + ev_i \perp \lambda_{i+1} \geqslant 0 \\ \\ q_{i+1} = q_i + hv_{i+1} \end{array}\right. \tag{5}$$

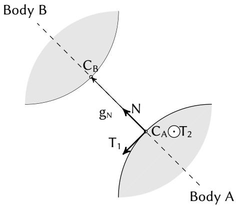
which is nothing else but solving a simple Linear complementarity systems (LCP) (or a quadratic program QP) at each step.

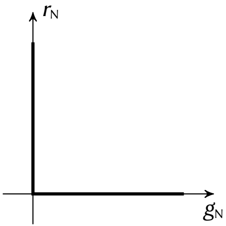
One–sided and Nonsmooth Mechanics. Pioneering work of J.J. Moreau

Irreversible processes in thermodynamics as convex subdifferentials

- Formulation of one-sided and threshold phenomena:
  - admissible (feasible) domains for the state, inequality constraints
- By duality (power), introduction of the force (multipliers):
  - set-valued laws derived from a convex potential thanks to subgradients,
  - potential with values in the completed real line $\mathbb{R}_+ \cup \{+\infty\}$,
  - variational inequalities (normal cone inclusion) $-F(z) \in N_C(z)$,
  - complementarity problems (C is a cone).
- Pseudo–potential of dissipation, $-A \in \partial\varphi(\dot{a})$, φ l.s.c. proper convex:
  - principle of maximum dissipation for friction
  - dual energy principles [Moreau, 1968, 1974].
- Gauss principle with unilateral constraints [Moreau, 1963, 1966]

Unilateral contact and Coulomb friction





- Signorini condition on the gap

$$0 \leqslant g_N \perp r_N \geqslant 0 \Leftrightarrow -r_N \in N_{\mathbb{R}_+}(g_N)$$

- Signorini condition on the velocity

$$\begin{cases} 0 \leqslant u_N \perp r_N \geqslant 0 & \text{if } g_N \leqslant 0 \\ r_N = 0 & \text{otherwise} \end{cases} \Leftrightarrow -\mathbf{r}_N \in N_{T_{\mathbb{R}_+}(g_N)}(u_N)$$

- Impact Law (Newton Impact law), e coefficient of restitution.

$$u_N^+ = -e\, u_N^-$$

- Moreau's Impact Law

$$\begin{cases} 0 \leqslant u_N^+ + e u_N^- \perp \iota_N \geqslant 0 & \text{if } g_N \leqslant 0 \\ \iota_N = 0 & \text{otherwise} \end{cases}$$

$$\Updownarrow$$

$$-\iota_N \in N_{T_{\mathbb{R}_+}(g_N)}(u_N^+ + e u_N^-)$$

Unilateral contact and Coulomb friction

Second order cone complementarity problem

De Saxcé [1992]; Acary and Brogliato [2008]; Acary et al. [2011, 2018].

- Coulomb friction $K = \{r \in \mathbb{R}^3 \mid \|r_{\mathsf{T}}\| \leqslant \mu r_n\}$
  - nonassociated character (loss of monotony) [De Saxcé, 1992]

$$-\hat{u} := -(u + \mu \|u_{\mathsf{T}}\| \mathsf{N}) \in N_K(r)$$

  - Second order cone complementarity condition[1].

$$K^\star \ni \hat{u} \perp r \in K$$

[1] The set $K^\star$ is the dual cone to K defined by $K^\star = \{u \in \mathbb{R}^3 \mid r^\top u \geqslant 0, \quad \text{for all } r \in K\}$.

Unilateral contact and Coulomb friction

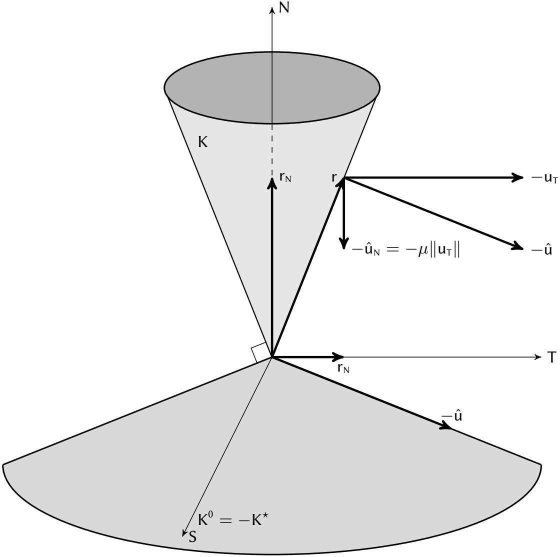


Figure: Coulomb friction and modified relative velocity $\hat{u}$. Sliding case.

Other applications of unilateral and nonsmooth mechanics

Non exhaustive list of applications

- Cavitation in fluids [Moreau, 1964]
  The pressure must be positive or higher than the vaporization pressure
- Plasticity and generalized standard materials [Moreau, 1974, 1976; Halphen and Nguyen, 1975]
  The stresses and strain hardening variables belong to a convex set
- Granular materials [Moreau, 1997, 2001]
- No tension materials and tension field modeling
- Fracture and damage (cohesive zone models)
- Non Newtonian Fluids
  - Quasi-brittle and visco-plastic fluids (Bingham, damage, ...)
  - Multiphase fluid flows,

Modeling for the environment and natural hazards

- Debris flows, avalanches, block falls, threshold fluids, complex rheology
- Coastal swell protection, ice pack modeling, ...

Nonsmooth mechanics

Motivated historically by theoretical mechanics, Convex analysis
is the appropriate tools for modeling and mathematical analysis

With mathematical programming and optimisation,
it paves the way to numerical efficient methods

Nonsmooth Dynamics

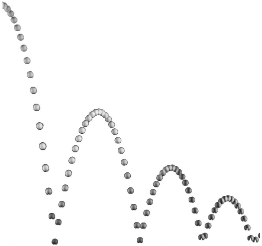
nonsmooth? Késako?

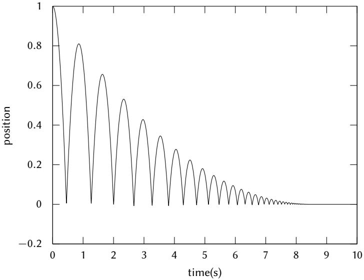
Lack of mathematical regularity of functions.
Everything that is not everywhere differentiable

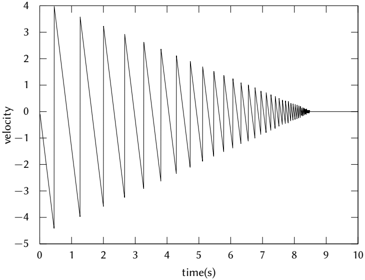
in Mechanics:

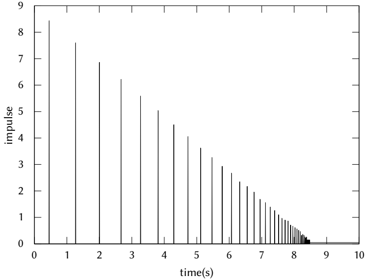
A non-smooth formulation of the laws of constitutive laws
(multi-valued function, inequalities, complementarity)
which can imply non-smooth solutions in time
(angular points, jumps, measures, distributions)

Nonsmooth Dynamics









Nonsmooth Dynamics

Nonsmooth formulation based on differential inclusion

- Writing quasi-static or dynamic evolutions in the form of differential inclusion (parallel research of J.J. Moreau, H. Brézis, M. Schatzman):
  - Second order Moreau's sweeping process
  - Measure differential inclusion
  - The state lies in the space of functions of bounded variations,,
    and its derivatives are differential measures
  - Impact laws as variational inequalities on differential measures

Nonsmooth Dynamics

Efficient numerical methods

- Numerical time integration schemes of these formulations
  - "Event-capturing time–stepping schemes"
  - The discrete variables are the velocities and impulses
  - Iterative solution methods at each time step of the non-smooth and non-convex variational problem based on optimization and mathematical programming techniques

One sided constraint as an inclusion

Definition (Dynamics with perfect one-sided constraints)

[Moreau, 1988]]

$$\begin{cases} \dot{q} = v \\ \\ M(q)\dfrac{dv}{dt} + F(q,v) = r \\ \\ -r \in N_{\mathcal{C}(t)}(q) \end{cases} \tag{6}$$

where r is the generalized reaction force.

- Extension of Lagrange equations with one-sided constraints
- Second order differential inclusion (relatice degree 2)
- The constraints are said to be perfect since their work is vanishes (Normality law in coordinates.)

Nonsmooth Lagrangian Dynamics

Fundamental assumptions

- The velocity $v = \dot{q}$ is a function of bounded variations. The unknown of the equation of motion is its right limit.

$$v^+ = \dot{q}^+ \tag{7}$$

- The coordinate q is an absolutely continuous function by the Lebesgue fundamental Theorem of integration:

$$q(t) = q(t_0) + \int_{t_0}^{t} v^+(t)\,dt \tag{8}$$

- The acceleration ($\dot{v} = \ddot{q}$ in the usual sense) is a differential measure associated with v such that

$$dv(]a,b]) = \int_{]a,b]} dv = v^+(b) - v^+(a) \tag{9}$$

Nonsmooth Lagrangian Dynamics

Definition (Nonsmooth Lagrangian Dynamics [Moreau, 1988])

$$\begin{cases} M(q)dv + F(q, v^+)dt = \iota \\ \\ v^+ = \dot{q}^+ \end{cases} \tag{10}$$

where ι is the generalized reaction measure

Advantages

- The formulation allows to take into account complex behaviors such as finite accumulations in time (Zenon phenomenon)
- The formulation is useful for mathematical analysis [Schatzman, 1973, 1978; Monteiro Marques, 1993; Ballard, 2000]
- The non-smooth dynamics contains both the impact equations and the equations of continuous motion

Impact Equations and equations of motion

Using the densities of the differential measures, with respect to the Lebesgue measure and the discrete measures, we obtain

Définition (Impact equations at any time)

$$M(q)(v^+ - v^-)d\nu = p d\nu, \qquad avec\ p = \frac{d\iota}{d\nu} \tag{11}$$

or, equivalently,

$$M(q(t_i))(v^+(t_i) - v^-(t_i)) = p_i, \tag{12}$$

Définition (Continuous dynamics almost-everywhere)

$$M(q)\dot{v}dt + F(q,v)dt = f dt \qquad avec\ f = \frac{d\iota}{dt} \tag{13}$$

or, equivalently,

$$M(q)\dot{v}^+ + F(q,v^+) \quad = \quad f^+ \quad [dt - a.e.] \tag{14}$$

Second order Moreau's sweeping process

Définition (Moreau [1983, 1988])

The keystone of the formulation is the inclusion of measures at the velocity level:

$$\begin{cases} M(q)dv + F(t, q, v^+)dt = \iota \\ \\ v^+ = \dot{q}^+ \\ \\ -\iota \in N_{T_C(q)}(v^+) \end{cases} \tag{15}$$

Comments

An inclusion that involves measures

A single framework for non-smooth dynamics with inelastic impacts.

➜ Foundations of the numerical scheme of Moreau-Jean

Principles of event–capturing schemes

1. A unified formulation

$$\begin{cases} -mdv + fdt = \iota \\ \dot{q} = v^+ \\ 0 \leqslant \iota \perp v^+ \geqslant 0 \text{ si } q \leqslant 0 \end{cases} \tag{16}$$

2. A consistent integration

$$\int_{]t_k,t_{k+1}]} mdv = \int_{]t_k,t_{k+1}]} m\,dv = m(v^+(t_{k+1}) - v^+(t_k)) \approx m(v_{k+1} - v_k) \tag{17}$$

3. An consistent approximation with the measure differential inclusion-measure

$$0 \leqslant \iota \perp v^+ \geqslant 0 \text{ si } q \leqslant 0 \quad \rightarrow \quad \begin{cases} p_{k+1} \approx \displaystyle\int_{]t_k,t_{k+1}]} \iota \\ 0 \leqslant p_{k+1} \perp v_{k+1} \geqslant 0 \quad \text{if } \tilde{q}_k \leqslant 0 \end{cases} \tag{18}$$

Moreau-Jean's scheme

[Jean and Moreau, 1987; Moreau, 1988; Jean, 1999]

$$
\begin{cases}
M(q_{k+\theta})(v_{k+1} - v_k) - hF_{k+\theta} = p_{k+1} = G(q_{k+\theta})P_{k+1}, \\
q_{k+1} = q_k + hv_{k+\theta}, \\
u_{k+1} = G^T(q_{k+\theta})\, v_{k+1} \\
\begin{array}{ll} 0 \leqslant u^\alpha_{k+1} + eU^\alpha_k \perp P^\alpha_{k+1} \geqslant 0 & \text{if} \quad \bar{g}^\alpha_{k,\gamma} \leqslant 0 \\ P^\alpha_{k+1} = 0 & \text{otherwise} \end{array}
\end{cases}
\tag{19}
$$

with

- $G(q) = \nabla_q g(q)$
- $\theta \in [0, 1]$
- $x_{k+\theta} = (1-\theta)x_{k+1} + \theta x_k$
- $F_{k+\theta} = F(t_{k+\theta}, q_{k+\theta}, v_{k+\theta})$
- $\bar{g}_{k,\gamma} = g_k + \gamma h U_k, , \gamma \geqslant 0$

An optimization problem is solved at each time–step.

Moreau-Jean's scheme

Advantages

a consistent and stable scheme that is robust
that satisfies some invariants in discrete time:
equilibrium, energy, dissipation, ...

Recent improvements

- Nonsmooth generalized–α schemes [Chen et al., 2013; Brüls et al., 2014]
- Time discontinuous Galerkin methods [Schindler and Acary, 2013; Schindler et al., 2015]
- Stabilized index-2 formulation [Acary, 2014, 2013]
- Stabilized index-1 formulation [Brüls et al., 2018]
- Discrete variational integrators, geometric and symplectic properties [Capobianco and R. Eugster, 2016; Capobianco and Eugster, 2018]

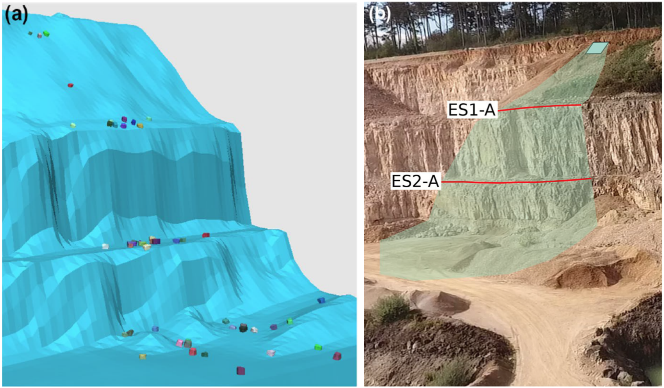
Applications in geomechanics. Gravity driven hazards.

Le village suisse de Blatten a été transformé en un amas de roches et un lac artificiel par l'éboulement du glacier - *© Alexandre Agrusti / AFP*

- Rock fall simulation and protective structures
- Granular flows for avalanches (snow, rock, debris)
- Nonassociated plasticity with contact and friction in Material Point Method (MPM)
- Fracture and Cohesive Zone Model (CZM), and rock permafrost instability.

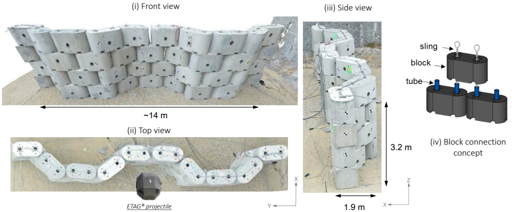
Rock fall simulations and protective structures

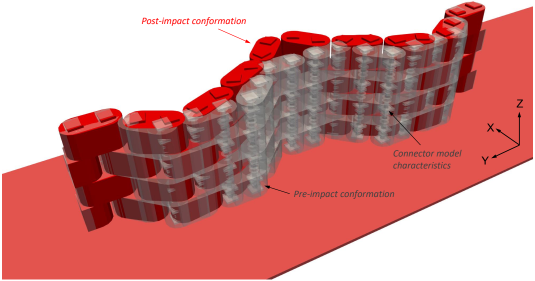
Joint work with Franck Bourrier (INRAe)



- 3D rigid bodies simulation with arbitrary shapes.
- Improvement of contact laws including rolling friction
- Calibration on real case studies
- Statistical and sensitivity analysis.

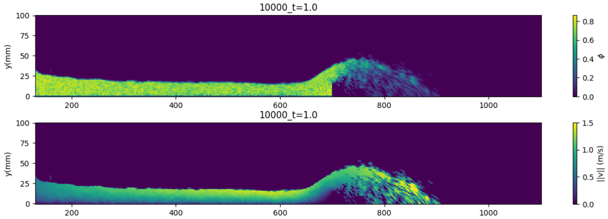
Rock fall simulations and protective structures





- Collaboration with Géolithe and INRAe

Smooth and nonsmoth DEM, validation of $\mu(I)$ model on obstacles with FEM



Collaboration with T. Faug., F. Bourrier. M. Oziol PhD

Objectives

- Modeling of heavy, wet and dense avalanches by DEM
- Impact on obstacles (protective structures) as an continuum media by FEM
- Comparison of smooth DEM (Yade) and nonsmooth DEM (Siconos): computation of stresses, strains, velocity profiles, porosity, forces on obstacles.
- Validation of $\mu(I)$, $\phi(I)$ model in the flow, near the obstacle (dead zone)
- 3D Simulation on real case obstacles.
- Learning of constitutive laws with TANNS (F. Masi)

Plasticity in the generalized standard materials framework

Simplest framework: Small perturbation, associative plasticity and linear hardening

- Small perturbations hypothesis with additive decomposition of the strain

$$\varepsilon = \varepsilon^e + \varepsilon^p. \tag{20}$$

- Linear elasticity and hardening laws

$$\boldsymbol{\sigma} = \mathbf{E} : \varepsilon^e \quad \text{and } \boldsymbol{a} = \mathbf{D} \cdot \boldsymbol{\alpha}. \tag{21}$$

- Generalized standard material (GSM) (associative plasticity)

$$\begin{pmatrix} \dot{\varepsilon}^p \\ \dot{\boldsymbol{\alpha}} \end{pmatrix} \in N_{\boldsymbol{C}} \begin{pmatrix} \boldsymbol{\sigma} \\ \boldsymbol{a} \end{pmatrix}. \tag{22}$$

$\boldsymbol{C}(\boldsymbol{\sigma}, \boldsymbol{a})$ a convex set of admissible stresses $\boldsymbol{\sigma}$ and hardening forces $\boldsymbol{a}$

- Clausius-Duhem dissipation inequality is automatically satisfied

$$d = \boldsymbol{\sigma} : \dot{\varepsilon}^p + \boldsymbol{a} \cdot \dot{\boldsymbol{\alpha}} \geqslant 0 \text{ if } 0 \in C. \tag{23}$$

- Extension to nonassociated materials with implicit standard material: bipotential approach (G. de Saxcé *et al.*)

Elasto-dynamics with plasticity, contact and impact.

A second order sweeping process

FEM or MPM discretization yields

$$\begin{cases} v^+ = \dot{q}^+ & \text{(velocity of bounded variations)} \\ M(q)dv + F(q, v^+)dt + B^\top \sigma dt = \iota & \text{(differential measure)} \\ \dot{\sigma} = E(Bv - \dot{\varepsilon}^p) & \text{(elasticity)} \\ \dot{\varepsilon}^p \in N_C(\sigma) & \text{(plasticity)} \\ -\iota \in N_{T_M(q)}(v^+ + ev^-) & \text{(impact and contact)} \end{cases}$$

→ Implicit monolithic solver based on optimisation

MPM, plasticity and contact for debris flows

Louis Guillet PhD (F. Bourrier, O. Goury)

Objectives

- Simulation of landslides and debris flows (elasto-plastic fluids + rocks + debris)
- Non-associative plasticity (Drucker–Prager, Mohr–Coulomb) with controlled dilatency
- Contact, impact and Coulomb's friction (non-associated)
- Transition from instability to flows
- Monolithic solver based on (non-monotone) variational inequalities and complementarity problems: semi-smooth Newton methods, interior point methods, first-order accelerated methods
- Existence, convergence, energy consistency

MPM, plasticity and contact for debris flows

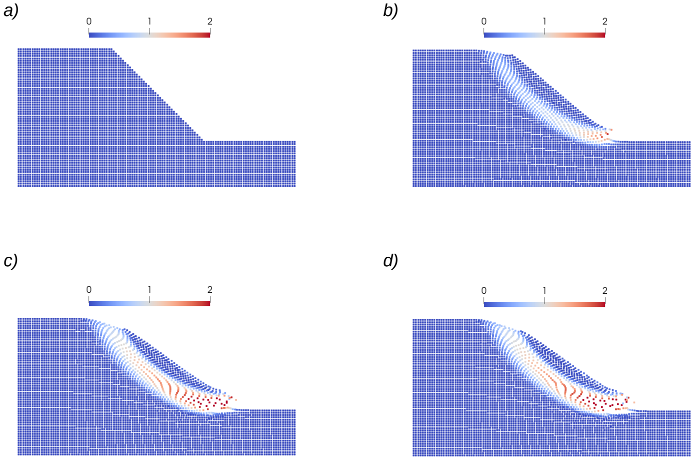


Figure: Cumulative plastic strains for an associated material ($\theta = \phi$) for different simulation times: a) $t = 0$ s, b) $t = 1.7$ s, c) $t = 3.3$ s, d) $t = 5$ s.

MPM, plasticity and contact for debris flows

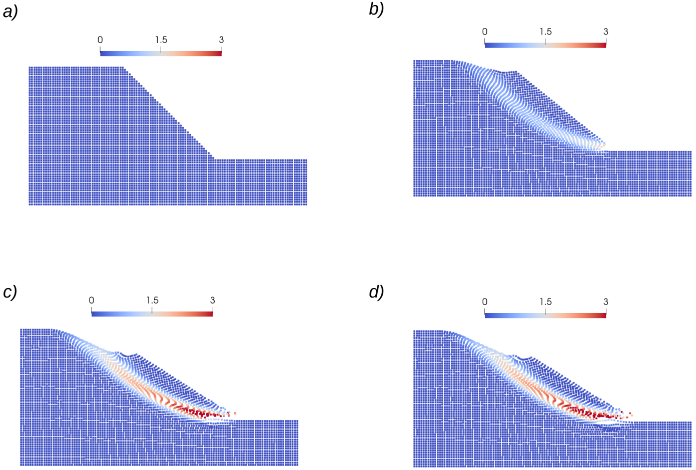
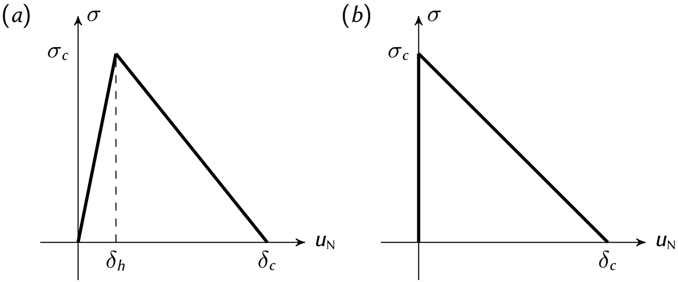


Figure: Cumulative plastic strains for a strongly non-associated material ($\theta = 0^\circ$) and for different simulation times: a) $t = 0$ s, b) $t = 1.7$ s, c) $t = 3.3$ s, d) $t = 5$ s.

Fracture. Extrinsic and intrinsic cohesive zone models

Joint work with N. Collins Craft and F. Bourrier.

An intrinsic CZM (a) and an extrinsic CZM (b)

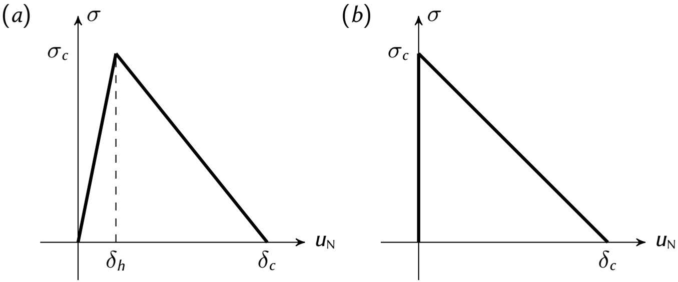


- Intrinsic models : initial stiffness in the interface.
  - σ is a function of u_{N}
  - difficulty to give a value to the initial stiffness
  - modify the elasticity of the material prior to the crack
  - the effect is worse with a lot of interfaces (FEM applications)
  - need to use high initial stiffness value that implies numerical difficulties (stiff ODE systems)

Fracture. Extrinsic and intrinsic cohesive zone models

Joint work with N. Collins Craft and F. Bourrier.

An intrinsic CZM (a) and an extrinsic CZM (b)



- Extrinsic models : initially rigid, perfect bond, bilateral constraint.
  - the model is set-valued (like unilateral contact)
  - keep the original elasticity of the material.
  - bilateral constraints rather penalty (no stiff ODE)

Fracture. An extrinsic cohesive zone model

Free energy potential

$$\Psi_s(u_{\mathrm{N}}, u_{\mathrm{T}}, \beta) = \underbrace{\beta\sigma_c u_{\mathrm{N}} + \beta\gamma\sigma_c|u_{\mathrm{T}}|}_{\text{potential energy}} + \underbrace{wf(\beta)}_{\text{fracture energy}} + \underbrace{\mathcal{I}_{[0,1]}(\beta)}_{\text{constraints on }\beta} + \underbrace{\mathcal{I}_{\mathbb{R}^+}(u_{\mathrm{N}})}_{\text{unilateral contact}}$$

- w is the free energy released by the decohesion
- $f(\beta)$ is a function that describes the “shape” of the cohesive law,
- γ is the ratio of critical traction in mode II to mode I

Remarks

- unilateral contact is contained in the model
- β is constrained to be $[0, 1]$
- the potential energy related to u_{N} and u_{T} is piece linear to avoid the introduction of elasticity

An extrinsic cohesive zone model

State laws, constitutive laws for reversible processes

$$
\left\{
\begin{array}{rcl}
-r^{\mathrm{r}}_{\mathrm{N}} & \in & \partial_{u_{\mathrm{N}}} \Psi_s(u_{\mathrm{N}}, u_{\mathrm{T}}, \beta) = \beta\sigma_c + \partial\mathcal{I}_{\mathbb{R}^+}(u_{\mathrm{N}}), \\
-r^{\mathrm{r}}_{\mathrm{T}} & \in & \partial_{u_{\mathrm{T}}} \Psi_s(u_{\mathrm{N}}, u_{\mathrm{T}}, \beta) = \beta\gamma\sigma_c \mathrm{sgn}(u_{\mathrm{T}}), \\
-A^{\mathrm{r}} & \in & \partial_{\beta} \Psi_s(u_{\mathrm{N}}, u_{\mathrm{T}}, \beta) = \sigma_c\,(u_{\mathrm{N}} + \gamma|u_{\mathrm{T}}|) + wf'(\beta) + \partial\mathcal{I}_{[0,1]}(\beta),
\end{array}
\right.
$$

where A is the thermodynamic driving force associated with the cohesion state β.

- unilateral contact with cohesion

$$-(r^{\mathrm{r}}_{\mathrm{N}} + \beta\sigma_c) \in \partial\mathcal{I}_{\mathbb{R}^+}(u_{\mathrm{N}}) \Longleftrightarrow 0 \leqslant r^{\mathrm{r}}_{\mathrm{N}} + \beta\sigma_c \perp u_{\mathrm{N}} \geqslant 0$$

- set-valued tangential cohesion

$$-r^{\mathrm{r}}_{\mathrm{T}} \in \beta\gamma\sigma_c \mathrm{sgn}(u_{\mathrm{T}})$$

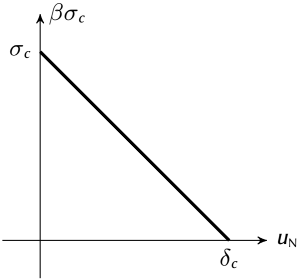
- cohesion state law

$$-(A^{\mathrm{r}} + \sigma_c\,(u_{\mathrm{N}} + \gamma|u_{\mathrm{T}}|) + wf'(\beta)) \in \partial\mathcal{I}_{[0,1]}(\beta)$$

An extrinsic cohesive zone model

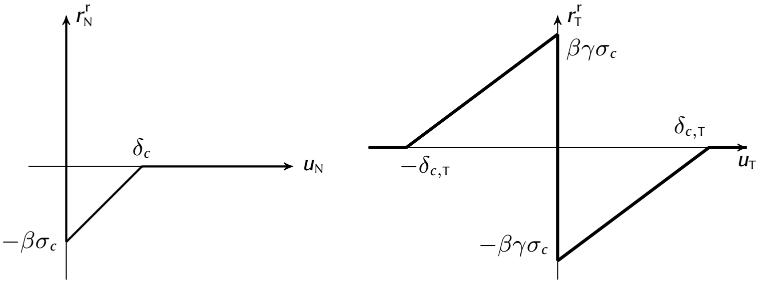
A simple triangle law as state cohesion law

$$f(\beta) = (\beta^2 - 1), \qquad w = \frac{\sigma_c \delta_c}{2}$$



w area under the curve, free energy earns by the system.

An extrinsic cohesive zone model



An extrinsic cohesive zone model

Irreversible process (2D)

$$\Phi(v_{\mathrm{N}}, v_{\mathrm{T}}, \dot{\beta}) = \underbrace{\mathcal{I}_{\mathbb{R}^-}(\dot{\beta})}_{\text{fracture irreversibility}} + \underbrace{\mu(r_{\mathrm{N}} + \beta\sigma_c)|v_{\mathrm{T}}|}_{\text{dissipation by friction}} \tag{24}$$

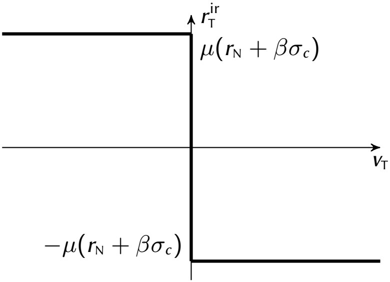
Comments

- the decohesion process is irreversible ($\dot{\beta} \leqslant 0$) but not dissipative and rate-independent.
- the friction threshold accounts for the cohesion force,

An extrinsic cohesive zone model

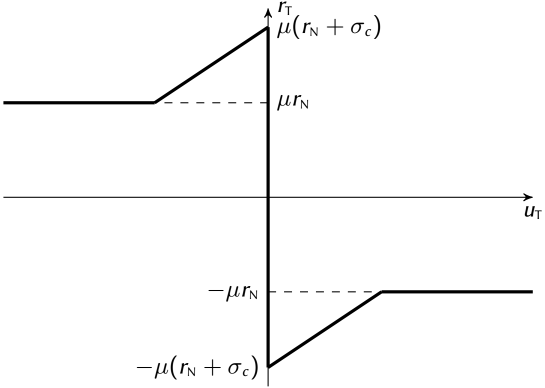
Irreversible process (2D). Constitutive laws

$$
\begin{aligned}
-r_{\mathrm{N}}^{\mathrm{ir}} &= \partial_{v_{\mathrm{N}}}\Phi(v_{\mathrm{N}}, v_{\mathrm{T}}, \dot{\beta}) = 0,\\
-r_{\mathrm{T}}^{\mathrm{ir}} &\in \partial_{v_{\mathrm{T}}}\Phi(v_{\mathrm{N}}, v_{\mathrm{T}}, \dot{\beta}) = \mu(r_{\mathrm{N}} + \beta\sigma_c)\mathrm{sgn}(v_{\mathrm{T}}),\\
-A^{\mathrm{ir}} &\in \partial_{\dot{\beta}}\Phi(v_{\mathrm{N}}, v_{\mathrm{T}}, \dot{\beta}) = \partial\mathcal{I}_{\mathbb{R}^-}(\dot{\beta}).
\end{aligned}
$$



An extrinsic cohesive zone model

The net tangential behaviour assuming $u_T = \pm v_T t$



Comments

The tangential depends on two separated terms: a cohesion forces that depends on displacement and a frictional forces that depends on the velocity.

Instability phenomena linked to warming in ice–filled permafrost rock

Rockfall at Mel de la Niva. Evolène, Switzerland, October 18, 2015.

Chloé Gergely PhD. (F. Bourrier)

Objectives

- Run-out of the granular flows with fragmentation.
- Particle breakage modeling cohesive zone model(CZM) and with unilateral contact and friction
- Characterize the shape and the volume of the “big blocks” after fragmentation
- understanding and quantifying the effect of temperature on the stability of permafrost rock mass
- Extrinsic CZM models taking into account the effect of heat (and temperature) on the mechanical properties of interface
- Coupling with heat equation in the rock mass
- Understanding whether other phenomena need to be added (freeze/thaw cycle, water flow and porous media, . . .)

Conclusions

Nonsmooth dynamics a framework :

- to model one-sided and threshold effects,
- to give a rigorous mathematical setting prone to results, and
- to enable the design of powerful numerical tools,

with relevant application to gravity flows.

Thank you for your attention

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